AdS_6 solutions of type IIB supergravity and their hidden symmetry

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The $\mathrm{AdS}_6/\mathrm{CFT}_5$ correspondence

Five-dimensional gauge theories are non-renormalizable.

SUSY fixed point theories can be obtained in the infinite gauge coupling limit of $\mathcal{N}=1$ supersymmetric $Sp(N)$ gauge theories with $N_f < 8$ fundamental and 1 antisymmetric hypermultiplets. [Seiberg 96]

This theory appears as the worldvolume gauge theory on the N D4-branes probing an O8-plane with N_f D8-branes.

The gravity dual is a warped product of AdS_6 and half-S^4 in massive type IIA supergravity. [Brandhuber, Oz 99]

Supersymmetric AdS_6 solutions

massive type IIA

- A warped $\mathrm{AdS}_6 \times_w S^4$ solution is known. [Brandhuber, Oz 99]
  This solution is proven to be unique. [Passias 12]

type IIB

- AdS_6 solutions are obtained by performing
  (1) abelian [Cvetic, Lu, Pope, Vanzquez-Poritz 00]
  (2) non-abelian [Lozano, O Colgain, Rodriguez-Gomez, Sfetsos 12]

  T-dual transformation of the Brandhuber-Oz solution.
- Systematic search using pure spinor approach
  [Apruzzi, Fazzi, Passias, Rosa, Tomasiello 14]

M-theory

- It is shown that there is no solution.

Supersymmetric AdS_6 solutions

Search for AdS$_6$ solutions in IIB supergravity

Metric ansatz : $$ds^2 = e^{2U} ds^2_{\mathrm{AdS}_6} + ds^2_{M_4}.$$

To preserve the symmetry of AdS$_6$, we set

- warp factor U, axion C and dilaton ϕ : functions on M_4,
- F_5 =0,
- complex G_3 on M_4 : They can be dualized to scalar fields f and g.

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Require that supersymmetry transformations of the gravitino and the dilatino vanish.

$$\delta_\epsilon \psi_M = 0, \qquad \delta_\epsilon \lambda = 0.$$

The metric

The metric of the four-dimensional Riemannian space M_4 is

$$ds_4^2 = \frac{1}{9m^2}\Big[e^{-6U}\,x^2\,ds^2(S^2) + \frac{e^{-2U}}{e^{8U+\phi} - e^{\phi}x^2 - e^{4U}y^2} \times \Big\{(e^{4U+\phi} - y^2)\,dx^2 + 9\,(e^{8U} - x^2)\,dy^2 + 6\,x\,y\,dx\,dy\Big\}\Big].$$

- S^2 fibration over $D = 2$ surface spanned by $(x,\, y)$.
- S^2 corresponds to $SU(2)_R$ R-symmetry in the dual field theory.
- The metric is written in terms of the warp factor U and the dilation ϕ, which are solutions to two coupled PDEs.

form fields

Two PDEs

$$
\begin{aligned}
4e^{\phi}x = 12\left(e^{8U+\phi} + e^{\phi}x^2 - 2e^{4U}y^2\right)\partial_x U + 8e^{\phi}xy\,\partial_y U \\
- 3e^{\phi}\left(e^{8U} - x^2\right)\partial_x \phi + 2e^{\phi}xy\,\partial_y \phi,
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
-4e^{4U+\phi}xy = 12e^{4U}y\left(e^{8U+\phi} - 3e^{\phi}x^2 - 2e^{4U}y^2\right)\partial_x U \\
+ e^{\phi}x\left(-e^{8U+\phi} + e^{\phi}x^2 + 2e^{4U}y^2\right)\partial_y \phi \\
+ 4e^{2\phi}x\left(e^{8U} + x^2\right)\partial_y U - 3ye^{4U+\phi}\left(e^{8U} - x^2\right)\partial_x \phi.
\end{aligned} \tag{2}
$$

- Once we are given the solutions to the PDEs, then the metric and the fluxes can be determined.

Two PDEs

- The two explicit solutions(Abelian/non-Abelian T-dual of the Brandhuber-Oz solution) can be reproduced as specific solutions to the PDEs.
- If we can solve the PDEs, then we can obtain new AdS_6 solutions.
- But it seems very hard to solve these non-linear coupled PDEs.

