Thermodynamic law from the Entanglement Entropy Bound

Chanyong Park (APCTP)

@ Workshop on Fields, Strings, and Gravity(KIAS,2016.02.22)

Based on

C. Park, Phys.Rev. D92 (2015) 126013[arXiv:1505.03951]
and arXiv:1511.02288

Contents

1. Review on the holographic entanglement entropy

Motivation

One of the most remarkable successes in the AdS/CFT correspondence is the microscopic derivation of the Bekenstein-Hawking entropy for a BPS black hole

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}$$

This idea relates the gravitational entropy to the degeneracy of the dual quantum field theory.

On the other hand, there exists another interesting entropy in quantum mechanical systems which is called the entanglement entropy and measures the entanglement between quantum states. It is an interesting quantity because

- it plays a role of an order parameter in a quantum phase transition,
- counts the degrees of freedom of the quantum system.
- In addition, it describes a higher-dimensional Zamoldchikov's c-theorem along the RG flow.

Holographic entanglement entropy (HEE) *[Ryu–Takayanagi]*

Due to this similarity between the entangling surface and the event horizon,

Ryu and Takayanagi proposed following the AdS/CFT correspondence that

the EE of a d–dimensional CFT can be evaluated by the area of the minimal surface in the d+1–dimensional dual gravity

$$\boxed{S = \frac{\text{Area of } \gamma_A}{4G_N^{d+1}}}$$

where γ_A is the d–1 dimensional static minimal surface in AdS_{d+1} whose boundary coincides with the entangling surface .

Checking the HEE proposal

For a 2–dim CFT, it has been known that the entanglement entropy is given by

$$S_A = \frac{c}{3} \cdot \log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right), \quad \textit{[Calabrese–Cardy]}$$

where l and L are the length of the subsystem A and the total system.

a is a UV cutoff (lattice space) and c is the central charge of the CFT.

The same result has been also reproduced by the HEE in a 3–dim. AdS space.

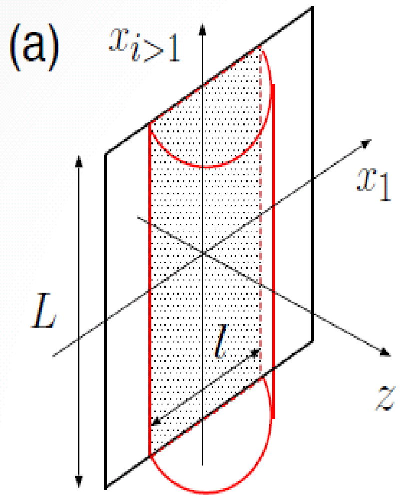
2. Thermodynamics-like law of the entanglement entropy

Bhattacharya, Nozaki, Takayanag and Ugajin found that

- the holographic entanglement entropy of excited states satisfies the thermodynamics-like relation if temperature is properly defined

- It is called the entanglement temperature which shows a universal behavior inversely proportional to the size of a subsystem

The holographic entanglement entropy a subsystem with a strip shape



$$0 < x_1 < l, \quad -L/2 < x_{2,3,\cdots,d-1} < L/2,$$

If one define the entanglement temperature proportional to the inverse of l.

$$T_{ent} = c \cdot l^{-1}$$

the entanglement entropy satisfies the thermodynamics-like relation

$$T_{ent} \cdot \Delta S_A = \Delta E_A$$

This thermodynamics-like law plays an important role in explaining the emergence of AdS.

Conjectures

1) This thermodynamics-like relation is universal.

2) The entanglement temperature has a universal property proportional to the inverse of the size.

What is the origin of a universal thermodynamics-like law of the entanglement entropy?

3. The entanglement entropy bound

Bekenstein bound

- A universal thermal entropy bound in black hole physics.

- When an object is absorbed into a black hole, the entropy of an object increases the black hole area due to the generalized second law of thermodynamics. In this case, the increased entropy is bounded by the absorbed energy

$$\Delta S \leq \lambda l \Delta E,$$

where l and λ are a typical size of the system and a non-universal numerical factor of order one.

