Mathematical Physics

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I. Matrix

1. Permutation and Parity

1.1 Permutation

Definition 1.1 Consider an ordered set $(a^1, a^2, \cdots, a^{n-1}, a^n)$ of n different numbers out of $\{1, 2, 3, \cdots, n\}$. ***Permutation*** σ is a mapping of an ordered set of numbers $(a^1, a^2, \cdots, a^{n-1}, a^n)$ to another

$$(a^1, a^2, \cdots, a^{n-1}, a^n) \xrightarrow{\sigma} (\sigma^{a^1}, \sigma^{a^2}, \cdots, \sigma^{a^n}). \tag{1.1}$$

Note that

$$\{a^1, a^2, \cdots, a^{n-1}, a^n\} = \{\sigma^{a^1}, \sigma^{a^2}, \cdots, \sigma^{a^{n-1}}, \sigma^{a^n}\} = \{1, 2, 3, \cdots, n\}. \tag{1.2}$$

1.2 Parity and Levi-Civita Symbol

Problem 1.2 Show that there are $n!$ distinct mappings for the permutations σ that permutes $(1, 2, \cdots, n)$.

Definition 1.3 The ***parity*** of a permutation σ of $(1, 2, \cdots, n)$ is called even (odd) if it is obtained by an even (odd) number of two-element exchanges. We define the parity operator ϵ such that

$$\epsilon(\sigma) = \begin{cases} +1, \text{ if } \sigma \text{ is even,} \\ -1, \text{ if } \sigma \text{ is odd.} \end{cases} \tag{1.3}$$

Show that

$$\epsilon(\sigma^2\sigma^1) = \epsilon(\sigma^2)\epsilon(\sigma^1). \tag{1.4}$$

Definition 1.4 We say that a permutation σ is the ***inversion*** if

$$\sigma(a^1, a^2, \cdots, a^{n-1}, a^n) = (a^n, a^{n-1}, \cdots, a^2, a^1). \tag{1.5}$$

Problem 1.5 Let us find the parity of the inversion for any positive integer $n \geq 2$.

1. Show that the number $N(n)$ of exchanges for the inversion of $(1, 2, \cdots, n)$ to obtain the permuatation $(n, n-1, \cdots, 2, 1)$ is

$$N(n) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}. \tag{1.6}$$

2. It is trivial to check $N(2) = 1$, $N(3) = 3$, $N(4) = 6$, $N(5) = 10$, $N(6) = 15$, $\cdots$. Therefore, the inversion changes the parity (even $\leftrightarrow$ odd) for $n = 2$ and 3. However, for $n = 4$ and 5, the parity is invariant under inversion.

   (a) Show that parity is invariant under inversion for $n = 4k$ or $4k + 1$ for k is a positive integer.

   (b) Show that the parity changes under inversion for $n = 4k + 2$ or $4k + 3$ for k is a non-negative integer.

3. Show that there are $n!/2$ even permutations of $(1, 2, \cdots, n)$ for $n \geq 2$.

4. Show that there are $n!/2$ odd permutations of $(1, 2, \cdots, n)$ for $n \geq 2$.

Problem 1.6 There is only a single way of exchanging two adjacent elements like

$$(\cdots, a, b, \cdots) \to (\cdots, b, a, \cdots). \tag{1.7}$$

Therefore, the exchange of two adjacent elements is well defined.

1. Let us consider the change of parity after exchanging two elements in a permutation

$$(\cdots, a, b, c, \cdots) \to (\cdots, c, b, a, \cdots). \tag{1.8}$$

   Show that the permutation can be decomposed into an odd number (3) of exchanges of two adjacent indices:

$$(\cdots, a, b, c, \cdots) \to (\cdots, b, a, c, \cdots) \to (\cdots, b, c, a, \cdots) \to (\cdots, c, b, a, \cdots). \tag{1.9}$$

2. Let us consider the exchange of two elements a and c

$$(\cdots, a, b^1, \cdots, b^n, c, \cdots) \to (\cdots, c, b^1, \cdots, b^n, a, \cdots). \tag{1.10}$$

   This permutation σ can be decomposed into

$$\sigma = \sigma^2 \sigma^1, \tag{1.11a}$$

$$\sigma^1(\cdots, a, b^1, \cdots, b^n, c, \cdots) = (\cdots, c, a, b^1, \cdots, b^n, \cdots), \tag{1.11b}$$

$$\sigma^2(\cdots, c, a, b^1, \cdots, b^n, \cdots) = (\cdots, c, b^1, \cdots, b^n, a, \cdots). \tag{1.11c}$$

Show that the parity of $\epsilon(\sigma^1) = (-1)^{n+1}$ and $\epsilon(\sigma^2) = (-1)^n$. Therefore, the parity of $\sigma = \epsilon(\sigma^2)\epsilon(\sigma^1) = -1$ that is independent of n.

Problem 1.7 Consider a permutation of an ordered set of two numbers $(1, 2)$.

1. Show that there are two permutations $\sigma(1, 2) = (1, 2)$ and $\sigma(1, 2) = (2, 1)$.

2. Show that the identity permutation $\sigma(1, 2) = (1, 2)$ is even.

3. Show that $\sigma(1, 2) = (2, 1)$ is odd.

Problem 1.8 Consider a permutation of an ordered set of three numbers $(1, 2, 3)$.

1. Show that there are 6 permutations.

2. Show that the permutations $\sigma(1, 2, 3) = (1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ are even.

3. Show that the permutations $\sigma(1, 2, 3) = (3, 2, 1)$, $(1, 3, 2)$, $(2, 1, 3)$ are odd.

Problem 1.9 Consider a permutation of an ordered set of four numbers $(1, 2, 3, 4)$.

1. Show that there are 24 permutations.

2. Show that the permutations all possible even permutations of the form $(a, b, c, 4)$ are $\sigma(1, 2, 3, 4) = (1, 2, 3, 4)$, $(2, 3, 1, 4)$, $(3, 1, 2, 4)$.

3. Show that the permutations all possible even permutations of the form $(a, b, 4, c)$ are $\sigma(1, 2, 3, 4) = (2, 1, 4, 3)$, $(3, 2, 4, 1)$, $(1, 3, 4, 2)$.

4. Show that the permutations all possible even permutations of the form $(a, 4, b, c)$ are $\sigma(1, 2, 3, 4) = (1, 4, 2, 3)$, $(2, 4, 3, 1)$, $(3, 4, 1, 2)$.

5. Show that the permutations all possible even permutations of the form $(4, a, b, c)$ are $\sigma(1, 2, 3, 4) = (4, 2, 1, 3)$, $(4, 3, 2, 1)$, $(4, 1, 3, 2)$.

6. Show that the permutations all possible odd permutations of the form $(a, b, c, 4)$ are $\sigma(1, 2, 3, 4) = (2, 1, 3, 4)$, $(2, 3, 1, 4)$, $(1, 3, 2, 4)$.

7. Show that the permutations all possible odd permutations of the form $(a, b, 4, c)$ are $\sigma(1, 2, 3, 4) = (1, 2, 4, 3)$, $(2, 3, 4, 1)$, $(3, 1, 4, 2)$.

8. Show that the permutations all possible odd permutations of the form $(a, 4, b, c)$ are $\sigma(1, 2, 3, 4) = (1, 4, 3, 2),\ (2, 4, 1, 3),\ (3, 4, 2, 1)$.

9. Show that the permutations all possible odd permutations of the form $(4, a, b, c)$ are $\sigma(1, 2, 3, 4) = (4, 2, 3, 1),\ (4, 3, 1, 2),\ (4, 1, 2, 3)$.

$\sigma(1, 2, 3, 4)$ can be generated by making use of the following MATHEMATICA command:

```
Permutations[{1, 2, 3, 4}]
```

1.3 Algebra involving Levi-Civita Symbols

Definition 1.10 An ***antisymmetric permutation symbol*** ϵ for an ordered set of numbers $(a^1, \cdots, a^n) = \sigma(1, 2, 3, \cdots, n)$ where $a^i \in \{1, 2, 3, \cdots, n\}$ is defined by

$$\epsilon(a^1, \cdots, a^n) = \begin{cases} +1, & \text{if } \sigma \text{ is even,} \\ -1, & \text{if } \sigma \text{ is odd,} \\ \ \ 0, & \text{if } \{a^1, \cdots, a^n\} \neq \{1, 2, 3, \cdots, n\}. \end{cases} \tag{1.12}$$

ϵ is also called the ***Levi-Civita symbol***. Note that the last case is that there exists at least one pair such that $a^i = a^j$ for $i \neq j$.

Problem 1.11 Let us consider the permutations of $(1, 2)$.

1. Show that

$$\epsilon_{11} = 0, \quad \epsilon_{12} = 1, \quad \epsilon_{21} = -1, \quad \epsilon_{22} = 0. \tag{1.13}$$

2. Show that

$$\sum_{i,j} \epsilon_{ij} = 0. \tag{1.14}$$

3. Show that

$$\epsilon_{11}^2 = 0, \quad \epsilon_{12}^2 = 1, \quad \epsilon_{21}^2 = 1, \quad \epsilon_{22}^2 = 0. \tag{1.15}$$

4. Show that

$$\sum_{i,j} \epsilon_{ij}^2 = 2!. \tag{1.16}$$

We use Einstein's convention that any repeated indices are assumed to be summed over:

$$\epsilon_{ij}\epsilon_{ij} = \sum_{i,j} \epsilon_{ij}\epsilon_{ij}. \tag{1.17}$$

Problem 1.12 Let us consider the permutation of $(1, 2, 3)$.

1. Show that the only non-vanishing elements ϵ_{ijk} are

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1, \tag{1.18a}$$

$$\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1, \tag{1.18b}$$

2. Show that

$$\sum_{i,j,k} \epsilon_{ijk} = 0. \tag{1.19}$$

3. Show that

$$\epsilon_{123}^2 = \epsilon_{231}^2 = \epsilon_{312}^2 = +1, \tag{1.20a}$$

$$\epsilon_{321}^2 = \epsilon_{132}^2 = \epsilon_{213}^2 = +1, \tag{1.20b}$$

4. Show that

$$\sum_{i,j,k} \epsilon_{ijk}^2 = 3!. \tag{1.21}$$

Problem 1.13 Let us consider the permutation of $(1, 2, 3)$.

