Workshop on Fields, Strings and Gravity, Seoul, February 22-23, 2016.

Multiphoton amplitude and generalized LKF transformation in scalar QED using worldline formalism

Naser Ahmadiniaz

Institute for Basic Science (IBS)
Center for Relativistic Laser Science (CoReLS), Gwangju, Korea

In collaboration with C. Schubert and A. Bashir

Outline

- History and introduction.
- Worldline formalism for scalar propagator.
- Multiphoton amplitud in scalar QED (master formula).
- Constructing one-loop correction to scalar propagator and its vertex.
- Gauge transformation for photons in configuration space (LKFT).
- Gauge transformation for internal photons in momentum space.
- Conclusion.

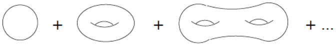
History and introduction

One of the main motivations for study of string theory is the fact that it reduces to quantum field theory in the limit where the tension along the string becomes infinite. In this limit all massive modes of the string get suppressed, and one remains with the massless modes. Those can identified with ordinary massless particles such as gauge bosons, gravitons, or massless spin-$\frac{1}{2}$ fermions.

The basic tool for the calculation of string scattering amplitudes is the Polyakov path integral. In the simplest case, the closed bosonic string propagating in flat spacetime, this integral is of the form

$$\langle V_1 V_2 \cdots V_N \rangle \sim \sum_{\mathrm{top}} \int \mathcal{D}h \int \mathcal{D}x(\sigma,\tau) V_1 V_2 \cdots V_N \; \mathrm{e}^{-S[x,h]}$$

This path integral corresponds to first quantization in the sense that it describes a single string propagating in a given background. The parameters σ, τ parametrize the world sheet surface swept out by the string in its motion, and the integral $\int \mathcal{D}x(\sigma,\tau)$ has to be performed over the space of all embeddings of the string world sheet with a fixed topology into spacetime. The integral $\int \mathcal{D}h$ is over the space of all world sheet metrics, and the sum over topologies $\sum_{\mathrm{top}}$ corresponds to the loop expansion in field theory, see the figure below:

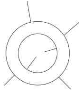


The loop expansion in string perturbation theory

In the case that the background is simply Minkowski spacetime the world sheet action is given by:

$$S[x,h] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu$$

where $\frac{1}{2\pi\alpha'}$ is the string tension. The vertex operators $V_1, \cdots, V_N$ represent the scattering string states. In the case of the open string, which is the more relevant one for our purpose, the world sheet has a boundary, and the vertex operators are inserted on this boundary. For instance, for the open oriented string at the one-loop level the world sheet is just an annulus, and a vertex operator may be integrated along either one of the two boundary components.



Vertex operators inserted on the boundary of the annulus

The vertex operators most relevant for us are of the form

$$V^A[k,\varepsilon,a] = \int d\tau\, T^a \varepsilon \cdot \dot{x}(\tau)\, \mathrm{e}^{ik\cdot x(\tau)}$$

They represent a scalar and a gauge boson particle with definite momentum k and polarization vector ε. T^a is a generator of the gauge group in some representation. The integration variable τ parametrizes the boundary in question. Since the action is Gaussian, $\int \mathcal{D}x(\tau)$ can be performed by **Wick contractions**,

$$\langle x^\mu(\tau_1) x^\nu(\tau_2)\rangle = G(\tau_1,\tau_2)\eta^{\mu\nu}$$

G denotes the **Green's function** for the Laplacian on the annulus, restricted to its boundary, and $\eta^{\mu\nu}$ the Lorentz metric, for a review see C. Schubert 2001.

String-inspired formalism

Bern-Kosower master formula (Z. Bern and D. Kosower 1991)
In their analysis of the N-gluon amplitude, Bern and Kosower therefore used, instead of the open string, a certain heterotic string model containing $SU(Nc)$ Yang-Mills theory in the infinite string tension limit. This allows for a consistent reduction to four dimensions, at the price of a more complicated representation of this amplitude. By an explicit analysis of the infinite string tension limit, they succeeded in deriving a novel type of parameter integral representation for the on-shell N - gluon amplitude in Yang-Mills theory, at the tree- and one-loop level. Moreover, they established a set of rules which allows one to construct this parameter integral, for any number of gluons and choice of helicities, without referring to string theory any more.

$$\begin{aligned}\Gamma^{a_1\cdots a_N}[p_1,\varepsilon_1;\ldots;p_N,\varepsilon_N] &= (-ig)^N \mathrm{tr}(T^{a_1}\ldots T^{a_N})\int_0^\infty dT(4\pi T)^{-D/2}e^{-m^2T}\\ &\times\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\ldots\int_0^{\tau_{N-2}}d\tau_{N-1}\\ &\times\exp\left\{\sum_{i,j=1}^N\left[\frac{1}{2}G_{Bij}p_i\cdot p_j - i\dot G_{Bij}\varepsilon_i\cdot p_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\right]\right\}\Big|_{\mathrm{lin}(\varepsilon_1\ldots\varepsilon_N)}\end{aligned}$$

As it stands, this is a parameter integral representation for the (color-ordered) N - gluon vertex, with momenta p_i and polarizations ε_i, induced by a scalar loop, in D dimensions. Here m and T are the loop mass and proper-time, τ_i the location of the ith gluon, and

$$\begin{aligned}G_{Bij} &= |\tau_i-\tau_j| - \frac{(\tau_i-\tau_j)^2}{T} \quad , \quad \dot G_B(\tau_1,\tau_2)=\mathrm{sign}(\tau_1-\tau_2)-2\frac{(\tau_1-\tau_2)}{T}\\ \ddot G_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2)-\frac{2}{T}\,.\end{aligned}$$

Let us just mention some advantages of the **Bern-Kosower Rules** as compared to the Feynman rules:

- Superior organization of gauge invariance.
- Absence of loop momentum, which reduces the number of kinematic invariants from the beginning, and allows for a particularly efficient use of the spinor helicity method.
- The method combines nicely with spacetime supersymmetry.
- Calculations of scattering amplitudes with the same external states but particles of different spin circulating in the loop are more closely related than usual.

Since the Bern-Kosower rules do not refer to string theory any more, the question naturally arises whether it should not be possible to re-derive them completely inside field theory. Obviously, such a re-derivation should be attempted starting from a first-quantized formulation of ordinary field theory, rather than from standard quantum field theory.