Four-dimensional effective action

Dimensional reduction on AdS_6

4 dimensional effective action from the equations of motion

$$\mathcal{L} = \sqrt{g_4}\Big[R - 24(\partial U)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial C)^2 + \frac{1}{2}e^{-12U-\phi}(\partial f)^2 + \frac{1}{8}e^{-12U+\phi}(\partial g + 2C\partial f)^2 - 30e^{-8U}\Big].$$

- We obtain a non-linear sigma model of 5 scalar fields coupled to gravity.
- There is a non-trivial scalar potential.
- The signs of the kinetic terms for dualized scalars are reversed.

D=5 target space

$$ds_5^2 = 48dU^2 + d\phi^2 + e^{2\phi}dC^2 - \frac{1}{4}e^{-12U+\phi}(dg + 2Cdf)^2 - e^{-12U-\phi}df^2.$$

- We have found that there are 8 Killing vectors.
- They generate $sl(3,\mathbb{R})$ algebra.

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- We have found that there are 8 Killing vectors.
- They generate $sl(3, \mathbb{R})$ algebra.
- The five-dimensional target space parametrize the coset $SL(3, \mathbb{R})/SO(2, 1)$.

Scalar kinetic terms : $SL(3,\mathbb{R})/SO(2,1)$

Construct a coset representative in Borel gauge

$$\mathcal{V} = e^{\frac{1}{\sqrt{2}}\phi H_1} e^{-2\sqrt{6}UH_2} e^{CE_{\alpha_1}} e^{fE_{\alpha_2}} e^{\frac{1}{2}gE_{\alpha_3}}.$$

Introduce a Lie algebra-valued 1-form as

$$\mathcal{V}_i{}^m \partial_\mu (\mathcal{V}^{-1})_m{}^k \eta_{kj} = P_{\mu(ij)} + Q_{\mu[ij]},$$

with $\eta_{ij} = \mathrm{diag}(1,1,-1)$.

The scalar kinetic terms in the Lagrangian become $\mathcal{L}_{\mathrm{kin}} = -\mathrm{Tr}(P_\mu P^\mu)$.

Scalar potential

The scalar potential $V = 30\, e^{-8U}$

- breaks the $SL(3,\mathbb{R})$ global symmetry into a nontrivial subalgebra,
- is invariant under the action of 5 Killing vectors, which form a certain algebra $A_{5,40} \cong sl(2,\mathbb{R}) \ltimes \mathbb{R}^2$. [Patera, Sharp, Winternitz, Zassenhaus 76]

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Goal : Write the scalar potential in terms of the coset representative.

Strategy : Study the integrability conditions with the Killing spinor equations.

Killing spinor equations

KSE Killing spinor equations can be written as

$$\begin{pmatrix} \delta\psi_{\mu+} \\ \delta\psi_{\mu-} \end{pmatrix} = \begin{pmatrix} \nabla_\mu + \frac{1}{4}Q_{\mu ij}\Gamma^{ij} & S\gamma_\mu \\ S\gamma_\mu & \nabla_\mu + \frac{1}{4}Q_{\mu ij}\bar{\Gamma}^{ij}_\mu \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix},$$

$$\begin{pmatrix} \delta\lambda_{i+} \\ \delta\lambda_{i-} \end{pmatrix} = \begin{pmatrix} M_{ij}\bar{\Gamma}^j & P_{\mu ij}\gamma^\mu\bar{\Gamma}^j \\ P_{\mu ij}\gamma^\mu\Gamma^j & M_{ij}\Gamma^j \end{pmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix},$$

where

$$S = \frac{3i}{2L}e^{-4U}, \qquad M_{ij} = \frac{2i}{L}e^{-4U}\mathrm{diag}(1,1,2).$$

Integrability conditions

The gravitino-gravitino integrability condition

$$\gamma_\mu{}^{\nu\rho}\,[\mathcal{D}_\nu, \mathcal{D}_\rho] \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