1. Show that the only non-vanishing elements ϵ_{ijk} are

$$\epsilon_{1234} = \epsilon_{2314} = \epsilon_{3124} = +1, \tag{1.22a}$$

$$\epsilon_{2143} = \epsilon_{3241} = \epsilon_{1342} = +1, \tag{1.22b}$$

$$\epsilon_{1423} = \epsilon_{2431} = \epsilon_{3412} = +1, \tag{1.22c}$$

$$\epsilon_{4213} = \epsilon_{4321} = \epsilon_{4132} = +1, \tag{1.22d}$$

$$\epsilon_{2134} = \epsilon_{2314} = \epsilon_{1324} = -1, \tag{1.22e}$$

$$\epsilon_{1243} = \epsilon_{2341} = \epsilon_{3142} = -1, \tag{1.22f}$$

$$\epsilon_{1432} = \epsilon_{2413} = \epsilon_{3421} = -1, \tag{1.22g}$$

$$\epsilon_{4231} = \epsilon_{4312} = \epsilon_{4123} = -1. \tag{1.22h}$$

2. Show that

$$\sum_{i,j,k,\ell} \epsilon_{ijk\ell} = 0. \tag{1.23}$$

3. Show that

$$\epsilon_{1234}^2 = \epsilon_{2314}^2 = \epsilon_{3124}^2 = +1, \tag{1.24a}$$

$$\epsilon_{2143}^2 = \epsilon_{3241}^2 = \epsilon_{1342}^2 = +1, \tag{1.24b}$$

$$\epsilon_{1423}^2 = \epsilon_{2431}^2 = \epsilon_{3412}^2 = +1, \tag{1.24c}$$

$$\epsilon_{4213}^2 = \epsilon_{4321}^2 = \epsilon_{4132}^2 = +1, \tag{1.24d}$$

$$\epsilon_{2134}^2 = \epsilon_{2314}^2 = \epsilon_{1324}^2 = +1, \tag{1.24e}$$

$$\epsilon_{1243}^2 = \epsilon_{2341}^2 = \epsilon_{3142}^2 = +1, \tag{1.24f}$$

$$\epsilon_{1432}^2 = \epsilon_{2413}^2 = \epsilon_{3421}^2 = +1, \tag{1.24g}$$

$$\epsilon_{4231}^2 = \epsilon_{4312}^2 = \epsilon_{4123}^2 = +1. \tag{1.24h}$$

4. Show that

$$\sum_{i,j,k,\ell} \epsilon_{ijk\ell}^2 = 4!. \tag{1.25}$$

Problem 1.14 By making use of mathematical induction, show that

$$\epsilon_{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n} = n!. \tag{1.26}$$

1.4 Application of Levi-Civita Symbols to Vector Analysis

Problem 1.15 Let us consider vectors defined in a 3-dimensional Euclidean space. By making use of Levi-Civita symbols, prove the following identities.

1. $BAC - CAB$ rule:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}). \tag{1.27}$$

2. Jacobi identity:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{0}. \tag{1.28}$$

3. Verify the identity and interpret the answer based on permutation and parity:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C}) \cdot (\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D}) \cdot (\boldsymbol{B} \cdot \boldsymbol{C}). \tag{1.29}$$

4. Verify the identity and interpret the answer based on trigonometry:

$$(\boldsymbol{A} \times \boldsymbol{B})^2 = \boldsymbol{A}^2 \boldsymbol{B}^2 - (\boldsymbol{A} \cdot \boldsymbol{B})^2. \tag{1.30}$$

5. Verify the identity and interpret the sign of each term based on permutation and parity:

$$\begin{aligned}(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}) &= (\boldsymbol{A}\cdot\boldsymbol{C}\times\boldsymbol{D})\boldsymbol{B}-(\boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{D})\boldsymbol{A}\\ &= (\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{D})\boldsymbol{C}-(\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C})\boldsymbol{D}.\end{aligned} \tag{1.31}$$

Problem 1.16 Consider a permutation of $(1,2,3)$ and its parity.

1. Show that the triple scalar product of three three-vectors can be expressed as

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=\epsilon_{ijk}a^i b^j C^k. \tag{1.32}$$

2. By making use of the parity properties of permutations, show that

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}=\boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A}=\boldsymbol{C}\cdot\boldsymbol{A}\times\boldsymbol{B}. \tag{1.33}$$

Problem 1.17 The curl of a vector field in a 3-dimensional Euclidean space is defined by

$$(\boldsymbol{\nabla}\times\boldsymbol{A})_i=\epsilon_{ijk}\frac{\partial}{\partial x^j}a^k. \tag{1.34}$$

The vector fields $\boldsymbol{A}$ and $\boldsymbol{B}$ and the scalar field ϕ are dependent on the position. By making use of the Levi-Civita symbol, verify the following formulas.

1.

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B})=(\boldsymbol{\nabla}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A}-(\boldsymbol{\nabla}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}, \tag{1.35}$$

2.

$$\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=\boldsymbol{\nabla}_{\boldsymbol{B}}(\boldsymbol{A}\cdot\boldsymbol{B})-(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B}, \tag{1.36}$$

where the gradient operator $\boldsymbol{\nabla}_{\boldsymbol{B}}$ with the subscript $\boldsymbol{B}$ acts only on $\boldsymbol{B}$.

3.

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}. \tag{1.37}$$

4.

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)=\boldsymbol{0}. \tag{1.38}$$

5.

$$\boldsymbol{\nabla}\times(\phi\boldsymbol{A})=(\boldsymbol{\nabla}\phi)\times\boldsymbol{A}+\phi(\boldsymbol{\nabla}\times\boldsymbol{A}). \tag{1.39}$$

2. Determinant

2.1 Definition

Definition 2.1 The ***determinant*** of an $n \times n$ square matrix A is defined by

$$\begin{aligned}
\mathfrak{Det}[A] &\equiv \begin{vmatrix} a^{11} & a^{12} & a^{13} & \cdots & a^{1n} \\ a^{21} & a^{22} & a^{23} & \cdots & a^{2n} \\ a^{31} & a^{32} & a^{33} & \cdots & a^{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & a^{n3} & \cdots & a^{nn} \end{vmatrix} \\
&\equiv \sum_{\sigma} \epsilon_{\sigma^1\sigma^2\sigma^3\cdots\sigma^n} a^{1\sigma^1} a^{2\sigma^2} \cdots a^{n-1\sigma^{n-1}} a^{n\sigma^n} \\
&= \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{n} a^{i\sigma^i}, \qquad (2.1)
\end{aligned}$$

where the sum is over $n!$ permuations σ of $(1, 2, \cdots, n)$ and

$$(\sigma^1, \sigma^2, \cdots, \sigma^n) = \sigma(1, 2, \cdots, n). \qquad (2.2)$$

The Levi-Civita symbol is defined by the parity of a permutation σ:

$$\epsilon_{\sigma^1\sigma^2\sigma^3\cdots\sigma^n} \equiv \epsilon[\sigma(1, 2, \cdots, n)] = \epsilon(\sigma). \qquad (2.3)$$

Problem 2.2 Let σ and τ be permutations of $(1, 2, \cdots, n)$. Show that

1. $\epsilon(\mathbb{1}) = 1$, where $\mathbb{1}$ is the identity permutation.

2. $\epsilon(\sigma^{-1}) = \epsilon(\sigma)$.

3. $\epsilon(\sigma\tau) = \epsilon(\sigma)\epsilon(\tau) = \epsilon(\tau\sigma)$.

Problem 2.3 Show that

$$\begin{aligned}
\mathfrak{Det}[A] &= \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{n} a^{i\sigma^i} \\
&= \frac{1}{n!} \sum_{\sigma,\tau} \epsilon(\tau\sigma) \prod_{i=1}^{n} a^{\tau_i\sigma^i} = \frac{1}{n!} \sum_{\sigma,\tau} \epsilon(\sigma\tau) \prod_{i=1}^{n} a^{\tau_i\sigma^i} = \frac{1}{n!} \sum_{\sigma,\tau} \epsilon(\tau\sigma) \prod_{i=1}^{n} a^{\sigma^i\tau_i} \\
&= \sum_{\sigma} \epsilon(\sigma) \prod_{i=1}^{n} a^{\sigma^i i}. \qquad (2.4)
\end{aligned}$$

Problem 2.4 Let us consider the determinant of a 2×2 matrix.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.5}$$

1. Show that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{2.6}$$

Problem 2.5 Let us compute the determinant of a 3×3 square matrix.

1. Show that

$$\begin{aligned}
\begin{vmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{vmatrix} &= +a^{11} \begin{vmatrix} a^{22} & a^{23} \\ a^{32} & a^{33} \end{vmatrix} - a^{12} \begin{vmatrix} a^{21} & a^{23} \\ a^{31} & a^{33} \end{vmatrix} + a^{13} \begin{vmatrix} a^{21} & a^{22} \\ a^{31} & a^{32} \end{vmatrix} \\
&= -a^{21} \begin{vmatrix} a^{12} & a^{13} \\ a^{32} & a^{33} \end{vmatrix} + a^{22} \begin{vmatrix} a^{11} & a^{13} \\ a^{31} & a^{33} \end{vmatrix} - a^{23} \begin{vmatrix} a^{11} & a^{12} \\ a^{31} & a^{32} \end{vmatrix} \\
&= +a^{31} \begin{vmatrix} a^{12} & a^{13} \\ a^{22} & a^{23} \end{vmatrix} - a^{32} \begin{vmatrix} a^{11} & a^{13} \\ a^{21} & a^{23} \end{vmatrix} + a^{33} \begin{vmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{vmatrix}.
\end{aligned} \tag{2.7}$$

2. Show that

$$\begin{aligned}
\begin{vmatrix} a^{11} & a^{12} & a^{13} \\ a^{21} & a^{22} & a^{23} \\ a^{31} & a^{32} & a^{33} \end{vmatrix} &= +a^{11} \begin{vmatrix} a^{22} & a^{23} \\ a^{32} & a^{33} \end{vmatrix} - a^{21} \begin{vmatrix} a^{12} & a^{13} \\ a^{32} & a^{33} \end{vmatrix} + a^{31} \begin{vmatrix} a^{12} & a^{13} \\ a^{22} & a^{23} \end{vmatrix} \\
&= +a^{12} \begin{vmatrix} a^{21} & a^{23} \\ a^{31} & a^{33} \end{vmatrix} - a^{22} \begin{vmatrix} a^{11} & a^{13} \\ a^{31} & a^{33} \end{vmatrix} + a^{32} \begin{vmatrix} a^{11} & a^{13} \\ a^{21} & a^{23} \end{vmatrix} \\
&= +a^{13} \begin{vmatrix} a^{21} & a^{22} \\ a^{31} & a^{32} \end{vmatrix} - a^{23} \begin{vmatrix} a^{11} & a^{13} \\ a^{31} & a^{33} \end{vmatrix} + a^{33} \begin{vmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{vmatrix}.
\end{aligned} \tag{2.8}$$

Problem 2.6 Show that for any n,

$$\mathfrak{Det}[\mathbb{1}] = 1, \tag{2.9}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix.

Problem 2.7 Provide the reason why the determinant of each of the following matrices vanishes.