Worldline formalism

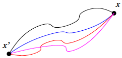
In 1948, Feynman developed the path integral approach to non-relativistic quantum mechanics (based on earlier work by Wentzel and Dirac). Two years later, he started his famous series of papers that laid the foundations of relativistic quantum field theory (essentially quantum electrodynamics at the time) and introduced Feynman diagrams. However, at the same time he also developed a representation of the QED S-matrix in terms of relativistic particle path integrals.

Why worldline formalism?

- No need to compute momentum integrals and Dirac traces.
- Worldline formalism works well for massive particles (on- and off-shell) not even at tree-level but at loop order too.

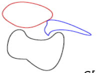
The difference between open line and loop:

- Dirichlet boundary conditions (topology of a line)



$$\langle x|e^{-HT}|x'\rangle = \int_{x(0)=x'}^{x(T)=x} Dx(\tau)\, e^{-S[x,G]}$$

- Periodic boundary conditions (topology of a closed line)



$$\int_{x(0)=x(T)} Dx(\tau) e^{-S[x,G]}$$

We have in mind three main purposes:

1. First, in on-shell amplitudes, the multi-photon generalizations of Compton scattering are becoming important these days for laser physics, for a review on high-intensity laser QED, see: A. Di Piazza et.al Rev. Mod. Phys **84**, 1177 (2012).

2. Second, for off-shell amplitudes, computation of form factors in Scalar QED where the main interest is in QCD and spinor QED, but it is always good to study scalar QED as the simplest nontrivial gauge theory in four dimensions.

3. And third, efficient ways of changing from one covariant gauge to another, independently for external and for internal photons (LKF transformation).

Worldline formalism for scalar propagator

In the following, we discuss our method which is based on the worldline formalism, initially developed by Feynman for scalar QED in Phys. Rev. **80**, 440 (1950) and spinor QED in Phys. Rev. **84**, 108 (1951). If we consider a scalar propagator in the presence of a background gauge filed which propagates from point x' to point x

$$\Gamma[x';x] = \int dTe^{-m^2T}\int_{x(0)=x'}^{x(T)=x}\mathcal{D}x(\tau)\, e^{-S_0-S_e-S_i}$$

where

$$\begin{aligned} S_0 &= \int_0^T d\tau \frac{1}{4}\dot{x}^2 \rightarrow \textit{describes the free propagation}\,, \\ S_e &= ie\int_0^T d\tau\, \dot{x}\cdot A(x(\tau)) \rightarrow \textit{the interaction of the scalar with the external field}\,, \\ S_i &= \frac{e^2}{2}\int_0^T d\tau_1\int_0^T d\tau_2\, \dot{x}_1^\mu D_{\mu\nu}(x_1-x_2)\dot{x}_2^\nu \rightarrow \textit{virtual photons exchanged along the scalar's trajectory}\,. \end{aligned} \tag{1}$$

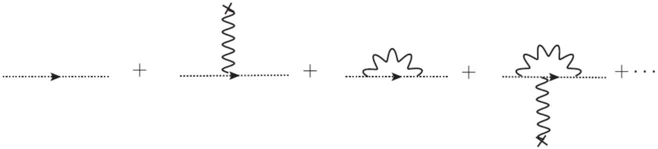
$D_{\mu\nu}$ is the x - space photon propagator in D dimensions. In an arbitrary covariant gauge, it is given by

$$D_{\mu\nu}(x) = \frac{1}{4\pi^{\frac{D}{2}}} \left\{ \frac{1+\xi}{2} \Gamma\left(\frac{D}{2} - 1\right) \frac{\delta_{\mu\nu}}{x^{2\frac{D}{2}-1}} + (1-\xi)\Gamma\left(\frac{D}{2}\right) \frac{x_\mu x_\nu}{x^{2\frac{D}{2}}} \right\}. \tag{2}$$

where

$$\begin{cases} \xi = 1 & \Rightarrow \quad \text{Feynman gauge} \\ \xi = 0 & \Rightarrow \quad \text{Landau gauge} \end{cases}$$

The expansion of the exponentials of the interaction terms Se and Si generates the Feynman diagrams depicted below



The external legs represent interactions with the field $A(x)$, and are converted into momentum-space photons by choosing $A(x)$ as a sum of plane waves,

$$A^\mu(x) = \sum_{i=1}^{N} \varepsilon_i^\mu \; \mathrm{e}^{ik_i \cdot x} \, . \tag{3}$$

Each external photon then gets represented by a *vertex operator*.

$$V^A_{\rm scal}[k,\varepsilon] \equiv \varepsilon_\mu \int_0^T d\tau \, \dot{x}^\mu(\tau) \, \mathrm{e}^{ik\cdot x(\tau)} \, . \tag{4}$$

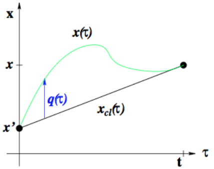
According to our convention, external photon momenta are ingoing.
Note that the integrand in S_i, as defined in Eq. (1), may also be written as

$$\frac{1}{4\pi^{\frac{D}{2}}}\left[\Gamma\Big(\frac{D}{2}-1\Big)\frac{\dot{x}_1\cdot\dot{x}_2}{[(x_1-x_2)^2]^{\frac{D}{2}-1}} - \frac{1-\xi}{4}\Gamma\Big(\frac{D}{2}-2\Big)\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}[(x_1-x_2)^2]^{2-\frac{D}{2}}\right], \tag{5}$$

which shows, already at this level, that a change of the covariant gauge parameter creates only a total derivative term.

The path integral is computed by splitting $x^\mu(\tau)$ into a "background" part $x^\mu_{\rm bg}(\tau)$, which encodes the boundary conditions, and a fluctuation part $q^\mu(\tau)$, which has **Dirichlet** boundary conditions at the endpoints $\tau = 0, T$:

$$
\begin{aligned}
x(\tau) &= x_{\rm bg}(\tau) + q(\tau)\,,\\
x_{\rm bg}(\tau) &= x' + \frac{(x-x')\tau}{T}\,,\\
\dot x(\tau) &= \frac{x-x'}{T} + \dot q(\tau)\,,\\
q(0) &= q(T) = 0\,.
\end{aligned}
$$



The path integral over the fluctuation variable $q(\tau)$ is gaussian, except for the denominators of the photon exchange terms S_i. A fully gaussian representation is achieved by the further introduction of a photon proper-time, rewriting

$$
\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}\Big([x(\tau_a)-x(\tau_b)]^2\Big)^\lambda} = \int_0^\infty d\bar T (4\pi\bar T)^{-\frac{D}{2}} \exp\Bigg[-\frac{\big(x(\tau_a)-x(\tau_b)\big)^2}{4\bar T}\Bigg]. \qquad (6)
$$