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$$\gamma_\mu{}^{\nu\rho}\left[\mathcal{D}_\nu, \mathcal{D}_\rho\right]\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\Downarrow$$

$$\begin{aligned}
&-\frac{1}{2}M^i{}_j\,\bar{\Gamma}^j\,\gamma_\mu\,\delta\lambda_{i+}+\frac{1}{2}P_\nu{}^i{}_j\,\gamma^\nu\,\Gamma^j\,\gamma_\mu\,\delta\lambda_{i-}\\
&+\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\left(P_{\mu ij}P_\nu{}^{ij}-\frac{1}{2}g_{\mu\nu}P_{\rho ij}P^{\rho ij}\right)+g_{\mu\nu}\left(-12S^2+\frac{1}{2}M_{ij}M^{ij}\right)\right)\gamma^\nu\xi_+\\
&-4\left(\partial_\nu S+\frac{1}{4}M_{ij}P_\nu{}^{ij}\right)\gamma^\nu{}_\mu\xi_-\\
&-\left(\left(SQ_{\nu ij}+\frac{1}{2}\left(P_\nu\eta M\right)_{ij}\right)-K_i{}^l\left(SQ_{\nu lk}+\frac{1}{2}\left(P_\nu\eta M\right)_{lk}\right)K^k{}_j\right)\Gamma^{ij}\gamma^\nu{}_\mu\xi_-\\
&-\frac{1}{2}\left(\left(P_\mu\eta M\right)_{ij}+\left(KP_\mu\eta MK\right)_{ij}\right)\Gamma^{ij}\xi_-,
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Einstein equation

Integrability conditions

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$$\begin{aligned}
&-\frac{1}{2}M^i{}_j\,\bar{\Gamma}^j\,\gamma_\mu\,\delta\lambda_{i+} + \frac{1}{2}P_\nu{}^i{}_j\,\gamma^\nu\,\Gamma^j\,\gamma_\mu\,\delta\lambda_{i-} \\
&+\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \left(P_{\mu ij}P_\nu{}^{ij} - \frac{1}{2}g_{\mu\nu}P_{\rho ij}P^{\rho ij}\right) + g_{\mu\nu}\left(-12S^2 + \frac{1}{2}M_{ij}M^{ij}\right)\right)\gamma^\nu\xi_+ \\
&-4\left(\partial_\nu S + \frac{1}{4}M_{ij}P_\nu{}^{ij}\right)\gamma^\nu{}_\mu\xi_- \\
&-\left(\left(SQ_{\nu ij} + \frac{1}{2}\left(P_\nu\eta M\right)_{ij}\right) - K_i{}^l\left(SQ_{\nu lk} + \frac{1}{2}\left(P_\nu\eta M\right)_{lk}\right)K^k{}_j\right)\Gamma^{ij}\gamma^\nu{}_\mu\xi_- \\
&-\frac{1}{2}\left(\left(P_\mu\eta M\right)_{ij} + \left(KP_\mu\eta MK\right)_{ij}\right)\Gamma^{ij}\xi_-,
\end{aligned}$$

Equations for S and M_{ij}

Integrability conditions

The integrability conditions with the Killing spinor equations

$$\gamma_\mu{}^{\nu\rho}\,[\mathcal{D}_\nu, \mathcal{D}_\rho]\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}, \qquad \gamma^\mu D_\mu \delta\lambda_{i-},$$

- the Einstein equation and the scalar equations of motion part
- the remaining part, which give a number of equations for S and M_{ij}.

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Result : The scalar potential is

$$V = M_{ij} M^{ij} - 24 S^2,$$

where

$$S = \frac{3}{4}\mathcal{V}_{11}\,\mathcal{V}_{22}, \qquad M_{ij} = \mathcal{V}_{11}\,\mathcal{V}_{22}\,\mathrm{diag}(1,1,2).$$

Summary

Via dimensional reduction on AdS_6, the problem of finding AdS_6 solutions of type IIB supergravity is reduced to a four-dimensional non-linear sigma model, i.e. a gravity theory coupled to 5 scalar fields with a non-trivial scalar potential.

$$\mathcal{L} = \sqrt{g_4}\left[R - \mathrm{Tr}\left(P_\mu P^\mu\right) - \left(M_{ij}M^{ij} - 24S^2\right)\right].$$

- The scalar kinetic terms parameterize $SL(3,\mathbb{R})/SO(2,1)$.
- The scalar potential breaks the global symmetry to a certain subalgebra.
- The Killing spinor equations are re-written in terms of coset representative $\mathcal{V}$.