1.

$$\mathfrak{Det}\begin{pmatrix} 1&1&1&1&1&1&1\\ 1&1&1&1&1&0&1\\ 1&1&1&1&0&1&1\\ 1&1&1&0&1&1&1\\ 1&1&0&1&1&1&1\\ 1&0&1&1&1&1&1\\ 0&0&0&0&0&0&0 \end{pmatrix} = 0. \tag{2.10}$$

2.

$$\mathfrak{Det}\begin{pmatrix} 1&1&1&1&1&1&-1\\ 2&1&1&1&1&0&-2\\ 3&1&1&1&0&1&-3\\ 4&1&1&0&1&1&-4\\ 5&1&0&1&1&1&-5\\ 6&0&1&1&1&1&-6\\ 7&0&0&0&0&0&-7 \end{pmatrix} = 0. \tag{2.11}$$

3.

$$\mathfrak{Det}\begin{pmatrix} 1&1&1&1&1&2&7\\ 2&0&1&1&1&1&6\\ 3&0&0&1&0&0&5\\ 4&0&0&0&1&0&4\\ 5&0&0&0&1&0&3\\ 6&0&0&0&1&0&2\\ 7&0&0&0&0&0&1 \end{pmatrix} = 0. \tag{2.12}$$

2.2 Expressions involving Levi-Civita Symbols

Problem 2.8 The tensor $\epsilon_{ij}\epsilon_{ab}$ must be antisymmetric under exchange of $i \leftrightarrow j$ and under exchange of $a \leftrightarrow b$.

1. Show that the only non-vanishing elements of the tensor $\epsilon_{ij}\epsilon_{ab}$ are

$$\epsilon_{12}\epsilon_{12} = \epsilon_{21}\epsilon_{21} = 1, \tag{2.13a}$$

$$\epsilon_{12}\epsilon_{21} = \epsilon_{21}\epsilon_{12} = -1. \tag{2.13b}$$

Therefore, the only non-vanishing cases are

$$\{i, j\} = \{a, b\} = \{1, 2\}, \tag{2.14}$$

$$\epsilon_{ij}\epsilon_{ab} = \begin{cases} +1, \ (ij) = (a, b) = (1, 2) \text{ or } (2, 1) \\ -1, \ (ij) = (b, a) = (1, 2) \text{ or } (2, 1) \end{cases} \tag{2.15}$$

2. Show that this condition is equivalent to

$$\epsilon_{ij}\epsilon_{ab} = \delta^{ia}\delta^{jb} - \delta^{ib}\delta^{ja} = \mathfrak{Det}\begin{pmatrix} \delta^{ia} & \delta^{ib} \\ \delta^{ja} & \delta^{jb} \end{pmatrix}. \tag{2.16}$$

3. By multiplying $\delta^{ia}\delta^{jb}$ to both sides and summing over repeated indices, show that the relation (2.16) is consistent in normalization:

$$2! = (2!)^2 - 2!. \tag{2.17}$$

Problem 2.9 The tensor $\epsilon_{ijk}\epsilon_{abc}$ is non-vanishing only if both $(i, j, k) = \sigma(1, 2, 3)$ and $(a, b, c) = \tau(1, 2, 3)$ are permutations of $(1, 2, 3)$.

1. Show that the only non-vanishing elements of the tensor $\epsilon_{ijk}\epsilon_{abc}$ are

$$\epsilon_{ijk}\epsilon_{abc} = \epsilon(\sigma)\epsilon(\tau) = \epsilon(\sigma\tau). \tag{2.18}$$

2. Show that this condition is equivalent to

$$\epsilon_{ijk}\epsilon_{abc} = \delta^{ia}\delta^{jb}\delta^{kc} + \delta^{ib}\delta^{jc}\delta^{ka} + \delta^{ic}\delta^{ja}\delta^{kb} - \delta^{ic}\delta^{jb}\delta^{ka} - \delta^{ia}\delta^{jc}\delta^{kb} - \delta^{ib}\delta^{ja}\delta^{kc} = \mathfrak{Det}\begin{pmatrix} \delta^{ia} & \delta^{ib} & \delta^{ic} \\ \delta^{ja} & \delta^{jb} & \delta^{jc} \\ \delta^{ka} & \delta^{kb} & \delta^{kc} \end{pmatrix}. \tag{2.19}$$

3. By multiplying $\delta^{ia}\delta^{jb}\delta^{kc}$ to both sides, show that the relation is consistent in normalization:

$$3! = 3^3 + 2 \times 3 - 3 \times 3^2. \tag{2.20}$$

The following REDUCE program reproduces this result.

```
vecdim 3;
vector i,j,k,p,q,r;
m:=mat((i.p,i.q,i.r),(j.p,j.q,j.r),(k.p,k.q,k.r));
f:=det(m);
index i,j,k,p,q,r;
ff:=f*i.p*j.q*k.r;
```

Problem 2.10 Based on mathematical induction, show that

$$\epsilon_{i_1 i_2 i_3 \cdots i_n}\epsilon_{j_1 j_2 j_3 \cdots j_n} = \mathfrak{Det}\begin{pmatrix} \delta^{i_1 j_1} & \delta^{i_1 j_2} & \ldots & \delta^{i_1 j_n} \\ \delta^{i_2 j_1} & \delta^{i_2 j_2} & \ldots & \delta^{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n j_1} & \delta^{i_n j_2} & \ldots & \delta^{i_n j_n} \end{pmatrix}. \tag{2.21}$$

2.3 Basic Properties of Determinant

Problem 2.11 Let us consider a matrix A,

$$A = (a^1\, a^2\, \cdots\, a^n), \quad a^i = \begin{pmatrix} a^{1i} \\ a^{2i} \\ \vdots \\ a^{ni} \end{pmatrix}, \tag{2.22}$$

where a^i is the ith column vector. Let B be the matrix that satisfies the following conditions:

$$\begin{aligned} B &= (b^1\, b^2\, \cdots\, b^n), \\ b^p &= a^q, \\ b^q &= a^p, \\ b^i &= a^i, \text{ for } i \neq p,\ q. \end{aligned} \tag{2.23}$$

Here, $p \neq q$. Show that

$$\mathfrak{Det}[B] = -\mathfrak{Det}[A]. \tag{2.24}$$

Problem 2.12 Let us consider a matrix A,

$$A = \begin{pmatrix} \tilde{A}^1 \\ \tilde{A}^2 \\ \vdots \\ \tilde{A}^n \end{pmatrix}, \quad \tilde{A}^i = \begin{pmatrix} a^{i1} & a^{i2} & \cdots & a^{in} \end{pmatrix}, \tag{2.25}$$

where $\tilde{A}^i$ is the ith row vector of A. Let C be the matrix that satisfies the following conditions:

$$\begin{aligned} C &= \begin{pmatrix} \tilde{C}^1 \\ \tilde{C}^2 \\ \vdots \\ \tilde{C}^n \end{pmatrix}, \\ \tilde{C}^p &= \tilde{A}^q, \\ \tilde{C}^q &= \tilde{A}^p, \\ \tilde{C}^i &= \tilde{A}^i, \text{ for } i \neq p,\ q. \end{aligned} \tag{2.26}$$

Here, $p \neq q$. Show that

$$\mathfrak{Det}[C] = -\mathfrak{Det}[A]. \tag{2.27}$$

Problem 2.13 Show that

$$\begin{aligned} \mathfrak{Det}(A) &= \frac{1}{n!} \sum_{\sigma} \sum_{\tau} \epsilon_{\sigma^1 \sigma^2 \cdots \sigma^n} \epsilon_{\tau_1 \tau_2 \cdots \tau_n} a^{\sigma^1 \tau_1} a^{\sigma^2 \tau_2} \cdots a^{\sigma^n \tau_n} \\ &= \frac{1}{n!} \sum_{\sigma} \sum_{\tau} \epsilon(\sigma) \epsilon(\tau) \prod_{i=1}^{n} a^{\sigma^i \tau_i}, \end{aligned} \tag{2.28}$$

where the sums are over two permutations σ and τ for $(1, 2, 3, \cdots, n)$.

Problem 2.14 Show that for a given $j \in \{1, 2, \cdots, n\}$

$$\mathfrak{Det}[A] = \sum_{i=1}^{n} a^{ij} C^{ij} = \sum_{i=1}^{n} a^{ji} C^{ji}, \tag{2.29}$$

$$C^{ij} = (-1)^{i+j} M^{(ij)}, \tag{2.30}$$

where C^{ij} and $M^{(ij)}$ are called the (ij) cofactor and the (ij) minor of A, respectively. The (ij)

minor $M^{(ij)}$ is the determinant of a submatrix of A in which ith row and jth column are deleted:

$$C^{ij} = (-1)^{i+j} M^{(ij)} = (-1)^{i+j} \begin{vmatrix} a^{11} & a^{12} & \cdots & a^{1\,j-1} & a^{1\,j+1} & \cdots & a^{1n} \\ a^{21} & a^{22} & \cdots & a^{2\,j-1} & a^{2\,j+1} & \cdots & a^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ a^{i-1\,1} & a^{i-1\,2} & \cdots & a^{i-1\,j-1} & a^{i-1\,j+1} & \cdots & a^{i-1\,n} \\ a^{i+1\,1} & a^{i+1\,2} & \cdots & a^{i+1\,j-1} & a^{i+1\,j+1} & \cdots & a^{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & \cdots & a^{n\,j-1} & a^{n\,j+1} & \cdots & a^{nn} \end{vmatrix}. \tag{2.31}$$

Problem 2.15 Verify the following statements for an $n \times n$ square matrix A.

1. If the columns (rows) of A are linearly dependent, then $\mathfrak{Rank}(A) < n$ and $\mathfrak{Det}(A) = 0$.
2. If the columns (rows) of A are linearly independent, then $\mathfrak{Rank}(A) = n$ and $\mathfrak{Det}(A) \neq 0$.
3. If $\mathfrak{Det}(A) = 0$, then $\mathfrak{Rank}(A) < n$ and A is not invertible: A^{-1} does not exist.
4. If A is not invertible, then $\mathfrak{Rank}(A) < n$ and $\mathfrak{Det}(A) = 0$.
5. If $\mathfrak{Det}(A) \neq 0$, then $\mathfrak{Rank}(A) = n$ and A is invertible: A^{-1} exists.
6. If A is invertible, then $\mathfrak{Rank}(A) = n$ and $\mathfrak{Det}(A) \neq 0$.
7. $\mathfrak{Det}(cA) = c^n \mathfrak{Det}(A)$, where c is a number.
8. $\mathfrak{Det}(A^T) = \mathfrak{Det}(A)$.
9. Let $B = (b^{ij})$ and $C = (C^{ij})$ be matrices such that
$$b^{ij} = a^{i\sigma^j}, \quad C^{ij} = a^{\sigma^i j} \tag{2.32}$$
where σ is a permutation of $(1, 2, \cdots, n)$. Show that
$$\mathfrak{Det}(B) = \mathfrak{Det}(C) = \epsilon(\sigma)\mathfrak{Det}(A). \tag{2.33}$$
10. Let $D = (d_{ij})$ be a matrix such that
$$d_{ij} = a^{\sigma^i \tau_j}, \tag{2.34}$$
where σ and τ are permutations of $(1, 2, \cdots, n)$. Show that
$$\mathfrak{Det}(D) = \epsilon(\sigma)\epsilon(\tau)\mathfrak{Det}(A). \tag{2.35}$$

2.4 Factorization of $\mathfrak{Det}[AB] = \mathfrak{Det}[A]\mathfrak{Det}[B]$

Problem 2.16 Let us prove

$$\mathfrak{Det}[AB] = \mathfrak{Det}[A]\mathfrak{Det}[B] \tag{2.36}$$

for 2×2 matrices.