The calculation of the path integral then requires only the knowledge of the free path integral normalization, which is

$$
\int \mathcal{D}q(\tau)\, \mathrm{e}^{-\int_0^T d\tau \frac14 \dot q^2} = (4\pi T)^{-\frac{D}{2}}\,, \qquad (7)
$$

and of the two-point correlator, given by

$$
\langle q^\mu(\tau_1) q^\nu(\tau_2)\rangle = -2\delta^{\mu\nu}\Delta(\tau_1,\tau_2) \qquad (8)
$$

with the worldline Green function $\Delta(\tau_i, \tau_j)$,

$$\Delta(\tau_1, \tau_2) \quad = \quad \frac{\tau_1 \tau_2}{T} + \frac{|\tau_1 - \tau_2|}{2} - \frac{\tau_1 + \tau_2}{2} \,. \tag{9}$$

We note that this Green function has a nontrivial coincidence limit

$$\Delta(\tau, \tau) \quad = \quad \frac{\tau^2}{T} - \tau \,, \tag{10}$$

and we will also need its following derivatives:

$$\begin{aligned}
{}^\bullet\Delta(\tau_1, \tau_2) &= \frac{\tau_2}{T} + \frac{1}{2}\mathrm{sign}(\tau_1 - \tau_2) - \frac{1}{2} \,, \\
\Delta^\bullet(\tau_1, \tau_2) &= \frac{\tau_1}{T} - \frac{1}{2}\mathrm{sign}(\tau_1 - \tau_2) - \frac{1}{2} \,, \\
{}^\bullet\Delta^\bullet(\tau_1, \tau_2) &= \frac{1}{T} - \delta(\tau_1 - \tau_2) \,.
\end{aligned} \tag{11}$$

Note that the mixed derivative ${}^\bullet\Delta^\bullet(\tau_1, \tau_2)$ contains a delta function which brings together two photon legs; this is how the seagull vertex arises in the worldline formalism. In the simplest case, for the free scalar propagator, we thus get the following standard proper-time representation in D dimensions:

$$\Gamma_{\mathrm{free}}[x, x'] = \int_0^\infty dT\, \mathrm{e}^{-m^2 T}\, (4\pi T)^{-\frac{D}{2}}\, \mathrm{e}^{-\frac{1}{4T}(x - x')^2} \,. \tag{12}$$

Multiphoton amplitude in scalar QED

Now, let's go back to the worldline formula for scalar propagator:

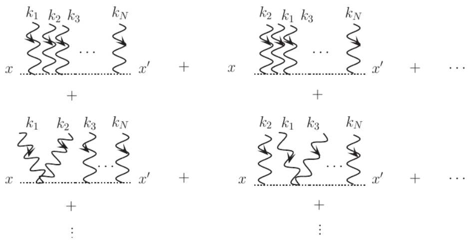
$$\Gamma[x';x] = \int dT e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau[\dot{x}^2+ie\dot{x}\cdot A(x(\tau))]}$$

After expanding the interaction term in the exponential we get

$$\Gamma[x';x;k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2T} \int_{x(0)=x'}^{x(T)=x} \mathcal{D}x(\tau) e^{-\frac{1}{4}\int_0^T d\tau \dot{x}^2} \times \int_0^T \prod_{i=1}^N d\tau_i V^A_{\mathrm{scal}}[k_1,\varepsilon_1]\cdots V^A_{\mathrm{scal}}[k_N,\varepsilon_N]$$

For our purpose, it will be convenient to formally rewrite the vertex operator as

$$V^A_{\mathrm{scal}}[k,\varepsilon] = \int_0^T d\tau \varepsilon\cdot\dot{x}(\tau)\, e^{ik\cdot x(\tau)} = \int_0^T d\tau e^{ik\cdot x(\tau)+\varepsilon\cdot\dot{x}(\tau)}\Big|_{\mathrm{lin}\ \varepsilon}$$


k_1 k_2 k_3 k_N
x x'
+
k_2 k_1 k_3 k_N
x x'
+ ⋯
+
k_1 k_2 k_3 k_N
x x'
+
k_2 k_1 k_3 k_N
x x'
+ ⋯
+
⋮

Substituting this vertex operator, and applying the split, one gets

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2T}\, \mathrm{e}^{-\frac{1}{4T}(x-x')^2} \int_{q(0)=q(T)=0} \mathcal{D}q(\tau)\, e^{-\frac{1}{4}\int_0^T d\tau \dot{q}^2}$$
$$\times \int_0^T \prod_{i=1}^N d\tau_i\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i\cdot\frac{(x-x')}{T} + \varepsilon_i\cdot\dot{q}(\tau_i) + ik_i\cdot(x-x')\frac{\tau_i}{T} + ik_i\cdot x' + ik_i\cdot q(\tau_i)\right)}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{13}$$

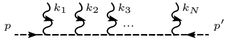
After completing the square in the exponential, we obtain the following tree-level "Bern-Kosower-type formula" in configuration space:

$$\Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N \int_0^\infty dT e^{-m^2T}\, \mathrm{e}^{-\frac{1}{4T}(x-x')^2} (4\pi T)^{-\frac{D}{2}}$$
$$\times \int_0^T \prod_{i=1}^N d\tau_i\, \mathrm{e}^{\sum_{i=1}^N \left(\varepsilon_i\cdot\frac{(x-x')}{T} + ik_i\cdot(x-x')\frac{\tau_i}{T} + ik_i\cdot x'\right)}\, \mathrm{e}^{\sum_{i,j=1}^N \left[\Delta_{ij} p_i\cdot p_j - 2i\,{}^\bullet\Delta_{ij}\varepsilon_i\cdot k_j - {}^\bullet\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}. \tag{14}$$

Now, we also Fourier transform the scalar legs of the master formula to momentum space,

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon N] = \int d^Dx \int d^Dx' \, \mathrm{e}^{ip\cdot x+ip'\cdot x'} \, \Gamma[x,x';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N]\,. \tag{15}$$

This gives a representation of the multi-photon Compton scattering diagram as depicted in FIG. **??** (together with all the permuted and "seagulled" ones).