1. Action of G

The coset representative $\mathcal{V}$ transforms as

$$\mathcal{V} \longrightarrow K\,\mathcal{V}\,G,$$

under global G and local K transformations.

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Our case : $G/K = SL(3,\mathbb{R})/SO(2,1)$

The scalar potential is invariant under the action of h_1, e_1, e_2, e_3 and f_1.

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$$\begin{aligned} e^{\phi} &\to e^{\phi} + 2aCe^{\phi}, \\ C &\to C + a\left(e^{-2\phi} - C^2\right), \\ f &\to f - \frac{a}{2}g. \end{aligned}$$

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This corresponds to $SL(2,\mathbb{R})$ transformation with $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ in type IIB supergravity.

1. Action of G

$h_1,\ e_1,\ f_1$ generate $SL(2,\mathbb{R})$ symmetry and e_2, e_3 generate trivial gauge symmetries. This is consistent with the algebra $A_{5,40} \cong sl(2,\mathbb{R}) \ltimes \mathbb{R}^2$ generated by 5 Killing vectors.

- The two known AdS_6 solutions can be re-written in terms of coset representative $\mathcal{V}$ to show the symmetry manifestly.
- By applying group transformations on $\mathcal{V}$ of this "seed solution", we could not obtain a new solution. The transformed solution can be obtained by $SL(2,\mathbb{R})$ symmetry of type IIB supergravity directly.

2. $SL(n,\mathbb{R})/SO(p,q)$ generalization

We have generalized this construction to $SL(n,\mathbb{R})/SO(p,q)$.

$$S = \frac{n}{4}\left(\prod_{i'=1}^{n-1}\mathcal{V}_{i'i'}\right), \qquad M_{ij} = \left(\prod_{i'=1}^{n-1}\mathcal{V}_{i'i'}\right)\tilde{M}_i{}^k\eta_{kj}.$$

$$\eta = \mathrm{diag}(\underbrace{1,\cdots,1}_{p},\underbrace{-1\cdots,-1}_{q}),$$

$$K_i{}^j = \mathrm{diag}(\underbrace{1,\cdots,1}_{n-1},-1),$$

$$\tilde{M}_i{}^j = \mathrm{diag}(\underbrace{1,\cdots,1}_{n-1},-(n-1)).$$

2. $SL(n,\mathbb{R})/SO(p,q)$ generalization

We can construct or classify some D=4 gravity theories

- coupled to non-linear sigmal model of $\frac{1}{2}\left(n^2+n-2\right)$ scalar fields
- admitting Killing spinor equations without supersymmetry.

Example

- $SL(2,\mathbb{R})/SO(1,1)$ case

$$\mathcal{L} = \sqrt{g_4}\Big[R - \frac{1}{2}(\partial A)^2 + \frac{1}{2}e^{-2A}(\partial B)^2 + 4\,a^2\,e^{-A}\Big].$$

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- $SL(3,\mathbb{R})/SO(2,1)$ case

$$\mathcal{L} = \sqrt{g_4}\Big[R - 24(\partial U)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\phi}(\partial C)^2 + \frac{1}{2}e^{-12U-\phi}(\partial f)^2 + \frac{1}{8}e^{-12U+\phi}(\partial g + 2C\partial f)^2 - 30e^{-8U}\Big].$$

3. AdS_6 in M-theory

The similar structure appears in AdS_6 solutions in M-theory.

There are two scalars : the warp factor U and a dualized scalar f.

We obtain a five-dimensional effective Lagrangian as

$$\mathcal{L} = \sqrt{g_5}\left(R - 18\,(\partial U)^2 + 18\,e^{-12U}\,(\partial f)^2 - 30\,e^{-6U}\right).$$

The target space parameterize the coset $SL(2,\mathbb{R})/SO(1,1)$.

the Killing spinor equations, the integrability conditions, $\cdots$.