1. Show that

$$\begin{aligned}
\mathfrak{Det}[AB] &= \begin{vmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{vmatrix} = C^{11}C^{22} - C^{12}C^{21} \\
&= (\tilde{A}^1 b^1)(\tilde{A}^2 b^2) - (\tilde{A}^1 b^2)(\tilde{A}^2 b^1) \\
&= \frac{1}{2}\epsilon_{ij}\epsilon_{k\ell}(\tilde{A}^i b^k)(\tilde{A}^j b^\ell) \\
&= \frac{1}{2}\epsilon_{ij}\epsilon_{k\ell}(A^{ip} b^{pk})(A^{jq} b^{q\ell}) \\
&= \frac{1}{2}(\epsilon_{ij} A^{ip} A^{jq})(\epsilon_{k\ell} b^{pk} b^{q\ell}) \\
&= \frac{1}{2} \times (A^{1p}A^{2q} - A^{2p}A^{1q}) \times (b^{p1}b^{q2} - b^{p2}b^{q1}),
\end{aligned} \tag{2.37}$$

   where $C^{ij} = \tilde{A}^i b^j$, $\tilde{A}^i$ is the ith row of A, and b^j is the jth column of B.

2. Show that $A^{1p}A^{2q} - A^{2p}A^{1q}$ is antisymmetric under exchange of $p \leftrightarrow q$. Therefore,

$$A^{1p}A^{2q} - A^{2p}A^{1q} = \frac{1}{2}\epsilon_{pq}\epsilon_{rs}(A^{1r}A^{2s} - A^{2r}A^{1s}) = \epsilon_{pq}\mathfrak{Det}(A). \tag{2.38}$$

3. Show also that $b^{p1}b^{q2} - b^{p2}b^{q1}$ is antisymmetric under exchange of $p \leftrightarrow q$. Therefore,

$$b^{p1}b^{q2} - b^{p2}b^{q1} = \frac{1}{2}\epsilon_{pq}\epsilon_{rs}(b^{1r}b^{2s} - b^{2r}b^{1s}) = \epsilon_{pq}\mathfrak{Det}(B). \tag{2.39}$$

4. Show that

$$\begin{aligned}
\mathfrak{Det}[AB] &= \frac{1}{2} \times \epsilon_{pq}\mathfrak{Det}(A) \times \epsilon_{pq}\mathfrak{Det}(B) \\
&= \mathfrak{Det}(A)\mathfrak{Det}(B).
\end{aligned} \tag{2.40}$$

Problem 2.17 Let us prove

$$\mathfrak{Det}[AB] = \mathfrak{Det}[A]\mathfrak{Det}[B] \tag{2.41}$$

for 3×3 matrices.

1. Show that

$$\begin{aligned}
\mathfrak{Det}[AB] &= \begin{vmatrix} C^{11} & C^{12} & C^{13} \\ C^{21} & C^{22} & C^{23} \\ C^{31} & C^{32} & C^{33} \end{vmatrix} \\
&= \frac{1}{3!}\epsilon_{ijk}\epsilon_{abc}(\tilde{A}^i b^a)(\tilde{A}^j b^b)(\tilde{A}^k b^c) \\
&= \frac{1}{3!}\epsilon_{ijk}\epsilon_{abc}(A^{ip} b^{pa})(A^{jq} b^{qb})(A^{kr} b^{rc}) \\
&= \frac{1}{3!}(\epsilon_{ijk}A^{ip}A^{jq}A^{kr})(\epsilon_{abc}b^{pa}b^{qb}b^{rc}),
\end{aligned} \tag{2.42}$$

where $C^{ij} = \tilde{A}^i b^j$, $\tilde{A}^i$ is the ith row of A, and b^j is the jth column of B.

2. Show that

$$\begin{aligned}
\epsilon_{ijk}A^{ip}A^{jq}A^{kr} &= \epsilon_{pqr}\epsilon_{ijk}A^{i1}A^{j2}A^{k3} \\
&= \epsilon_{pqr} \times \frac{1}{3!}\epsilon_{ijk}\epsilon_{xyz}A^{ix}A^{jy}A^{kz} \\
&= \epsilon_{pqr}\mathfrak{Det}(A).
\end{aligned} \tag{2.43}$$

3. Show also that

$$\begin{aligned}
\epsilon_{abc}b^{pa}b^{qb}b^{rc} &= \epsilon_{pqr}\epsilon_{abc}b^{1a}b^{2b}b^{3c} \\
&= \epsilon_{pqr} \times \frac{1}{3!}\epsilon_{xyz}\epsilon_{abc}A^{xa}A^{yb}A^{zc} \\
&= \epsilon_{pqr}\mathfrak{Det}(B).
\end{aligned} \tag{2.44}$$

4. Show that

$$\begin{aligned}
\mathfrak{Det}[AB] &= \frac{1}{3!} \times \epsilon_{pqr}\mathfrak{Det}(A) \times \epsilon_{pqr}\mathfrak{Det}(B) \\
&= \mathfrak{Det}(A)\mathfrak{Det}(B).
\end{aligned} \tag{2.45}$$

Problem 2.18 Let us prove that

$$\mathfrak{Det}(AB) = \mathfrak{Det}(A)\mathfrak{Det}(B), \tag{2.46}$$

for $n \times n$ based on mathematical induction. We have shown that the relation is true for $n = 2$. Let us assume that the relation is true for $n = k$ and test if the relation is true for $n = k + 1$.

1. By definition, $\mathfrak{Det}[AB]$ is expressed as

$$\begin{aligned}
\mathfrak{Det}[AB] &= \frac{1}{(k+1)!}\epsilon_{i_1 i_2 \cdots i_{k+1}}\epsilon_{j_1 j_2 \cdots j_{k+1}}(\tilde{A}^{i_1} b^{j_1})(\tilde{A}^{i_2} b^{j_2})\cdots(\tilde{A}^{i_{k+1}} b^{j_{k+1}}) \\
&= \frac{1}{(k+1)!}\epsilon_{i_1 i_2 \cdots i_{k+1}}\epsilon_{j_1 j_2 \cdots j_{k+1}}(A^{i_1 x^1} b^{x^1 j_1})(A^{i_2 x^2} b^{x^2 j_2})\cdots(A^{i_{k+1} x^{k+1}} b^{x^{k+1} j_{k+1}}) \\
&= \frac{1}{(k+1)!}(\epsilon_{i_1 i_2 \cdots i_{k+1}} A^{i_1 x^1} A^{i_2 x^2}\cdots A^{i_{k+1} x^{k+1}})(\epsilon_{j_1 j_2 \cdots j_{k+1}} b^{x^1 j_1} b^{x^2 j_2}\cdots b^{x^{k+1} j_{k+1}}),
\end{aligned} \tag{2.47}$$

where $(AB)_{ij} = \tilde{A}^i b^j$, $\tilde{A}^i$ is the ith row of A, and b^j is the jth column of B.

2. By mathematical induction, show that

$$\begin{aligned}
\epsilon_{i_1 i_2 \cdots i_{k+1}} A^{i_1 x^1} A^{i_2 x^2}\cdots A^{i_{k+1} x^{k+1}} &= \epsilon_{i_1 i_2 \cdots i_{k+1}}\epsilon_{x^1 x^2 \cdots x^{k+1}} A^{i_1 1} A^{i_2 2}\cdots A^{i_{k+1}\, k+1} \\
&= \epsilon_{x^1 x^2 \cdots x^{k+1}} \times \frac{1}{(k+1)!}\epsilon_{i_1 i_2 \cdots i_{k+1}}\epsilon_{j_1 j_2 \cdots j_{k+1}} A^{i_1 j_1} A^{i_2 j_2}\cdots A^{i_{k+1} j_{k+1}} \\
&= \epsilon_{x^1 x^2 \cdots x^{k+1}}\mathfrak{Det}(A).
\end{aligned} \tag{2.48}$$

3. Show also that

$$\begin{aligned}
\epsilon_{j_1 j_2 \cdots j_{k+1}} b^{x^1 j_1} b^{x^2 j_1}\cdots b^{x^{k+1} j_{k+1}} &= \epsilon_{x^1 x^2 \cdots x^{k+1}}\epsilon_{j_1 j_2 \cdots j_{k+1}} b^{1 j_1} b^{2 j_2}\cdots b^{k+1\, j_{k+1}} \\
&= \epsilon_{x^1 x^2 \cdots x^{k+1}} \times \frac{1}{(k+1)!}\epsilon_{i_1 i_2 \cdots i_{k+1}}\epsilon_{j_1 j_2 \cdots j_{k+1}} b^{i_1 j_1} b^{i_2 j_2}\cdots b^{i_{k+1} j_{k+1}} \\
&= \epsilon_{x^1 x^2 \cdots x^{k+1}}\mathfrak{Det}(B).
\end{aligned} \tag{2.49}$$

4. Show that

$$\begin{aligned}
\mathfrak{Det}[AB] &= \frac{1}{(k+1)!} \times \epsilon_{x^1 x^2 \cdots x^{k+1}}\mathfrak{Det}(A) \times \epsilon_{x^1 x^2 \cdots x^{k+1}}\mathfrak{Det}(B) \\
&= \mathfrak{Det}(A)\mathfrak{Det}(B).
\end{aligned} \tag{2.50}$$

Problem 2.19 We can carry out the previous proof in a compact way.

1. For $n \times n$ matrices A and B, show that

$$\begin{aligned}
\mathfrak{Det}(AB) &= \epsilon(\sigma)\prod_{i=1}^{n}(AB)^{\sigma^i i} \\
&= \frac{1}{n!}\epsilon(\sigma)\epsilon(\tau)\prod_{i=1}^{n}(AB)^{\sigma^i \tau_i} \\
&= \frac{1}{n!}\epsilon(\sigma)\epsilon(\tau)\prod_{i=1}^{n} a^{\sigma^i k_i} b^{k_i \tau_i},
\end{aligned} \tag{2.51}$$

where the sums are over two permutations σ and τ for $(1, 2, 3, \cdots, n)$. Each of n dummy variables k_i are summed from 1 to n.