Changing the integration variables x, x' to

$$x - x' = x_- \quad \text{and} \quad x + x' = 2x_+\,,$$

the integral over x_+ just produces the usual energy-momentum conservation factor:

$$\begin{aligned}\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] \quad &= \quad (-ie)^N(2\pi)^D\delta^D\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\,\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\int d^Dx_-\,\mathrm{e}^{-\frac{1}{4T}x_-^2} \\ &\times \int_0^T\prod_{i=1}^N d\tau_i\,\mathrm{e}^{ix_-\cdot\left(p+\sum_{i=1}^N\frac{k_i\tau_i}{T}\right)}\,\mathrm{e}^{\sum_{i=1}^N\frac{\varepsilon_i\cdot x_-}{T}}\,\mathrm{e}^{\sum_{i,j=1}^N\left[\Delta_{ij}p_i\cdot p_j-2i{}^\bullet\Delta_{ij}\varepsilon_i\cdot k_j-{}^\bullet\Delta^\bullet_{ij}\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}\,.\end{aligned} \tag{16}$$

After performing also the x_- integral, and some rearrangements, one arrives at

$$\Gamma[p;p';k_1,\varepsilon_1;\cdots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta^D\Big(p+p'+\sum_{i=1}^N k_i\Big)\int_0^\infty dT\,\mathrm{e}^{-T(m^2+p^2)}\int_0^T\prod_{i=1}^N d\tau_i$$

$$\times\,\mathrm{e}^{\sum_{i=1}^N(-2k_i\cdot p\tau_i+2i\varepsilon_i\cdot p)+\sum_{i,j=1}^N\left[(\frac{|\tau_i-\tau_j|}{2}-\frac{\tau_i+\tau_j}{2})p_i\cdot p_j-i(\mathrm{sign}(\tau_i-\tau_j)-1)\varepsilon_i\cdot p_j+\delta(\tau_i-\tau_j)\varepsilon_i\cdot\varepsilon_j\right]}\Big|_{\mathrm{lin}(\varepsilon_1\varepsilon_2\cdots\varepsilon_N)}\,.$$

This is our final representation of the N - propagator in momentum space. It is important to mention that it gives the *untruncated* propagator, including the final scalar propagators on both ends. On-shell it corresponds to multi-photon Compton scattering, while off-shell it can be used for constructing higher-loop amplitudes by sewing.

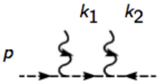
Loop correction

One-loop correction to the scalar propagator and its vertex has been studied long ago. The non-perturbative studies of this vertex has been challenging for decades for quantum chromodynamics (QCD) as well as for simpler case as quantum electrodynamics (QED).

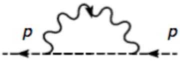
- A systematic studies of the spinor QED was initiated more than three decades ago by Ball and Chiu (PRD **22**, 2542 (1980)), they decomposed the vertex into **longitudinal** and **transversal** parts $\Rightarrow$ Form factors in Feynman gauge.
- In 1995, Pennington et.al (PRD **52**, 1242 (1995)) extended their results to an arbitrary covariant gauge.
- For massive and massless QED_3 case the results was obtained by several authors, (Adkins et.al in 1994, Bashir et.al, in 1999, 2000, 2001).
- In 2007, Bashir et.al (PRD **76**, 065009 (2007)) have extend the Ball and Chiu form factor decomposition to an **arbitrary spacetime dimensions** D and **covariant gauge** ξ for scalar QED.
- For QCD, Ball and Chiu first studied the gluon loop in Feynman gauge (PRD **22**, 2550 (1980)).
- In 2000, Davydychev et.al extended their results to an arbitrary spacetime dimensions (PRD **63**, 014022 (2000))

Constructing one-loop correction to scalar propagator and its vertex

One-loop correction to the scalar propagator is obtained from $N = 2$ by sewing two photons using the Feynman gauge.



$$\varepsilon_1^{\mu}\varepsilon_2^{\nu} \to \frac{\eta^{\mu\nu}}{q^2}$$

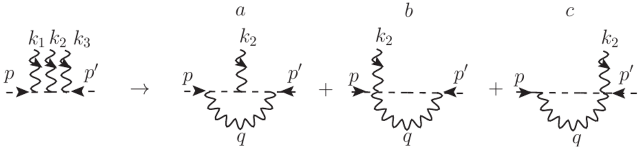


$$\begin{aligned}\Gamma_{\rm Feyn}[p] &= e^2(m^2+p^2)^2\int_0^{\infty} dT T^2\, e^{-T(m^2+p^2)}\int_0^1 du_1\int_0^{u_1} du_2\int\frac{d^Dq}{(2\pi)^D}\\ &\times \left[4p^{\mu}p^{\nu}+2(p^{\mu}q^{\nu}+p^{\nu}q^{\mu})+q^{\mu}q^{\nu}\right]\frac{\eta^{\mu\nu}}{q^2}\, e^{-Tu_1(q^2+2p\cdot q)+Tu_2(q^2+2p\cdot q)}\end{aligned}$$

After some algebra and performing all parameter integrals, one-loop correction to scalar propagator in the Feynman gauge can be written as

$$\boxed{\Gamma_{\rm Feyn}[p] = -\frac{e^2}{(4\pi)^{\frac{D}{2}}}(m^2)^{\frac{D}{2}-1}\Gamma\left(1-\frac{D}{2}\right)\left[2\frac{(m^2-p^2)}{m^2}\,{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right)-1\right]}$$

Now let us look at the scalar-photon vertex which can be obtained from $N = 3$, we have the following diagrams by sewing photon 1 and photon 3 (for the standard ordering $\tau_1 \geq \tau_2 \geq \tau_3$)



$$
\begin{aligned}
\Gamma_{\mathrm{vertex}}[p'; p; k_2, \varepsilon_2] \overset{\tau_1>\tau_2>\tau_3}{=} & \Gamma_a[p'; p; k_2, \varepsilon_2] + \Gamma_b[p'; p; k_2, \varepsilon_2] + \Gamma_c[p'; p; k_2, \varepsilon_2] \\
& = -e^3(m^2 + p'^2)(m^2 + p^2) \int_0^\infty dT \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-T(m^2+p^2)} \\
& \times \int \frac{d^D q}{(2\pi)^D} \left\{ \frac{(l_1 \cdot l_3)(l_2 \cdot \varepsilon_2)}{q^2} - \frac{(l_3 \cdot \varepsilon_2)}{q^2} \delta(\tau_1 - \tau_2) + \frac{(l_1 \cdot \varepsilon_2)}{q^2} \delta(\tau_2 - \tau_3) \right\} \\
& \times e^{-(-2q\cdot p+q^2)\tau_1 - (2k_2\cdot p + k_2^2 - 2q\cdot k_2)\tau_2 - (-q^2 + 2q\cdot(p+k_2))\tau_3}
\end{aligned}
$$

$$
l_1 = -q + 2p \quad , \quad l_2 = k_2 + 2(-q + p) \quad , \quad l_3 = q + 2p'
$$