Concluding remarks

- Searching for new supersymmetric AdS_6 solutions by solving the two coupled PDEs directly.
- Constructing the dual field theories for IIB AdS_6 solutions.

Thank you!!!

Appendix

The form fields

$$dC = \frac{e^{-2U-\phi}}{y\sqrt{e^{8U+\phi} - e^{\phi}x^2 - e^{4U}y^2}}\Big[2(e^{8U+\phi} + e^{\phi}x^2)dU - \frac{1}{2}(e^{8U+\phi} - e^{\phi}x^2 - 2e^{4U}y^2)d\phi - \frac{2}{3}e^{\phi}xdx\Big],$$

$$*\mathrm{Re}\,G = -\frac{2}{y}e^{-6U-\phi/2}\Big[(e^{8U+\phi} + e^{\phi}x^2 + 2e^{4U}y^2)dU - \frac{1}{4}(e^{8U+\phi} - e^{\phi}x^2)d\phi - \frac{1}{3}e^{\phi}xdx - 2e^{4U}ydy\Big],$$

$$*\mathrm{Im}\,G = 2\frac{e^{-4U-\phi/2}}{\sqrt{e^{8U+\phi} - e^{\phi}x^2 - e^{4U}y^2}}\Big[(3e^{8U+\phi} - e^{\phi}x^2 - 2e^{4U}y^2)dU + \frac{1}{4}(e^{8U+\phi} - e^{\phi}x^2)d\phi + \frac{1}{3}e^{\phi}xdx + 2e^{4U}ydy\Big].$$

The fluxes are written in terms of the warp factor U and the dilation ϕ.

return

KIlling spinor equations

return

$$
\begin{aligned}
2i\delta\tilde{\lambda}_{\pm}=&\Big(-\partial_\mu\phi-ie^{\phi}\partial_\mu C\Big)\gamma^\mu\xi_{2\mp}+\Big(\tfrac{1}{4}e^{-6U+\phi/2}(\partial_\mu g+2C\partial_\mu f)+\tfrac{i}{2}e^{-6U-\phi/2}\partial_\mu f\Big)\gamma^\mu\gamma_5\xi_{1\mp},\\
2i\delta\hat{\lambda}_{\pm}=&\Big(-\partial_\mu\phi+ie^{\phi}\partial_\mu C\Big)\gamma^\mu\xi_{1\mp}+\Big(\tfrac{1}{4}e^{-6U+\phi/2}(\partial_\mu g+2C\partial_\mu f)-\tfrac{i}{2}e^{-6U-\phi/2}\partial_\mu f\Big)\gamma^\mu\gamma_5\xi_{2\mp},\\
\tfrac{4}{3}e^{-3U}\delta\tilde{\chi}_{\pm}=&\tfrac{4i}{L}e^{-4U}\xi_{1\pm}+4\partial_\mu U\gamma^\mu\xi_{1\mp}+\Big(\tfrac{1}{4}e^{-6U+\phi/2}(\partial_\mu g+2C\partial_\mu f)+\tfrac{i}{2}e^{-6U-\phi/2}\partial_\mu f\Big)\gamma^\mu\gamma_5\xi_{2\mp},\\
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\delta\tilde{\psi}_{\mu\pm}=&\nabla_\mu\xi_{1\pm}-\tfrac{3}{2}\partial_\nu U\gamma_\mu\gamma^\nu\xi_{1\pm}+\tfrac{i}{4}e^{\phi}\partial_\mu C\xi_{1\pm}\\
&-\tfrac{3}{8}\Big(\tfrac{1}{4}e^{-6U+\phi/2}(\partial_\nu g+2C\partial_\nu f)+\tfrac{i}{2}e^{-6U-\phi/2}\partial_\nu f\Big)\gamma_\mu\gamma^\nu\gamma_5\xi_{2\pm}\\
&+\tfrac{1}{2}\Big(\tfrac{1}{4}e^{-6U+\phi/2}(\partial_\mu g+2C\partial_\mu f)+\tfrac{i}{2}e^{-6U-\phi/2}\partial_\mu f\Big)\gamma_5\xi_{2\pm},\\
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\end{aligned}
$$