2. Show that, before summation over k_i for $i = 1, \cdots, n$,

$$\begin{aligned}\epsilon(\sigma)\prod_{i=1}^{n} a^{\sigma^i k_i} &= \epsilon(\sigma)\frac{1}{n!}\epsilon(\alpha)\prod_{i=1}^{n} a^{\sigma^i \alpha^i} \\ &= \epsilon(\alpha)\mathfrak{Det}[A], \qquad (2.52)\\ \epsilon(\tau)\prod_{i=1}^{n} b^{k_i \tau_i} &= \epsilon(\tau)\frac{1}{n!}\epsilon(\alpha)\prod_{i=1}^{n} b^{\alpha^i \tau_i} \\ &= \epsilon(\alpha)\mathfrak{Det}[B]. \qquad (2.53)\end{aligned}$$

3. Show that

$$\begin{aligned}\mathfrak{Det}(AB) &= \frac{1}{n!}\epsilon(\alpha)\mathfrak{Det}[A]\epsilon(\alpha)\mathfrak{Det}[B] \\ &= \mathfrak{Det}[A]\mathfrak{Det}[B]. \qquad (2.54)\end{aligned}$$

Problem 2.20 Prove, for an invertible matrix A, that

$$\mathfrak{Det}[A^{-1}] = \frac{1}{\mathfrak{Det}[A]}. \qquad (2.55)$$

Problem 2.21 Verify the following identities:

1. $\mathfrak{Det}[AB] = \mathfrak{Det}[BA]$.

2. $\mathfrak{Det}[A^{-1}BA] = \mathfrak{Det}[ABA^{-1}] = \mathfrak{Det}[B]$.

Problem 2.22 Consider a 2×2 invertible matrix A

$$A = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}. \qquad (2.56)$$

The ***trace*** of the matrix is defined by

$$\mathrm{Tr}[A] = \sum_i a^{ii} = a^{11} + a^{22}. \qquad (2.57)$$

We consider an ***eigenvalue problem***

$$AX^{(\lambda)} = \lambda X^{(\lambda)}, \qquad (2.58)$$

where the number λ is an ***eigenvalue*** of A and a column vector $X^{(\lambda)}$ is the corresponding ***eigenvector***.

1. Verify the following identity:

$$\lambda^2 - \lambda\mathrm{Tr}[A] + \mathfrak{Det}[A] = 0. \qquad (2.59)$$

2. Show that the eigenvalues are

$$\lambda^{\pm} = \frac{1}{2}\left[\mathrm{Tr}[A] \pm \sqrt{(\mathrm{Tr}[A])^2 - 4\mathfrak{Det}[A]}\right]. \tag{2.60}$$

3. Show that

$$\lambda^{+} + \lambda^{-} = \mathrm{Tr}[A]. \tag{2.61}$$

4. Show that

$$\lambda^{+}\lambda^{-} = \mathfrak{Det}[A]. \tag{2.62}$$

5. Show that the corresponding eigenvectors are

$$\lambda = \lambda^{+} : X^{(\lambda^{+})} = c\begin{pmatrix} a^{12} \\ \lambda^{+} - a^{11} \end{pmatrix}, \tag{2.63a}$$

$$\lambda = \lambda^{-} : X^{(\lambda^{-})} = c'\begin{pmatrix} a^{12} \\ \lambda^{-} - a^{11} \end{pmatrix}, \tag{2.63b}$$

   where c and c' are arbitrary constants.

2.5 Volume element and Jacobian

Problem 2.23 Consider a Cartesian coordinate system that describes spatial points in the n-dimensional Euclidean space.

1. We introduce a set of orthonormal vectors

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \hat{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{2.64}$$

   Let us construct a square matrix E such that

$$E = \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \cdots & \hat{e}_n \end{pmatrix}. \tag{2.65}$$

   Show that $E = \mathbb{1}$.

2. Show that $\mathfrak{Det}[E] = 1$.

3. Let us consider another case that

$$A = \left(C^1\hat{e}_1 \ C^2\hat{e}_2 \ C^3\hat{e}_3 \ \cdots \ C^n\hat{e}_n \right). \tag{2.66}$$

Show that $\mathfrak{Det}[A] = C^1C^2\cdots C^n$.

4. Explain how to make use of the determinant operator to compute the volume of the region defined by

$$0 < \quad x^1 \quad < C^1, \tag{2.67a}$$
$$0 < \quad x^2 \quad < C^2, \tag{2.67b}$$
$$\vdots \tag{2.67c}$$
$$0 < x^{n-1} < C^{n-1}, \tag{2.67d}$$
$$0 < \quad x^n \quad < C^n, \tag{2.67e}$$

where $(x^1, x^2, \cdots, x^n)$ is the Cartesian coordinates of a point in the region.

5. Consider a set of points in

$$\boldsymbol{X} = \alpha^1\boldsymbol{A}^1 + \alpha^2\boldsymbol{A}^2 + \cdots + \alpha^n\boldsymbol{A}^n, \tag{2.68}$$

where each of real parameters α^i is constrained as

$$0 \le \alpha^i \le 1. \tag{2.69}$$

Show that the volume V of this region is

$$V = \mathfrak{Det}[A], \quad A = \left(\boldsymbol{A}^1 \ \boldsymbol{A}^2 \ \cdots \ \boldsymbol{A}^n \right), \tag{2.70}$$

where $\boldsymbol{A}^i$ is the ith column of the matrix argument of the determinant function. List all possible cases that result in $V = 0$.

6. Suppose that

$$\boldsymbol{A}^i = A^{1i}\hat{\boldsymbol{e}}_1 + A^{2i}\hat{\boldsymbol{e}}_2 + \cdots A^{ni}\hat{\boldsymbol{e}}_n. \tag{2.71}$$

Show that the matrix representation of A is

$$A = \begin{pmatrix} A^{11} & A^{12} & \cdots & A^{1n} \\ A^{21} & A^{22} & \cdots & A^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A^{n1} & A^{n2} & \cdots & A^{nn} \end{pmatrix} = (a^{ij}). \tag{2.72}$$

Problem 2.24 Consider a two-dimensional Cartesian coordinate system described by the coordinates (x^1, x^2). The differential volume element dV in the region,

$$C^1 \leq x^1 \leq C^1 + dx^1, \quad C^2 \leq x^2 \leq C^2 + dx^2, \tag{2.73}$$

is

$$dV = dx^1 dx^2, \tag{2.74}$$

where (C^1, C^2) are the Cartesian coordinates of a fixed point at which the volume element is defined. dx^i is an infinitesimal displacement of x^i.

We can find the transformation rules from this coordinates into a polar coordinates as

$$x^1 = r\cos\theta, \tag{2.75a}$$

$$x^2 = r\sin\theta. \tag{2.75b}$$

1. Show that the basis vectors for the polar coordinates are expressed as

$$\hat{\boldsymbol{e}}_r = \hat{\boldsymbol{e}}_1 \cos\theta + \hat{\boldsymbol{e}}_2 \sin\theta, \tag{2.76a}$$

$$\hat{\boldsymbol{e}}_\theta = -\hat{\boldsymbol{e}}_1 \sin\theta + \hat{\boldsymbol{e}}_2 \cos\theta. \tag{2.76b}$$

   Note that $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ are dependent on θ and independent of r, while $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ are both independent of position.

2. Show that

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 \cdot \hat{\boldsymbol{e}}_2 \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_r & \hat{\boldsymbol{e}}_r \cdot \hat{\boldsymbol{e}}_\theta \\ \hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{e}}_r & \hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{e}}_\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.77}$$

   Therefore, each coordinate system has a set of orthonormal basis vectors at each point.

3. Show that the position vector $\boldsymbol{x}$ can be expressed as

$$\boldsymbol{x} = x^1\hat{\boldsymbol{e}}_1 + x^2\hat{\boldsymbol{e}}_2 = r\hat{\boldsymbol{e}}_r. \tag{2.78}$$

4. Show that the infinitesimal volume element for the region $C^1 \leq x^1 \leq C^1 + dx^1$ and $C^2 \leq x^2 \leq C^2 + dx^2$ is dx^1dx^2.

5. Show that the infinitesimal volume element for the region $r_0 \leq r \leq r_0 + dr$ and $\theta^0 \leq \theta \leq \theta^0 + d\theta$ is $r_0 dr d\theta$.

6. Find the physical meaning of the following vectors:

$$dr\hat{\boldsymbol{e}}_r = \frac{\partial \boldsymbol{x}}{\partial r}dr = \hat{\boldsymbol{e}}_1\frac{\partial x^1}{\partial r}dr + \hat{\boldsymbol{e}}_2\frac{\partial x^2}{\partial r}dr, \tag{2.79a}$$

$$rd\theta\hat{\boldsymbol{e}}_\theta = \frac{\partial \boldsymbol{x}}{\partial \theta}d\theta = \hat{\boldsymbol{e}}_1\frac{\partial x^1}{\partial \theta}d\theta + \hat{\boldsymbol{e}}_2\frac{\partial x^2}{\partial \theta}d\theta. \tag{2.79b}$$

7. Show that the volume element in the 2-dimensional polar coordinate system is

$$|dr\hat{\boldsymbol{e}}_r \times rd\theta\hat{\boldsymbol{e}}_\theta| = rdrd\theta = \mathfrak{Det}\begin{pmatrix} \frac{\partial x^1}{\partial r} & \frac{\partial x^2}{\partial r} \\ \frac{\partial x^1}{\partial \theta} & \frac{\partial x^2}{\partial \theta} \end{pmatrix} drd\theta. \tag{2.80}$$

This determinant is called the ***Jacobian*** that is the conversion factor of a volume element of a coordinate system into another. The new coordinate system does not have to be orthonormal as long as it spans the same space.

