Exact Duality of The Dissipative Hofstadter Model on a Triangular Lattice: T-Duality and Non-commutative Algebra

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- Ref.:T. Lee, (2016) arXiv:1601.07757

I. Introduction

- Dissipative Hofstadter model
  = Open string theory in the background of the NS-B field and the tachyon potential

$$
\begin{aligned}
S &= \frac{1}{4\pi\alpha'}\int d\tau d\sigma E_{ab}(\partial_\tau + \partial_\sigma)X^a(\partial_\tau - \partial_\sigma)X^b \\
&\quad + \frac{g\pi}{2}\int_{\partial M}\frac{d\sigma}{2\pi}\sum_a\left(e^{iX^a} + e^{-iX^a}\right)
\end{aligned}
$$

where $E_{ab} = (g + 2\pi\alpha' B)_{ab}$ and $\alpha = 1/\alpha', \quad \beta = 2\pi B$

Dissipative Hofstadter model

- Dissipative Hofstadter model on a Triangular Lattice

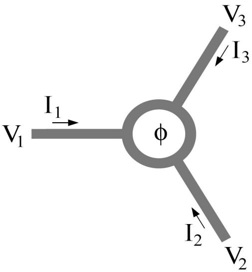
$$
\begin{aligned}
S &= \frac{\eta}{4\pi\hbar}\int_{-\beta_T/2}^{\beta_T/2} dt dt' \frac{(\boldsymbol{X}(t)-\boldsymbol{X}(t'))^2}{(t-t')^2} \\
&+ \frac{ieB_H}{2\hbar c}\int_{-\beta_T/2}^{\beta_T/2} dt \sum_{a,b=1}^{2} \epsilon^{ab}\partial_t X^a X^b \\
&+ \frac{V_0}{\hbar}\int_{-\beta_T/2}^{\beta_T/2} dt \sum_{a=1}^{2} \cos\frac{2\pi \boldsymbol{k}^a\cdot\boldsymbol{X}}{l},
\end{aligned}
$$

where $\beta_T = 1/T$ and

$$
\boldsymbol{k}_1 = (\frac{1}{2},\frac{\sqrt{3}}{2}), \quad \boldsymbol{k}_2 = (\frac{1}{2},-\frac{\sqrt{3}}{2}), \quad \boldsymbol{k}_3 = (-1,0)\,.
$$

Dissipative Hofstadter model

- Dissipative Hofstadter model and Quantum Wires



- Luttinger liquid and String theory: TL parameter, Regge slope and friction coefficient

$$\alpha = 1/\alpha' = \eta/2\pi.$$

Dissipative Hofstadter model

- DHM on triangular lattice and Y-Junction of quantum wires
- Hopping interaction between wires

$$\sum_{a=1}^{3} \left(\psi_L^{a\dagger} \psi_L^{a+1} - \psi_R^{a\dagger} \psi_R^{a+1} \right)$$

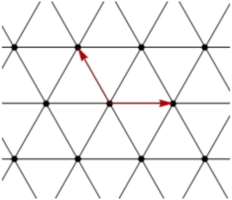
where $\psi_{L/R}^4 = \psi_{L/R}^1$

- Bosonization of hopping interaction

$$\sum_{a=1}^{3} \left(e^{i \frac{\phi^a - \phi^{a+1}}{\sqrt{2}}} + e^{-i \frac{\phi^a - \phi^{a+1}}{\sqrt{2}}} \right) = \sum_{a=1}^{3} \left(e^{i \boldsymbol{k}^a \cdot \boldsymbol{X}} + e^{-i \boldsymbol{k}^a \cdot \boldsymbol{X}} \right)$$

Non-Abelian Statistics and Quantum Computation

- Non-commutativity and Q-Deformation
- Triangular Lattice:



Elementary excitations of the system

$\Longrightarrow$ $SU(3)$ Non-Abelian Statistics

$\Longrightarrow$ Quantum Computation

Target Space Dual Transformation: $O(2,2;\mathbf{R})$

- Boundary condition in commutative basis

$$
\begin{aligned}
X^a &= X^a_L + X^a_R, \\
X^a_L &= \frac{1}{\sqrt{2}} x^a_L + \frac{1}{\sqrt{2}} p^a_L \sigma + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\alpha^a_n}{n} e^{-ni\sigma}, \\
X^a_R &= \frac{1}{\sqrt{2}} x^a_R - \frac{1}{\sqrt{2}} p^a_R \sigma + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}^a_n}{n} e^{ni\sigma},
\end{aligned}
$$

$$
\left(E_{ab}\alpha^b_{-n} + E^t_{ab}\tilde{\alpha}^b_n\right)|B_E\rangle = 0, \quad p^b|B_E\rangle = 0, \;\; a, b = 1, 2.
$$

Target Space Dual Transformation: $O(2,2;\mathbf{R})$

- $O(2,2;\mathbf{R})$ T-dual transformation

$$
\begin{aligned}
T &= \begin{pmatrix} I & 0 \\ \theta/(2\pi) & I \end{pmatrix}, \quad T^t\, J\, T = J, \\
\theta/(2\pi) &= \frac{1}{E}(2\pi B)\frac{1}{E^t} = \frac{\beta}{\alpha^2+\beta^2}\epsilon, \\
\alpha^a_n &= \left(G(E)^{-1}\right)^a{}_b\, \beta^b_n, \\
\tilde{\alpha}^a_n &= \left(G(E^t)^{-1}\right)^a{}_b\, \tilde{\beta}^b_n \\
G &= E^t g^{-1} E = \left(\frac{\alpha^2+\beta^2}{\alpha}\right) I.
\end{aligned}
$$

- Boundary condition in non-commutative basis :

$$
\left(\beta^a_{-n} + \tilde{\beta}^a_n\right)|B_E\rangle = 0, \quad a = 1,2.
$$

Boundary State and Magic Circles

- Non-Commutativity

$$\left[X^a(\sigma_1), X^b(\sigma_2)\right] = i\,\theta^{ab}.$$

$$X^a(\sigma,0) = Z^a(\sigma,0) + \frac{i}{\sqrt{2}}\frac{\beta}{\alpha}\sum_{n\neq 0}\frac{1}{n}\epsilon^{ab}\left(\beta^b_n + \tilde{\beta}^b_{-n}\right)e^{in\sigma},$$

$$Z^a(\sigma,0) = x^a + \omega^a\sigma + i\frac{1}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left[\beta^a_n e^{in\sigma} + \tilde{\beta}^a_n e^{-in\sigma}\right],$$

- Boundary State

$$Z = \langle 0|B\rangle,$$

$$\langle T\mathcal{O}_1\ldots\mathcal{O}_n\rangle = \langle 0| : \mathcal{O}_1\ldots\mathcal{O}_n : |B\rangle.$$

- Boundary State of DHM on Triangular Lattice

$$|B\rangle = \boldsymbol{T} \exp\left[\frac{V_0}{2} \int d\sigma \sum_{a=1}^{3} \left(e^{i\boldsymbol{k}^a \cdot \boldsymbol{X}} + e^{-i\boldsymbol{k}^a \cdot \boldsymbol{X}} \right) \Big|_{\tau=0} \right] |B_E\rangle .$$

$$\sum_{a=1}^{3} \left(e^{i\boldsymbol{k}^a \cdot \boldsymbol{X}} + e^{-i\boldsymbol{k}^a \cdot \boldsymbol{X}} \right) = \sum_{a=1}^{3} \Bigg\{ \exp\left(i \sum_{b=1}^{2} \sqrt{\frac{3}{2}} R_{ab} X^b \right) + \exp\left(-i \sum_{b=1}^{2} \sqrt{\frac{3}{2}} R_{ab} X^b \right) \Bigg\}$$

where R_{ab}, for $a = 1, 2, 3$ and $b = 1, 2$, are the components of 3×2 submatrix of an $SO(3)$ rotation matrix (R)

$$(R) = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad (R)^t (R) = (R)(R)^t = I.$$