The final result for diagram *a* becomes

$$
\begin{aligned}
\Gamma_a^\mu[p,p';k_2] \quad &= \quad -\frac{e^3}{(2\pi)^D}\Big\{(p'^\mu - p^\mu)K^{(0)} + 2K_\mu^{(1)} + 2(p^\nu - p'^\nu)\big[(p^\mu - p'^\mu)J_\nu^{(1)} - 2J_{\mu\nu}^{(2)}\big] \\
&\quad +4(p\cdot p')\big[(p^\mu - p'^\mu)J^{(0)} - 2J_\mu^{(1)}\big]\Big\}\,. \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
K^{(0)} &= \int d^D q \frac{1}{[m^2+(p-q)^2][m^2+(q+p')^2]}\,, \\
K_\mu^{(1)} &= \int d^D q \frac{q_\mu}{[m^2+(p-q)^2][m^2+(q+p')^2]}\,, \\
J^{(0)} &= \int d^D q \frac{1}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}\,, \\
J_\mu^{(1)} &= \int d^D q \frac{q_\mu}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}\,, \\
J_{\mu\nu}^{(2)} &= \int d^D q \frac{q_\mu q_\nu}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]}\,. \qquad (18)
\end{aligned}
$$

and diagram b

$$\Gamma_b^\mu(p') = \frac{1}{2}\frac{e^3 m^{D-4} p'^\mu}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\left\{\left(\frac{m^2}{p'^2}-3\right){}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p'^2}{m^2}\right)-\frac{m^2}{p'^2}\right\}.$$

Diagram c is obtained from diagram b simply by the replacement

$$\Gamma_c^\mu = \Gamma_b^\mu(p' \to -p)\,. \tag{19}$$

LKF transformation

Landau and Khalatnikov (Sov. Phys. JETP **2**, 69 (1956)) and independently Fradkin (Zh. Eksp. Teor. Fiz. **29**, 258 (1955)) had derived a series of transformations (LKF) which in QED they transform the Green functions in a specific manner under a variation of gauge. These transformations have been derived later by Johnson and Zumino by means of functional methods (PRL **3**, 351 (1959)). These transformations are **nonperturbative** and they are written in coordinate space, they can be used to predict higher-loop terms from lower-loop ones.

- the first work to prove that the longitudinal Ball-Chiu vertex was not sufficient to ensure the LKF transformation law for the fermion propagator when implemented into the fermion Schwinger-Dyson equation was done by Curtis and Pennington in PRD **42** (1990) 4165.
- Later the same problem was treated by Roberts et.al in PLB **333**, 536 (1994) with a different ansatz for the transverse vertex to achieve the same goal.
- LKF transformation law was implemented on a more general basis than the work by Curtis and Pennington by Bashir et.al (PRD **57** (1998) 1242), later this law for the fermion propagator and multiplicative renormalizability for the photon propagator was implemented by Pennington et.al (PRD **79** (2009) 125020).
- Simultaneously LKF transformation law was implemented for the fermion propagator and also ensures the gauge invariance of the critical coupling above which chiral symmetry is dynamically broken, Bashir, Roberts et.al, in PRC **85** (2012) 045205.
- In the following, we will apply the worldline formalism to the efficient construction of multi-photon and multi-loop amplitudes in Scalar QED, on-shell and off-shell.

Gauge transformation for photons, x-space

A gauge transformation on any photon produces (at most) an exponential factor, outside of the path integral, and those various factors can be collected and combined at the end; thus it is completely sufficient to consider, one at a time:

1 . a gauge transformation of one external photon, result $\rightarrow$ vertex operator collapses to a total derivative

2 . a gauge transformation of an internal photon with one or both ends on a loop, result $\rightarrow$ zero

3 . a gauge transformation on an internal photon with both ends on the same line, result: $\rightarrow$ LKF

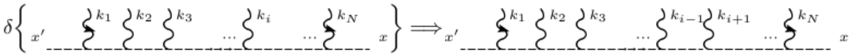
4 . a gauge transformation of an internal photon with both ends on different lines, result: $\rightarrow$ generalized LKF

A gauge transformation of an external photon:

$$\varepsilon_i \to \varepsilon_i + \xi k_i$$

$$V_{\rm scal}[\varepsilon_i, k_i] = \int_0^T d\tau_i \varepsilon_{i\mu} \dot{x}_i^{\mu}\, e^{ik_i\cdot x(\tau_i)} \to V_{\rm scal}[\varepsilon_i, k_i] - i\xi \int_0^T \frac{\partial}{\partial\tau_i} e^{ik_i\cdot x(\tau_i)} = V_{\rm scal}[\varepsilon_i, k_i] - i\xi\left(e^{ik_i\cdot x} - e^{ik_i\cdot x'}\right)$$

$$\delta V_{\rm scal}[\varepsilon_i, k_i] = -i\xi\left(e^{ik_i\cdot x} - e^{ik_i\cdot x'}\right)$$

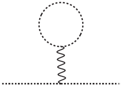


It means that under this gauge transformation the ith-photon is removed and new diagram has one leg less and it is understood that the amplitude is gauge dependent, otherwise under the transformation it would vanish.

More interesting is the case of a change of gauge for all the internal photons. Each internal photon is represented by a factor of $-S_i$, we can see that a change in the gauge parameter ξ by $\Delta\xi$ will change S_i by

$$\Delta_\xi S_i = \Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}} \Gamma\Big(\frac{D}{2} - 2\Big) \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\partial}{\partial\tau_1} \frac{\partial}{\partial\tau_2} [(x_1 - x_2)^2]^{2-\frac{D}{2}} \, . \tag{20}$$

Since the integrand is a total derivative in both variables, if the photon at least on one end sits on a closed loop, the result will vanish.