Problem 2.25 Compute the Jacobian for the spherical polar coordinate system to find that

$$|dr\hat{\boldsymbol{e}}_r \cdot rd\theta\hat{\boldsymbol{e}}_\theta \times r\sin\theta d\phi\hat{\boldsymbol{e}}_\phi| = r^2dr\sin\theta d\theta d\phi = \mathfrak{Det}\begin{pmatrix} \frac{\partial x^1}{\partial r} & \frac{\partial x^2}{\partial r} & \frac{\partial x^3}{\partial r} \\ \frac{\partial x^1}{\partial \theta} & \frac{\partial x^2}{\partial \theta} & \frac{\partial x^3}{\partial \theta} \\ \frac{\partial x^1}{\partial \phi} & \frac{\partial x^2}{\partial \phi} & \frac{\partial x^3}{\partial \phi} \end{pmatrix} drd\theta d\phi. \tag{2.81}$$

3. Inverse Matrix

3.1 Cramer's Rule

Theorem 3.1 Let us consider a linear equation

$$AX = B, \tag{3.1}$$

where A is an invertible ($\mathfrak{Det}[A] \neq 0$) $n \times n$ matrix and the unknown X and known B are column $n \times 1$ column vectors. ***Cramer rule*** is that

$$x^i = \frac{\mathfrak{Det}[A^{[i]}(B)]}{\mathfrak{Det}[A]} \quad \longrightarrow \quad X = \frac{1}{\mathfrak{Det}[A]}\begin{pmatrix} \mathfrak{Det}[A^{[1]}(B)] \\ \mathfrak{Det}[A^{[2]}(B)] \\ \vdots \\ \mathfrak{Det}[A^{[n]}(B)] \end{pmatrix}, \tag{3.2}$$

where x^i is the ith element of X and $A^{[i]}(B)$ is a matrix whose elements are the same as A except that the ith column is replaced with b:

$$A = \begin{pmatrix} a^{11} & a^{12} & \cdots & a^{1\,i-1} & a^{1\,i} & a^{1\,i+1} & \cdots & a^{1n} \\ a^{21} & a^{22} & \cdots & a^{1\,i-1} & a^{2\,i} & a^{2\,i+1} & \cdots & a^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^{n1} & a^{n2} & \cdots & a^{n\,i-1} & a^{n\,i} & a^{n\,i+1} & \cdots & a^{nn} \end{pmatrix}, \tag{3.3}$$

$$A^{[i]}(B) = \begin{pmatrix} a^{11} & a^{12} & \cdots & a^{1\,i-1} & b^1 & a^{1\,i+1} & \cdots & a^{1n} \\ a^{21} & a^{22} & \cdots & a^{1\,i-1} & b^2 & a^{2\,i+1} & \cdots & a^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^{n1} & a^{n2} & \cdots & a^{n\,i-1} & b^n & a^{n\,i+1} & \cdots & a^{nn} \end{pmatrix}, \tag{3.4}$$

$$B = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{pmatrix}, \quad X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix}. \tag{3.5}$$

Problem 3.2 Let A^{-1} be the inverse of A such that $A^{-1}A = AA^{-1} = \mathbb{1}$.

1. Show that

$$A^{-1}B = X, \quad A^{-1}a^i = E^i, \tag{3.6}$$

where a^i is the ith column of A:

$$A = \begin{pmatrix} a^1 & a^2 & \cdots & a^n \end{pmatrix}, \tag{3.7}$$

and E^i is the ith column of $\mathbb{1}$.

2. Consider a matrix

$$x^1 = \begin{pmatrix} X & E^2 & E^3 & \cdots & E^n \end{pmatrix} = \begin{pmatrix} x^1 & 0 & 0 & \cdots & 0 \\ x^2 & 1 & 0 & \cdots & 0 \\ x^3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x^n & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.8}$$

In a similar manner, we can define x^i as

$$x^i = \begin{pmatrix} E^1 & \cdots & E^{i-1} & X & E^{i+1} & \cdots & E^n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & x^1 & \cdots & 0 \\ 0 & 1 & \cdots & x^2 & \cdots & 0 \\ 0 & 0 & \cdots & x^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^n & \cdots & 1 \end{pmatrix}. \tag{3.9}$$

Show for all i that

$$x^i = \mathfrak{Det}[x^i]. \tag{3.10}$$

3. Show for all i that

$$A^{-1}A^{[i]}(B) = x^i. \tag{3.11}$$

4. Show for all i that

$$x^i = \mathfrak{Det}[x^i] = \mathfrak{Det}[A^{-1}A^{[i]}(B)] = \frac{\mathfrak{Det}[A^{[i]}(B)]}{\mathfrak{Det}[A]}. \tag{3.12}$$

This completes the proof of Cramer's rule.

Problem 3.3 Let us solve the linear equation

$$AX = B, \tag{3.13}$$

where A, B, and X are given by

$$A = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}. \tag{3.14}$$

1. Show that

$$a^1 = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a^2 = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad a^4 = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

2. Show that

$$\mathfrak{Det}[A] = 6, \quad \mathfrak{Det}[a^1] = 6, \quad \mathfrak{Det}[a^2] = 0, \quad \mathfrak{Det}[a^3] = 0, \quad \mathfrak{Det}[a^4] = -6. \tag{3.16}$$

3. Show that

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{3.17}$$

The following REDUCE code confirms the above calculation:

```
aa:=mat((1,0,3,0),(1,2,0,1),(1,0,3,1),(1,0,0,1));
 b:=mat((1),(0),(0),(0));
a1:=mat((1,0,3,0),(0,2,0,1),(0,0,3,1),(0,0,0,1));
a2:=mat((1,1,3,0),(1,0,0,1),(1,0,3,1),(1,0,0,1));
a3:=mat((1,0,1,0),(1,2,0,1),(1,0,0,1),(1,0,0,1));
a4:=mat((1,0,3,1),(1,2,0,0),(1,0,3,0),(1,0,0,0));
dd:=det(aa);
x1:=det(a1)/dd;
x2:=det(a2)/dd;
x3:=det(a3)/dd;
x4:=det(a4)/dd;
xx:=mat((x1),(x2),(x3),(x4));
xf:=aa^(-1)*b;
xx-xf;
```

3.2 Calculation of Inverse Matrix

Problem 3.4 Let us solve the matrix equation by making use of Cramer's rule:

$$AX = \mathbb{1}, \tag{3.18}$$

where A is an $n \times n$ invertible matrix, X is an unknown $n \times n$ matrix, and $\mathbb{1}$ is the $n \times n$ identity matrix. We define

$$A = \left(a^1 \ a^2 \ \cdots \ a^{i-1} \ a^i \ a^{i+1} \ \cdots \ a^n\right), \tag{3.19a}$$

$$X = \left(x^1 \ x^2 \ \cdots \ x^{i-1} \ x^i \ x^{i+1} \ \cdots \ x^n\right), \tag{3.19b}$$

$$\mathbb{1} = \left(E^1 \ E^2 \ \cdots \ E^{i-1} \ E^i \ E^{i+1} \ \cdots \ E^n\right), \tag{3.19c}$$

$$A^{[i]}(E^i) \equiv \left(a^1 \ a^2 \ \cdots \ a^{i-1} \ E^i \ a^{i+1} \ \cdots \ a^n\right), \tag{3.19d}$$

where M_i is the ith columns of an $n \times n$ matrix M.

1. By making use of Cramer's rule, show that

$$x^1 = \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} \mathfrak{Det}[A^{[1]}(E^1)] \\ \mathfrak{Det}[A^{[2]}(E^1)] \\ \vdots \\ \mathfrak{Det}[A^{[n]}(E^1)] \end{pmatrix}. \tag{3.20}$$

2. For any $i = 1, 2, \cdots, n$, show that

$$x^i = \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} \mathfrak{Det}[A^{[1]}(E^i)] \\ \mathfrak{Det}[A^{[2]}(E^i)] \\ \vdots \\ \mathfrak{Det}[A^{[n]}(E^i)] \end{pmatrix}. \tag{3.21}$$

3. Show that

$$\begin{aligned} X &= \left(x^1 \ x^2 \ \cdots \ x^{i-1} \ x^i \ x^{i+1} \ \cdots \ x^n\right) \\ &= \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} \mathfrak{Det}[A^{[1]}(E^1)] & \mathfrak{Det}[A^{[1]}(E^2)] & \cdots & \mathfrak{Det}[A^{[1]}(E^n)] \\ \mathfrak{Det}[A^{[2]}(E^1)] & \mathfrak{Det}[A^{[2]}(E^2)] & \cdots & \mathfrak{Det}[A^{[2]}(E^n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{Det}[A^{[n]}(E^1)] & \mathfrak{Det}[A^{[n]}(E^2)] & \cdots & \mathfrak{Det}[A^{[n]}(E^n)] \end{pmatrix}. \end{aligned} \tag{3.22}$$

4. Show that the (ij) element of X is

$$x^{ij} = \frac{\mathfrak{Det}[A^{[i]}(E^j)]}{\mathfrak{Det}[A]}. \tag{3.23}$$

5. Show that the solution X also satisfies the linear equation

$$XA = \mathbb{1}. \tag{3.24}$$

Therefore, we have shown that

$$A^{-1} = \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} \mathfrak{Det}[A^{[1]}(E^1)] & \mathfrak{Det}[A^{[1]}(E^2)] & \cdots & \mathfrak{Det}[A^{[1]}(E^n)] \\ \mathfrak{Det}[A^{[2]}(E^1)] & \mathfrak{Det}[A^{[2]}(E^2)] & \cdots & \mathfrak{Det}[A^{[2]}(E^n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{Det}[A^{[n]}(E^1)] & \mathfrak{Det}[A^{[n]}(E^2)] & \cdots & \mathfrak{Det}[A^{[n]}(E^n)] \end{pmatrix}, \tag{3.25}$$

$$(A^{-1})_{ij} = \frac{\mathfrak{Det}[A^{[i]}(E^j)]}{\mathfrak{Det}[A]}. \tag{3.26}$$

Theorem 3.5

$$A^{-1} = \frac{C^{\mathrm{T}}}{\mathfrak{Det}[A]} = \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} C^{11} & C^{21} & \cdots & C^{n1} \\ C^{12} & C^{22} & \cdots & C^{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C^{1n} & C^{2n} & \cdots & C^{nn} \end{pmatrix}, \tag{3.27}$$

where C^{ij} is the (ij) cofactor of the matrix A:

$$C^{ij} = (-1)^{i+j} M^{(ij)} = (-1)^{i+j} \begin{vmatrix} a^{11} & a^{12} & \cdots & a^{1\,j-1} & a^{1\,j+1} & \cdots & a^{1n} \\ a^{21} & a^{22} & \cdots & a^{1\,j-1} & a^{2\,j+1} & \cdots & a^{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ a^{i-1\,1} & a^{i-1\,2} & \cdots & a^{i-1\,j-1} & a^{i-1\,j+1} & \cdots & a^{i-1\,n} \\ a^{i+1\,1} & a^{i+1\,2} & \cdots & a^{i+1\,j-1} & a^{i+1\,j+1} & \cdots & a^{i+1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & \cdots & a^{n\,j-1} & a^{n\,j+1} & \cdots & a^{nn} \end{vmatrix}. \tag{3.28}$$

$M^{(ij)}$ is the (ij) minor of A.