- The Bekenstein bound is universal in that it is universally proportional to l independent of microscopic details. All information about the microscopic theory are included in a non-universal constant λ.

*Entanglement entropy bound (*generalization of the Bekenstein bound*)*

Which can be *applied to both quantum and thermal systems*.

In order to get the entanglement entropy bound,

let us first define ***a relative entropy***

When two quantum states are in the same Hilbert space, **the relative entropy** gives rise to a fundamental statistical measure of their distance

$$S(\rho_1|\rho_0) \equiv \mathrm{Tr}\,(\rho_1 \log \rho_1) - \mathrm{Tr}\,(\rho_1 \log \rho_0).$$

where ρ_0 is a reduced density matrix of a ground or thermal state and ρ_1 is that for a quantumly or thermally excited state.

If there exists a parameter connecting two reduced density matrices such that

$$\rho_1 = \rho_1(\lambda) \text{ and } \rho_0 = \rho_1(0),$$

the relative entropy usually has a non-negative value

$$S(\rho_0|\rho_0) = 0 \quad \text{and} \quad S(\rho_1|\rho_0) > 0 \text{ for } \rho_0 \neq \rho_1.$$

ρ_0 lies in a minimum point.

By using the definition of the entanglement entropy, the relative entropy can be reexpressed as

$$S(\rho_1|\rho_0) = \Delta \langle K \rangle - \Delta S,$$

Furthermore, the non-negativity of the relative entropy leads to the following **entanglement entropy bound**

$$\Delta \langle K \rangle \geq \Delta S$$

which has been regarded as a generalized Bekenstein bound holding for any region in QFT.

Following the simple dimension counting,

the modular Hamiltonian $\langle K \rangle$ of a relativistic QFT should be related to the energy as follows

$$\Delta \langle K \rangle = \lambda\, l \Delta E$$

where l is a typical size of the system and λ indicates a numerical number,

When the entanglement entropy bound is saturated, this result shows that the entanglement entropy bound is reduced to the thermodynamics-like law with a universal entanglement temperature

$$\frac{\Delta S}{\Delta E} \leq \frac{1}{T_E} \qquad \text{with} \qquad T_E = \frac{1}{\lambda l}.$$

Consider a charged black hole with a scalar hair

dual to a fermion system with a Fermi sea.

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[\mathcal{R} - \frac{1}{4}e^{4\phi}F_{\mu\nu}F^{\mu\nu} - 12\partial_\mu\phi\partial^\mu\phi + \frac{1}{R^2}\left(8e^{2\phi} + 4e^{-4\phi}\right)\right]$$

Geometric solution *(asymptotically AdS sapce)*

$$\begin{aligned} ds^2 &= r^2 e^{2A(r)}\left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{e^{2B(r)}}{r^2 f(r)}dr^2, \\ A &= A_t dt, \end{aligned}$$

with

$$\begin{aligned} \phi(r) &= \frac{1}{6}\log\left(1 + \frac{Q^2}{8mr^2}\right), \\ A(r) &= \frac{1}{3}\log\left(1 + \frac{Q^2}{8mr^2}\right), \\ B(r) &= -\frac{2}{3}\log\left(1 + \frac{Q^2}{8mr^2}\right), \\ f(r) &= 1 - \frac{m}{r^4\left(1 + \frac{Q^2}{8mr^2}\right)^2}, \\ A_t &= 2\kappa^2\mu - \frac{Q}{2r^2\left(1 + \frac{Q^2}{8mr^2}\right)} \end{aligned}$$