Therefore, the gauge transformation properties of an amplitude are determined by the photons exchanged between two scalar lines, or along one scalar line. Thus, in the study of the gauge parameter dependence, we can disregard external photons as well as closed scalar loops, and it therefore suffices to study the quenched 2n scalar amplitude.

$$A^{\mathrm{qu}}(x_1, \ldots, x_n; x_1', \ldots, x_n' | \xi) = \sum_{\pi \in S_n} A_\pi^{\mathrm{qu}}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}' | \xi) \, , \tag{21}$$

where in the partial amplitude $A_\pi^{\mathrm{qu}}(x_1, \ldots, x_n; x_{\pi(1)}', \ldots, x_{\pi(n)}' | \xi)$ it is understood that the line ending at x_i starts at $x_{\pi(i)}'$.

$$A_\pi^{\rm qu}(x_1,\dots,x_n;x'_{\pi(1)},\dots,x'_{\pi(n)}|\xi) = \prod_{l=1}^n \int_0^\infty dT_l\,{\rm e}^{-m^2T_l}\int_{x_l(0)=x'_{\pi(l)}}^{x_l(T_l)=x_l}\mathcal{D}x_l(\tau_l)\,{\rm e}^{-\sum_{l=1}^n S_0^{(l)}-\sum_{k,l=1}^n S_{i\pi}^{(k,l)}}\,. \qquad (22)$$

Here, $S_0^{(l)}$ is the free worldline Lagrangian for the path integral representing line l:

$$S_0^{(l)} = \int_0^{T_l} d\tau_l\,\frac{1}{4}\dot{x}_l^{\,2}\,, \qquad (23)$$

and $S_{i\pi}^{(k,l)}$ generates all the photons connecting lines k and l:

$$S_{i\pi}^{(k,l)} = \frac{e^2}{2}\int_0^{T_k} d\tau_k\int_0^{T_l} d\tau_l\,\dot{x}_k^\mu D_{\mu\nu}(x_k-x_l)\dot{x}_l^\nu\,. \qquad (24)$$

Thus, after a gauge change,

$$\begin{aligned} A_\pi^{\rm qu}(x_1,\dots,x_n;x'_{\pi(1)},\dots,x'_{\pi(n)}|\xi+\Delta\xi) &= \prod_{l=1}^n \int_0^\infty dT_l\,{\rm e}^{-m^2T_l}\int_{x_l(0)=x'_{\pi(l)}}^{x_l(T_l)=x_l}\mathcal{D}x_l(\tau_l) \\ &\times\ {\rm e}^{-\sum_{l=1}^n S_0^{(l)}-\sum_{k,l=1}^n\left(S_{i\pi}^{(k,l)}+\Delta_\xi S_{i\pi}^{(k,l)}\right)}\,, \qquad (25)\end{aligned}$$

where

$$\begin{aligned}\Delta_\xi S_{i\pi}^{(k,l)} &= \Delta\xi\,\frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\Big(\frac{D}{2}-2\Big)\Big\{\big[(x_k-x_l)^2\big]^{2-D/2}-\big[(x_k-x'_{\pi(l)})^2\big]^{2-D/2} \\ &\quad -\big[(x'_{\pi(k)}-x_l)^2\big]^{2-D/2}+\big[(x'_{\pi(k)}-x'_{\pi(l)})^2\big]^{2-D/2}\Big\}\,.\end{aligned}$$

Since this depends only on the endpoints of the scalar trajectories, we can pull the factors involving $\Delta\xi$ out of the path integration, leading to

$$A_\pi^{\mathrm{qu}}(x_1,\dots,x_n;x'_{\pi(1)},\dots,x'_{\pi(n)}|\xi+\Delta\xi) = T_\pi A_\pi^{\mathrm{qu}}(x_1,\dots,x_n;x'_{\pi(1)},\dots,x'_{\pi(n)}|\xi)\,, \tag{26}$$

where

$$T_\pi \quad \equiv \quad \prod_{k,l=1}^{N} \mathrm{e}^{-\Delta_\xi S_{i\pi}^{(k,l)}}\,. \tag{27}$$

This is an exact D-dimensional result. When using it in dimensional regularization around $D=4$, one has to take into account that the full non-perturbative A^{qu} in Scalar QED has poles in ϵ to arbitrary order, so that also the prefactor T_π, although regular, needs to be kept to all orders. Here we will consider only the leading constant term of this prefactor. Thus, we compute

$$\lim\nolimits_{D\to 4} \mathrm{e}^{-\Delta_\xi S_{i\pi}^{(k,l)}} = \left(r_\pi^{(k,l)}\right)^c\,, \tag{28}$$

where we have introduced the constant

$$c \equiv \Delta\xi\,\frac{e^2}{32\pi^2}\,, \tag{29}$$

and the conformal cross ratio $r_\pi^{(k,l)}$ associated to the four endpoints of the lines k and l,

$$r_\pi^{(k,l)} \equiv \frac{(x_k-x_l)^2(x'_{\pi(k)}-x'_{\pi(l)})^2}{(x'_{\pi(k)}-x_l)^2(x_k-x'_{\pi(l)})^2}\,. \tag{30}$$

Thus, at the leading order, the prefactor turns into

$$T_\pi = \left(\prod_{k,l=1}^{N} r_\pi^{(k,l)} \right)^c + O(\epsilon) \,. \tag{31}$$

We note that for the case of a single propagator, $s = k = l = 1$, Eq. (31) degenerates into

$$T = \left[\frac{(x - x)^2 (x' - x')^2}{((x - x')^2)^2} \right]^c . \tag{32}$$

Therefore, if we replace the vanishing numerator $(x - x)^2(x' - x')^2$ by the cutoff $(x_{\mathrm{min}}^2)^2$, and $\Delta\xi$ by ξ, we recuperate the original LKFT.

The generalized LKFT in perturbation theory

As in the case of the original LKFT, one would like to know how the non-perturbative gauge transformation formula works out in perturbation theory. A gauge parameter change will affect all the photons, except the ones ending on a loop, and convert a photon connecting lines k and l into a factor $-\Delta_\xi S_{i\pi}^{(k,l)}$. Thus, the difference between gauges involves only lower-loop diagrams such as shown below. If the gauge transformation of the whole set of diagrams is called $\Delta_\xi\, \mathrm{Fig}\, 8$, we can write

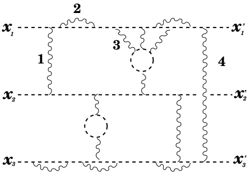
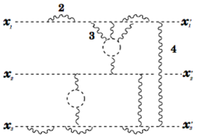


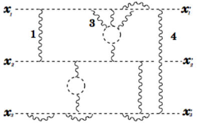
FIG7: Feynman diagram representing a class of contributions to the six - scalar amplitude at twelve loops.