Problem 3.6 Let us compute the inverse matrix of A, where

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}. \tag{3.29}$$

1. Show that

$$A^{[1]}(E^1) = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A^{[1]}(E^2) = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A^{[1]}(E^3) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$
$$A^{[2]}(E^1) = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad A^{[2]}(E^2) = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad A^{[2]}(E^3) = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix},$$
$$A^{[3]}(E^1) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{[3]}(E^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{[3]}(E^3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{3.30}$$

2. Show that

$$\begin{aligned}
\mathfrak{Det}[A] &= 1, \\
\mathfrak{Det}[A^{[1]}(E^1)] &= 2, \quad \mathfrak{Det}[A^{[1]}(E^2)] = 0, \quad \mathfrak{Det}[A^{[1]}(E^3)] = -3, \\
\mathfrak{Det}[A^{[2]}(E^1)] &= 0, \quad \mathfrak{Det}[A^{[2]}(E^2)] = 1, \quad \mathfrak{Det}[A^{[2]}(E^3)] = 0, \\
\mathfrak{Det}[A^{[3]}(E^1)] &= -1, \quad \mathfrak{Det}[A^{[3]}(E^2)] = 0, \quad \mathfrak{Det}[A^{[3]}(E^3)] = 2.
\end{aligned} \tag{3.31}$$

3. Show that

$$A^{-1} = \frac{1}{\mathfrak{Det}[A]} \begin{pmatrix} \mathfrak{Det}[A^{[1]}(E^1)] & \mathfrak{Det}[A^{[1]}(E^2)] & \mathfrak{Det}[A^{[1]}(E^3)] \\ \mathfrak{Det}[A^{[2]}(E^1)] & \mathfrak{Det}[A^{[2]}(E^2)] & \mathfrak{Det}[A^{[2]}(E^3)] \\ \mathfrak{Det}[A^{[3]}(E^1)] & \mathfrak{Det}[A^{[3]}(E^2)] & \mathfrak{Det}[A^{[3]}(E^3)] \end{pmatrix} = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}. \tag{3.32}$$

4. Show that $A^{-1}A = AA^{-1} = \mathbb{1}$:

$$\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.33a}$$
$$\begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.33b}$$

We could have computed A^{-1} by making use of MATHEMATICA:

```
A = {{2, 0, 3}, {0, 1, 0}, {1, 0, 2}};
```

```
AInverse = Inverse[A]

A.AInverse

AInverse.A
```

II. Euclidean Space

Test

4. Polar coordinate system in n-dimensional Euclidean space

4.1 3-dimensional polar coordinates

Exercise 4.1 One of the ways to evaluate the gaussian integral,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, \tag{4.1}$$

is to compute the double integral

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy. \tag{4.2}$$

1. We introduce 2-dimensional polar coordinate system

$$r = \sqrt{x^2 + y^2}, \tag{4.3a}$$

$$\theta = \arctan \frac{y}{x}, \quad 0 \le \theta \le 2\pi, \tag{4.3b}$$

where θ is the polar angle and ϕ is the azimuthal angle. Show that

$$x = r\cos\theta, \tag{4.4a}$$

$$y = r\sin\theta. \tag{4.4b}$$

2. Show that, for an arbitrary function $f(x, y)$,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x, y) = \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta\, J\left(\frac{x, y}{r, \theta}\right) f(r\cos\theta, r\sin\theta), \tag{4.5}$$

where the Jacobian $J\left(\frac{x,y}{r,\theta}\right)$ is defined by

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r. \tag{4.6}$$

We assume that the definite integral $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x, y)$ converges.

3. Show that

$$I^2 = \int_0^\infty re^{-r^2} dr \int_0^{2\pi} d\theta = \pi. \tag{4.7}$$

4. Show that I must be real and positive. This leads to

$$I = \sqrt{\pi}. \tag{4.8}$$

5. Show that, for any positive real numbers a and any real number b,

$$\int_{-\infty}^\infty e^{-a(x-b)^2} dx = \sqrt{\frac{\pi}{a}}. \tag{4.9}$$

Problem 4.2 Let us consider the case in three dimensions:

$$I^3 = \int_{-\infty}^\infty e^{-(x^2+y^2+z^2)} dx\, dy\, dz. \tag{4.10}$$

By making use of the previous result $I = \sqrt{\pi}$, we know $I^3 = \pi^{3/2}$. Let us evaluate the integral directly in the spherical polar coordinate system.

1. We introduce the spherical polar coordinate system,

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{4.11a}$$

$$\theta = \arccos \frac{z}{r}, \quad 0 \le \theta \le \pi, \tag{4.11b}$$

$$\phi = \arctan \frac{y}{x}, \quad 0 \le \phi \le 2\pi. \tag{4.11c}$$

Show that

$$x = r \sin\theta \cos\phi, \tag{4.12a}$$

$$y = r \sin\theta \sin\phi, \tag{4.12b}$$

$$z = r \cos\theta. \tag{4.12c}$$

2. Show that, for an arbitrary function $f(x, y)$,

$$\int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \int_{-\infty}^\infty dz\, f(x, y, z) = \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, J\left(\frac{x, y, z}{r, \theta, \phi}\right) \times f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta), \tag{4.13}$$

where the Jacobian $J\left(\frac{x,y,z}{r,\theta,\phi}\right)$ is defined by

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} = r^2 \sin\theta. \tag{4.14}$$

We assume that the definite integral $\int_{-\infty}^\infty dx \int_{-\infty}^\infty dy \int_{-\infty}^\infty dz\, f(x, y, z)$ converges.

3. Show that

$$I^3 = \Omega_3 \int_0^\infty r^2 e^{-r^2} dr, \tag{4.15}$$

where the 3-dimensional solid-angle is

$$\Omega_3 = \int d\Omega_3 = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta = \int_0^{2\pi} d\phi \int_{-1}^1 d\cos\theta = 4\pi. \tag{4.16}$$

4. Show that I must be real and positive. This leads to

$$\int_0^\infty r^2 e^{-r^2} dr = \frac{\sqrt{\pi}}{4}. \tag{4.17}$$

Therefore,

$$I^3 = \pi^{3/2}. \tag{4.18}$$

We can check this result by making use of the following MATHEMATICA code:

```
Integrate[x^2 E^(-x^2),{x,0,Infinity}]
```

Exercise 4.3 Let us consider Γ function. For any natural number n,

$$\Gamma(n) = (n-1)!, \quad n = 1, 2, 3, \cdots. \tag{4.19}$$

For any integer $n > 0$, $\Gamma(n)$ has the recurrence relation

$$\Gamma(n+1) = n\Gamma(n). \tag{4.20}$$

In order to evaluate the radial integral (4.25), it is convenient to make use of the following integral definition of the Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0. \tag{4.21}$$

1. Show that the definition (4.21) satisfies $\Gamma(n) = (n-1)!$

2. Show that the definition (4.21) satisfies $n\Gamma(n) = \Gamma(n+1)$.

3. Show that the integral converges for any real number x such that $0 < x < 1$.

4. Show that the relation $n\Gamma(n) = \Gamma(n+1)$ can be generalized into $x\Gamma(x) = \Gamma(x+1)$ to define $\Gamma(x)$ for any real number x except for $x = 0$ and negative integers.

5. Show that the relation $x\Gamma(x) = \Gamma(x+1)$ can be generalized into $z\Gamma(z) = \Gamma(z+1)$ to define $\Gamma(z)$ for any complex number z except for $z = 0$ and negative integers by making use of analytic continuation.

Problem 4.4 By making use of the definition of Γ function, we can generalize the result for the integral I to the n-dimensional case easily:

$$I^n = \prod_{i=1}^{n} \int_{-\infty}^{\infty} dx_i \, e^{-x_i^2} = \pi^{\frac{n}{2}}. \tag{4.22}$$

In this case, we define the radius

$$r = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{4.23}$$

and define the appropriate polar and azimuthal angles to express x_i.

1. Show that

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \cdots \int_{-\infty}^{\infty} dx_n f\left(\sqrt{\sum_{i=1}^{n} x_i^2}\right) = \Omega_n \int_0^{\infty} r^{n-1} dr f(r), \tag{4.24}$$

where Ω_n is the solid angle in the n-dimensional Euclidean space. We postpone to define the polar and azithumal angles in n dimensions. Instead, we want to find the solid angle in n dimensions.

2. Show that

$$\int_0^{\infty} r^{n-1} e^{-r^2} dr = \frac{\Gamma(n/2)}{2}. \tag{4.25}$$

3. Show that

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{4.26}$$

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
Ω_n	2	2π	4π	$2\pi^2$	$\frac{8}{3}\pi^2$	π^3	$\frac{16}{15}\pi^3$	$\frac{\pi^4}{3}$	$\frac{32}{105}\pi^4$	$\frac{\pi^5}{12}$	$\cdots$

(4.27)

We can check this table by making use of the following MATHEMATICA code:

```
o[n_] := 2 Pi^(n/2)/Gamma[n/2]
Do[Print[n, "=", Simplify[o[n]]], {n, 0, 10}]
```

4. Show that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{4.28}$$

Problem 4.5 We have shown that the solid angle of the n-dimensional Euclidean space is $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

1. Show that the area A_n of the surface of a sphere with radius R defined in the n-dimensional Euclidean space is

$$A_n = \Omega_n R^{n-1}. \tag{4.29}$$

2. Show that the volume V_n of a sphere with radius R defined in the n-dimensional Euclidean space is

$$V_n = \frac{\Omega_n}{n} R^n. \tag{4.30}$$

Problem 4.6 It is trivial to evaluate the following integral,

$$1 = \int_0^\infty e^{-t} dt. \tag{4.31}$$

1. By rescaling the integral by $t \to \lambda t$, show that

$$\frac{1}{\lambda} = \int_0^\infty e^{-\lambda t} dt. \tag{4.32}$$

We take the $(n-1)$th derivative of the above expression:

$$\left(-\frac{\partial}{\partial \lambda}\right)^{n-1} \frac{1}{\lambda} = \frac{(n-1)!}{\lambda^n} = \int_0^\infty t^{n-1} e^{-\lambda t} dt. \tag{4.33}$$

Therefore, the Gamma function for any positive integer n can be expressed as

$$\Gamma(n) = (n-1)! = \lambda^n \left(-\frac{\partial}{\partial \lambda}\right)^{n-1} \int_0^1 e^{-\lambda t} dt. \tag{4.34}$$

2. Show that

$$\lambda^n \left(-\frac{\partial}{\partial \lambda}\right)^{n-1} \int_0^1 e^{-\lambda t} dt = \int_0^1 t^{n-1} e^{-t} dt. \tag{4.35}$$

We can check this formula by making use of the following MATHEMATICA code:

```
F=D[(-1)^(n-1)/x,{x,n-1}]
```

4.2 4-dimensional polar coordinates

Problem 4.7 Let us construct the spherical polar coordinate system defined in 4-dimensional Euclidean space. We can introduce a Cartesian coordinates $x_i \in \mathbb{R}$ to describe the position of a point. The distance r between a point $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and the origin $\mathbf{0} = (0, 0, 0, 0)$ is defined by

$$r = \sqrt{\boldsymbol{x}^2} = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{(x_i)^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}. \tag{4.36}$$

1. We define the polar angle θ_1 with respect to x_4 axis. Show that

$$x_4 = r\cos\theta_1, \tag{4.37a}$$

$$\sqrt{x_1^2 + x_2^2 + x_3^2} = r\sin\theta_1, \quad 0 \le \theta_1 \le \pi. \tag{4.37b}$$

   Provide the reason why $0 \le \theta_1 \le \pi$.