Thermodynamics

$$
\begin{aligned}
E &= \frac{3\pi^4 V_3}{2\kappa^2} T_H^4 + 6\pi^2\kappa^2 V_3 T_H^2 \mu^2 + \frac{14}{3}\kappa^6 V_3 \mu^4, \\
P &= \frac{\pi^4}{2\kappa^2} T_H^4 + 2\pi^2\kappa^2 T_H^2 \mu^2 + \frac{10}{3}\kappa^6 \mu^4, \\
S_{BH} &= \frac{2\pi^4 V_3}{\kappa^2} T_H^3 + 4\pi^2\kappa^2 V_3 T_H \mu^2, \\
\frac{N}{V_3} &= 4\pi^2\kappa^2 T_H^2 \mu + \frac{40}{3}\kappa^6 \mu^3.
\end{aligned}
$$

Even though the asymptotic geometry is given by the AdS space, the trace of the stress tensor does not vanish

$$
T^\mu{}_\mu = E - 3PV_3 = \frac{16}{3}\kappa^6 V_3 \mu^4.
$$

Therefore, we can guess that the dual matter is not conformal.

Moduar Hamiltonian in a ball-shaped region

- In general, the modular Hamiltonian is not known except several simple cases.
- One of them is the case with a spherical entangling surface

$$K = 2\pi\Omega_2 \int_{\rho\leq l} d\rho\, \rho^2 \frac{l^2-\rho^2}{2l} T_{00}$$

The modular Hamiltonian for the previous fermion system is

$$\langle K \rangle = \frac{2\pi}{5} l E,$$

where the energy contained in the ball-shaped region is given by

$$E = \Omega_2 \int_{\rho\leq l} d\rho\, \rho^2 \langle T_{00} \rangle.$$

As mentioned before, this result shows the expected relation between the modular Hamiltonian and the energy. Due to this relation, furthermore, the entanglement entropy bound reduces to the Bekenstein bound

$$\Delta S \leq \lambda l \Delta E$$

When μ is fixed, the increased modular Hamiltonian is

$$\Delta \langle K \rangle = \frac{\pi^5 l^4 \Omega_2}{5\kappa^2} T_H^4 + \frac{4\pi^3 \kappa^4 l^4 \Omega_2}{5\kappa^2} \mu^2 T_H^2 .$$

which is exact (no higher order corrections).

After the holographic calculation,

the entanglement entropy in a UV region becomes

$$\begin{aligned} S &= \frac{\pi l^2 \Omega_2}{\kappa^2 \epsilon^2} + \frac{\pi \Omega_2}{\kappa^2} \log\left(\frac{\epsilon}{l}\right) - \frac{\pi \Omega_2}{2\kappa^2}\left(1 + 2\log 2\right) \\ &\quad + \frac{4\pi l^2 \Omega_2}{3\kappa^2} \kappa^4 \mu^2 - \frac{8\pi l^4 \Omega_2}{45\kappa^2} \kappa^8 \mu^4 + \frac{\pi l^4 \Omega_2}{5\kappa^2}\left(\pi^2 T_H^2 + 2\kappa^4 \mu^2\right)^2 + \cdots \end{aligned}$$

Up to l^4 order, the increased entanglement entropy

$$\Delta S = \frac{\pi^5 l^4 \Omega_2}{5\kappa^2} T_H^4 + \frac{4\pi^3 \kappa^4 l^4 \Omega_2}{5\kappa^2} \mu^2 T_H^2$$

Ignoring the higher order correction, we finally obtain the almost saturated entanglement entropy bound

$$\Delta \langle K \rangle = \Delta S = \frac{\Delta E}{T_E} \qquad \text{with} \qquad T_E = \frac{5}{2\pi l}.$$

which accounts for the universality of the entanglement temperature.

4. Conclusion

- The entanglement entropy (EE) is an important and useful concept to understand quantum phenomena but it is hard to calculate EE of interacting QFTs.

- In this situation, the holographic technique can shed light on studying the EE of interacting QFTs even in the strong coupling regime.

- By using the entanglement entropy bound,

  we can understand the origin of the thermodynamics-like law for the entanglement entropy and the universality of the entanglement temperature.

Thank you !