$$
\begin{aligned}
\Delta_\xi \,\mathrm{Fig}\, 7 \;=\; & \big(-2\Delta_\xi S_{i\pi}^{(1,2)}\big)\,\mathrm{Fig}\, 8(a) + \big(-\Delta_\xi S_{i\pi}^{(1,1)}\big)\,\mathrm{Fig}\, 8(b) + \big(-2\Delta_\xi S_{i\pi}^{(1,3)}\big)\,\mathrm{Fig}\, 8(c) + \cdots \\
& + \big(-2\Delta_\xi S_{i\pi}^{(1,2)}\big)\big(-\Delta_\xi S_{i\pi}^{(1,1)}\big)\,\mathrm{Fig}\, 8(d) + \cdots \\
& + \big(-2\Delta_\xi S_{i\pi}^{(1,2)}\big)\big(-\Delta_\xi S_{i\pi}^{(1,1)}\big)\big(-2\Delta_\xi S_{i\pi}^{(1,3)}\big)\,\mathrm{Fig}\, 8(e) + \cdots \\
& + \cdots .
\end{aligned}
$$

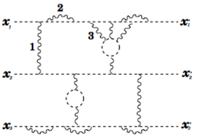
Here, on the right-hand side the first line is for gauge transformation of each photon, one by one. In the second line, we have simultaneous gauge transformations of all possible pairs of photons, etc.



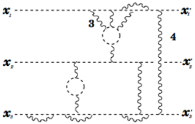
(a) Gauge transformation of photon 1.



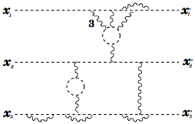
(b) Gauge transformation of photon 2.



(c) Gauge transformation of photon 4.

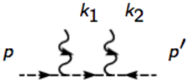


(d) Simultaneous gauge transformation of photons 1 and 2.

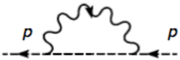


(e) Simultaneous gauge transformation of photons 1, 2 and 4.

Gauge transformation for internal photons in momentum space



$$\varepsilon_1^\mu \varepsilon_2^\nu \to \frac{\eta^{\mu\nu} q^2 - (1-\xi) q^\mu q^\nu}{q^4}$$



If we repeat our previous calculations but in covariant gauge we get

$$\begin{aligned} \Gamma_{\mathrm{propagator}}[p] &= \Gamma_{\mathrm{Feynman}} + \Gamma_\xi = -e^2(m^2+p^2)^2 \int_0^\infty dT T^2\, e^{-T(m^2+p^2)} \int_0^1 du_1 \int_0^{u_1} du_2 \int \frac{d^D q}{(2\pi)^D} \\ &\times \left[4p^\mu p^\nu + 2(p^\mu q^\nu + p^\nu q^\mu) + q^\mu q^\nu\right] \left[\frac{\eta^{\mu\nu}}{q^2} + (\xi - 1)\frac{q^\mu q^\nu}{q^4}\right] e^{-Tu_1(q^2+2p\cdot q) + Tu_2(q^2+2p\cdot q)} \end{aligned} \tag{33}$$

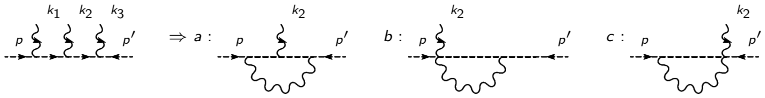
Note that the gauge part can be written as a second derivative of the exponential as

$$\Big[4p^\mu p^\nu + 2(p^\mu q^\nu + p^\nu q^\mu) + q^\mu q^\nu\Big](\xi - 1)\frac{q^\mu q^\nu}{q^4}e^{-T(q^2+2p\cdot q)(u_1-u_2)}$$
$$= (\xi - 1)\left[\frac{2p\cdot q}{q^2} + 1\right]^2 e^{-T(q^2+2p\cdot q)(u_1-u_2)} = -\frac{(\xi-1)}{T^2q^4}\frac{\partial^2}{\partial u_1\partial u_2}e^{-T(q^2+2p\cdot q)(u_1-u_2)}$$

Finally

$$\Gamma_{\rm propagator}(p) = \Gamma_{\rm Feynman} + \Gamma_\xi = \frac{e^2}{m^2}\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}}\Gamma\left(1-\frac{D}{2}\right)\Bigg\{1 - 2\frac{(m^2-p^2)}{m^2}\,{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right)$$
$$+(1-\xi)\,\frac{(m^2+p^2)^2}{m^4}\,{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right)\Bigg\}. \quad (34)$$

Now for the vertex



$$\Gamma_{\rm vertex}[p';p;k_2,\varepsilon_2] \overset{\tau_1>\tau_2>\tau_3}{=} \Gamma_a[p';p;k_2,\varepsilon_2]+\Gamma_b[p';p;k_2,\varepsilon_2]+\Gamma_c[p';p;k_2,\varepsilon_2]$$

$$= -e^3(m^2+p'^2)(m^2+p^2)\int_0^\infty dT\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\int_0^{\tau_2}d\tau_3\, e^{-T(m^2+p^2)}$$

$$\times\int\frac{d^Dq}{(2\pi)^D}\Big\{\Big[\frac{(l_1\cdot l_3)}{q^2}-(1-\xi)\frac{(l_1\cdot q)(l_3\cdot q)}{q^4}\Big](l_2\cdot\varepsilon_2)$$

$$-\delta(\tau_1-\tau_2)\Big[\frac{l_3\cdot\varepsilon_2}{q^2}-(1-\xi)\frac{(l_3\cdot q)(\varepsilon_2\cdot q)}{q^4}\Big]+\delta(\tau_2-\tau_3)\Big[\frac{(l_1\cdot\varepsilon_2)}{q^2}-(1-\xi)\frac{(l_1\cdot q)(\varepsilon_2\cdot q)}{q^4}\Big]\Big\}$$

$$\times e^{-(-2q\cdot p+q^2)\tau_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)\tau_2-(-q^2+2q\cdot(p+k_2))\tau_3}$$

where

$$(l_1\cdot q)(l_3\cdot q) = -q^4+2q^2q\cdot(p-p')+4(p'\cdot q)(p\cdot q)$$

Again, for this case the gauge part can be reproduced by two total derivatives, for example for diagram a

$$
\begin{aligned}
&(1-\xi)\frac{l_1\cdot q l_3\cdot q}{q^4}\, e^{(2q\cdot p-q^2)Tu_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)Tu_2+(q^2+2q\cdot p')Tu_3}\\
&=(1-\xi)\frac{1}{T^2q^4}\left(\frac{\partial^2}{\partial u_1\partial u_3}\right)\left[e^{(2q\cdot p-q^2)Tu_1-(2k_2\cdot p+k_2^2-2q\cdot k_2)Tu_2+(q^2+2q\cdot p')Tu_3}\right]
\end{aligned}
$$