- Boundary state in the non-commutative basis

$$
\begin{aligned}
|B\rangle \;=\; & \sum_{n^1,n^2,n^3} \frac{1}{n^1!n^2!n^3!}\left(\frac{V}{2}\right)^{n^1+n^2+n^3} \int \prod_{i=1}^{n^1} d\sigma^1{}_i \prod_{i=1}^{n^2} d\sigma^2{}_i \prod_{i=1}^{n^3} d\sigma^3{}_i \\
& \exp\left\{-i\frac{3}{2}\theta \sum_{a,d=1}^{3}\sum_{b,c=1}^{2}\sum_{i=1}^{n^a}\sum_{\sigma^a{}_i>\sigma^b{}_j}^{n^b} e^a{}_i R_{ab}\,\epsilon_{bc}\,(R^t)_{cd}\, e^d{}_j\right\} \\
& \boldsymbol{T}\exp\left\{i\sqrt{\frac{3}{2}}\sum_{a=1}^{3}\sum_{b=1}^{2}\sum_{i=1}^{n^a} R_{ab}Z^b(\sigma^a{}_i)\right\}|B_E\rangle
\end{aligned}
$$

$e^a{}_i = \pm 1$ for $a = 1, 2, 3$ and $i = 1, \ldots, n^a$.

- Non-commutative Phase

$$\frac{3}{2}\theta \sum_{a,b=1}^{3} \sum_{i=1}^{n^a} \sum_{\sigma^a{}_i > \sigma^b{}_j}^{n^b} \frac{1}{\sqrt{3}} e^a{}_i (N)_{ab} e^b{}_j,$$

where

$$(N) = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

- Equivalence Relation

$$\frac{\sqrt{3}}{2}\theta = \frac{\sqrt{3}}{2}\widehat{\theta} + 2\pi n, \quad n \in \mathbf{Z}.$$

Equivalence Relations

- Critical Circle: All the points on the circle

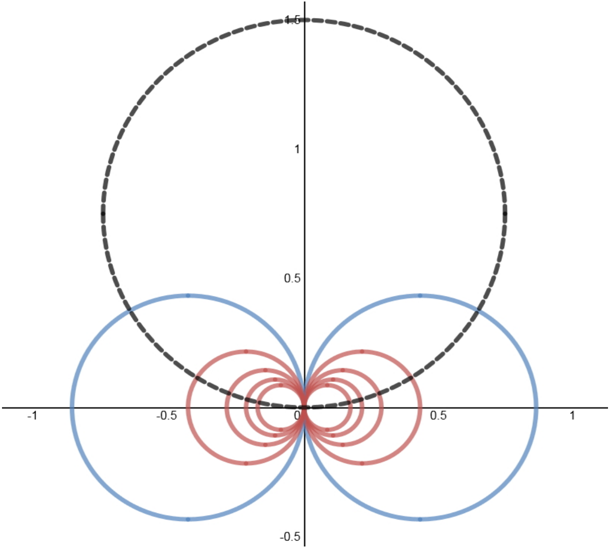
$$\left(\alpha - \frac{\sqrt{\det G}}{2}\right)^2 + \beta^2 = \left(\frac{\sqrt{\det G}}{2}\right)^2,$$

  have the same closed string metric $G_{ab} = \sqrt{\det G}\,\delta_{ab}$.
- Magic Circles: All the points on the circles

$$\alpha^2 + \left(\beta - \frac{1}{2\left(\theta + 2n/\sqrt{3}\right)}\right)^2 = \left(\frac{1}{2\left(\theta + 2n/\sqrt{3}\right)}\right)^2, \quad n \in \mathbf{Z}$$

  share the same non-commutativity parameter θ. When $\theta = 0$, magic circles.

Critical Circles and Magic Circles



Conclusions

- The exact $O(2,2;R)$ duality of the DHM on a triangular lattice: Phase Diagram
- The perturbation analysis of the DHM on a triangular lattice (to be done): RG exponent
- Particle-Kink Duality (to be done)
- Q-deformed Algebra (to be done)
- Realization of non-Abelian Statistics (to be done)