And finally the diagram a in a covariant gauge can be written as

$$
\begin{aligned}
\Gamma^{\mu}_{a}[p,p';k_2]=-\frac{e^3}{(2\pi)^D}\Bigg\{&(p'^{\mu}-p^{\mu})K^{(0)}+2K^{(1)}_{\mu}+2(p^{\nu}-p'^{\nu})\left[(p^{\mu}-p'^{\mu})J^{(1)}_{\nu}-2J^{(2)}_{\mu\nu}\right]\\
&+4p\cdot p'\left[(p^{\mu}-p'^{\mu})J^{(0)}-2J^{(1)}_{\mu}\right]\\
-(\xi-1)(p'^2+m^2)(p^2+m^2)\Bigg\{&\Bigg[\frac{\pi^{\frac{D}{2}}(p'^{\mu}p^2+p^{\mu}m^2)}{p^2(p'^2+m^2)}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-3}\,{}_2F_1\left(3-\frac{D}{2},2;\frac{D}{2};-\frac{p^2}{m^2}\right)\\
&-(p\leftrightarrow p')\Bigg]\\
&-\left[\frac{(\pi)^{\frac{D}{2}}p^{\mu}}{p^2(p'^2+m^2)}\Gamma\left(1-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-2}\,{}_2F_1\left(2-\frac{D}{2},1;\frac{D}{2};-\frac{p^2}{m^2}\right)-(p\leftrightarrow p')\right]\\
&+(p^{\mu}-p'^{\mu})I^{(0)}-2I^{(1)}_{\mu}\Bigg\}\Bigg\},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
K^{(0)} &= \int d^D q \frac{1}{[m^2+(p-q)^2][m^2+(q+p')^2]} \quad , \quad K^{(1)}_{\mu} = \int d^D q \frac{q^{\mu}}{[m^2+(p-q)^2][m^2+(q+p')^2]} \\
J^{(0)} &= \int d^D q \frac{1}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]} \quad , \quad J^{(1)}_{\mu} = \int d^D q \frac{q^{\mu}}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]} \\
J^{(2)}_{\mu\nu} &= \int d^D q \frac{q^{\mu} q^{\nu}}{q^2[m^2+(p-q)^2][m^2+(q+p')^2]} \quad , \quad I^{(0)} = \int d^D q \frac{1}{q^4[m^2+(p-q)^2][m^2+(q+p')^2]} \\
I^{(1)}_{\mu} &= \int d^D q \frac{q^{\mu}}{q^4[m^2+(p-q)^2][m^2+(q+p')^2]}
\end{aligned}
\tag{36}
$$

Similarly, diagram b yields:

$$
\begin{aligned}
\Gamma^{\mu}_{b}(p') \quad &= \quad \frac{1}{2} \frac{e^3 m^{D-4} \Gamma\left(1-\frac{D}{2}\right) p'^{\mu}}{(4\pi)^{\frac{D}{2}}} \left\{ \left(\frac{m^2}{p'^2} - 3 \right) {}_2F_1\left(2-\frac{D}{2}, 1; \frac{D}{2}; -\frac{p'^2}{m^2}\right) - \frac{m^2}{p'^2} \right. \\
&\left. -(\xi-1)\left(\frac{p'^2+m^2}{p'^2}\right)\left[{}_2F_1\left(2-\frac{D}{2}, 1; \frac{D}{2}; -\frac{p'^2}{m^2}\right) - \left(\frac{p'^2+m^2}{m^2}\right) {}_2F_1\left(3-\frac{D}{2}, 2; \frac{D}{2}; -\frac{p'^2}{m^2}\right)\right] \right\} .
\end{aligned}
\tag{37}
$$

Comparison with previous studies

Now we can compare our final results with the previous findings in Bashir et.al (PRD **76**, 065009 (2007)). . For the scalar propagator, is in complete agreement with the results quoted in Bashir et.al (PRD **76**, 065009 (2007))., after taking into account the conventions of momentum flow. The same is true for the scalar-photon 3-point vertex. Notice that in Bashir et.al (PRD **76**, 065009 (2007))., this result is expressed in terms of nine inequivalent vector and tensor integrals which are $K^{(0)}$, $J^{(0)}$, $I^{(0)}$, $K^{(1)}_{\mu}$, $J^{(1)}_{\mu}$, $I^{(1)}_{\mu}$, $J^{(2)}_{\mu\nu}$, $I^{(2)}_{\mu\nu}$ and $I^{(3)}_{\mu\nu\alpha}$. In our analysis, the use of total derivative terms has allowed us to reduce the number of independent integrals by two, i.e., we do not require $I^{(2)}_{\mu\nu}$ and $I^{(3)}_{\mu\nu\alpha}$ to express the vertex.

$$\begin{aligned}
I^{(2)}_{\mu\nu} &= \int d^D q \, \frac{q_\mu q_\nu}{q^4[m^2+(q+p)^2][m^2+(q+p')^2]} \\
I^{(3)}_{\mu\nu\alpha} &= \int d^D q \, \frac{q_\mu q_\nu q_\alpha}{q^4[m^2+(q+p)^2][m^2+(q+p')^2]}
\end{aligned} \tag{38}$$

Conclusions

- We have rederived the momentum-space Bern-Kosower type master formula for the tree-level scalar propagator dressed by an arbitrary number of photons, starting directly from the worldline path integral representation of this amplitude. We have also generalized this master formula to the x - space propagator. (N. A, A. Bashir, C. Schubert, to appear in PRD, arXiv: 1511.05087)
- We have used the master formula for constructing, by sewing in Feynman gauge, the one-loop scalar propagator and the one-loop vertex in arbitrary dimension.
- These momentum-space results were extended to an arbitrary covariant gauge in a relatively simple way, observing that the difference terms involve only total derivatives under the worldline integrals. We have checked that the result agrees with the earlier calculation.
- In x-space, the implementation of changes of the gauge parameter through total derivatives has allowed us to obtain, in a very simple way, an explicit non-perturbative formula for the effect of such a gauge parameter change on an arbitrary amplitude summed to all loop orders. This formula generalizes the LKFT and contains it as a special case. At leading order in the ϵ - expansion it can be written in terms of conformal cross ratios.
- We have illustrated with an example how this non-perturbative transformation works diagrammatically in perturbation theory.
- Extending our work to spinor QED is under study.
- We have succesfully obtained the master formula for non-abelian (gluons), N. A, O. Corradini, F. Bastianelli, PRD **93**, 025035 (2016), arXiv: 1508.05144
- Graviton-photon case

Thanks for your attention