2. Now we consider the three-dimensional vector $(x_1, x_2, x_3, 0)$ which is perpendicular to the x_4 axis. We define the polar angle θ_2 with respect to x_3 axis in the three dimensional space spanned by $(x_1, x_2, x_3, 0)$. Show that

$$x_3 = \sqrt{x_1^2 + x_2^2 + x_3^2}\cos\theta_2, \tag{4.38a}$$

$$\sqrt{x_1^2 + x_2^2} = \sqrt{x_1^2 + x_2^2 + x_3^2}\sin\theta_2, \quad 0 \le \theta_2 \le \pi. \tag{4.38b}$$

   Provide the reason why $0 \le \theta_2 \le \pi$.

3. As the last step, we define the azimuthal angle ϕ to define x_1 and x_2. Show that

$$x_1 = \sqrt{x_1^2 + x_2^2}\cos\phi, \tag{4.39a}$$

$$x_2 = \sqrt{x_1^2 + x_2^2}\sin\phi, \quad 0 \le \phi \le 2\pi. \tag{4.39b}$$

4. As a result, in the spherical polar coordinate system in the 4-dimensional Euclidean space, we need two polar angles and one azimuthal angle. Show that

$$x_1 = r\sin\theta_1\sin\theta_2\cos\phi, \tag{4.40a}$$

$$x_2 = r\sin\theta_1\sin\theta_2\sin\phi, \tag{4.40b}$$

$$x_3 = r\sin\theta_1\cos\theta_2, \tag{4.40c}$$

$$x_4 = r\cos\theta_1. \tag{4.40d}$$

Problem 4.8 We can make use of the parametrization (4.40) to find the volume element in terms of polar coordinates.

1. Show that the volume element of the 4-dimensional Euclidean space is expressed as

$$\begin{aligned}\int dx_1 \int dx_2 \int dx_3 \int dx_4 &= \int dr\, r^3 \int d\theta_1 \int d\theta_2 \int d\phi\, J\left(\frac{x_1, x_2, x_3, x_4}{r, \theta_1, \theta_2, \phi}\right) \\ &= \int dr\, r^3 \int d\theta_1 \sin^2\theta_1 \int d\theta_2 \sin\theta_2 \int d\phi. \end{aligned} \tag{4.41}$$

2. Show that the solid angle is obtained as

$$\begin{aligned}\Omega_4 &= \int_0^\pi d\theta_1 \sin^2\theta_1 \int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\phi \\ &= \int_{-1}^1 \sqrt{1-\cos^2\theta_1} d\cos\theta_1 \int_{-1}^1 d\cos\theta_2 \int_0^{2\pi} d\phi \\ &= \frac{\pi}{2} \times 2 \times 2\pi = 2\pi^2. \end{aligned} \tag{4.42}$$

   This is equivalent to $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

The following REDUCE program computes the Jacobian for the polar coordinate system in the 4-dimensional Euclidean space:

```
%r:=sqrt(x1^2+x2^2+x3^2+x4^2);
x1:=r*sin(t1)*sin(t2)*cos(ph);
x2:=r*sin(t1)*sin(t2)*sin(ph);
x3:=r*sin(t1)*cos(t2);
x4:=r*cos(t1);

mm:=mat(       (df(x1,r) ,df(x2,r ),df(x3,r ),df(x4,r )),
               (df(x1,t1),df(x2,t1),df(x3,t1),df(x4,t1)),
               (df(x1,t2),df(x2,t2),df(x3,t2),df(x4,t2)),
               (df(x1,ph),df(x2,ph),df(x3,ph),df(x4,ph)) );
let sin(t1)^2=1-cos(t1)^2;
let sin(t2)^2=1-cos(t2)^2;
let sin(ph)^2=1-cos(ph)^2;

dd:=det(mm);
```

4.3 $\boldsymbol{n}$-dimensional polar coordinates

Problem 4.9 By making use of mathematical induction, verify the following.

1. The spherical polar coordinates in the n-dimensional Euclidean space consist of radius r, $(n-2)$ polar angles, and a single azimuthal angle. The Cartesian coordinates are then expressed as

$$
\begin{aligned}
x_1 &= r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\sin\theta_{n-2}\cos\phi, && (4.43a)\\
x_2 &= r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\sin\theta_{n-2}\sin\phi, && (4.43b)\\
x_3 &= r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-3}\cos\theta_{n-2}, && (4.43c)\\
x_4 &= r\sin\theta_1\sin\theta_2\sin\theta_3\cdots\cos\theta_{n-3}, && (4.43d)\\
&\vdots && (4.43e)\\
x_{n-2} &= r\sin\theta_1\sin\theta_2\cos\theta_3, && (4.43f)\\
x_{n-1} &= r\sin\theta_1\cos\theta_2, && (4.43g)\\
x_n &= r\cos\theta_1. && (4.43h)
\end{aligned}
$$

2. Show that the solid angle is

$$
\begin{aligned}
\Omega_n = &\int_0^\pi d\theta_1\sin^{n-2}\theta_1\int_0^\pi d\theta_2\sin^{n-3}\theta_2\int_0^\pi d\theta_3\sin^{n-4}\theta_3\\
&\cdots\times\int_0^\pi d\theta_{n-3}\sin^2\theta_{n-3}\int_0^\pi d\theta_{n-2}\sin\theta_{n-2}\int_0^{2\pi}d\phi. \qquad (4.44)
\end{aligned}
$$

3. For any complex number a and b such that $a,\ b\neq 0,\ -1,\ -2,\ \cdots$, and for any complex number n except for $-1,\ -2,\ -3\cdots$, show that

$$
\int_0^1 x^{a-1}(1-x)^{b-1}dx = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad (4.45)
$$

$$
\int_{-1}^1 (1-x^2)^n dx = B(n+1,\tfrac{1}{2}) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}. \qquad (4.46)
$$

4. Show that

$$
\int_0^\pi d\theta\sin^n\theta = \int_{-1}^1(1-x^2)^{\frac{n-1}{2}}dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}. \qquad (4.47)
$$

5. Show that

$$
\prod_{k=1}^{n-2}\int_0^\pi d\theta\sin^k\theta = \frac{\pi^{\frac{n}{2}-1}}{\Gamma(n/2)}. \qquad (4.48)
$$

6. Show that

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{4.49}$$

This agrees with the result that we have obtained by making use of gaussian integrals.

5. Anglular Integrals

5.1 Angle Average

Problem 5.1 We have shown that

$$\begin{aligned}\Omega_n &= \int_0^\pi d\theta_1 \sin^{n-2}\theta_1 \int_0^\pi d\theta_2 \sin^{n-3}\theta_2 \int_0^\pi d\theta_3 \sin^{n-4}\theta_3 \cdots \\ &\quad \times \int_0^\pi d\theta_{n-3} \sin^2\theta_{n-3} \int_0^\pi d\theta_{n-2} \sin\theta_{n-2} \int_0^{2\pi} d\phi \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)}.\end{aligned} \tag{5.1}$$

1. Show that

$$\frac{\Omega_{n-1}}{\Omega_n} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \times \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}. \tag{5.2}$$

```
omega[n_] := 2 Pi^(n/2)/Gamma[n/2];
Simplify[omega[n - 1]/omega[n]]
```

2. By integrating over the angles except for θ_1, show that

$$\Omega_n = \Omega_n \times \frac{\Omega_{n-1}}{\Omega_n} \int_0^\pi d\theta_1 \sin^{n-2}\theta_1 = \frac{\Omega_n \Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^1 (1-x_1^2)^{\frac{n-3}{2}} dx_1, \tag{5.3}$$

where $x_1 = \cos\theta_1$. Note that

$$\int_{-1}^1 (1-x_1^2)^{\frac{n-3}{2}} dx_1 = \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}. \tag{5.4}$$

We can check this result by making use of the following MATHEMATICA code:

```
Gamma[n/2]/(Sqrt[Pi] Gamma[(n - 1)/2])*
Integrate[(1 - x^2)^((n - 3)/2), {x, -1, 1}]
```

3. By integrating over the angles except for θ_1 and θ_2, show that

$$\begin{aligned}\Omega_n &= \int_0^\pi d\theta_1 \sin^{n-2}\theta_1 \int_0^\pi d\theta_2 \sin^{n-3}\theta_2 \Omega_{n-2} \\ &= \Omega_n \frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n-2}{2})} \int_{-1}^{1} (1-x_1^2)^{\frac{n-3}{2}} dx_1 \int_{-1}^{1} (1-x_2^2)^{\frac{n-4}{2}} dx_2, \end{aligned} \tag{5.5}$$

where $x_i = \cos\theta_i$. Note that

$$\int_{-1}^{1} (1-x_1^2)^{\frac{n-3}{2}} dx_1 = \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}, \tag{5.6a}$$

$$\int_{-1}^{1} (1-x_2^2)^{\frac{n-4}{2}} dx_2 = \frac{\sqrt{\pi}\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})}, \tag{5.6b}$$

$$\int_{-1}^{1} (1-x_1^2)^{\frac{n-3}{2}} dx_1 \int_{-1}^{1} (1-x_2^2)^{\frac{n-4}{2}} dx_2 = \frac{\pi\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n}{2})}. \tag{5.6c}$$

```
Gamma[n/2]/(Pi Gamma[(n - 2)/2])*
Integrate[(1 - x^2)^((n - 3)/2), {x, -1, 1}]*
Integrate[(1 - x^2)^((n - 4)/2), {x, -1, 1}]
```

Problem 5.2 Consider an integral involving n-dimensional Euclidean vectors:

$$I_1 = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \boldsymbol{a}\cdot\hat{\boldsymbol{n}}, \tag{5.7}$$

where $\boldsymbol{a}$ is a constant vector, $d\Omega_{\hat{\boldsymbol{n}}}$ is the solid-angle element of the unit vector $\hat{\boldsymbol{n}}$, and $\Omega = 2\pi^{n/2}/\Gamma(n/2)$ is the solid angle in n dimensions. It is convenient to choose the polar coordinate system whose x_n axis is along $\boldsymbol{a}$. Then the coordinates of $\hat{\boldsymbol{n}}$ are given by $\hat{\boldsymbol{n}} = (\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_n)$, where

$$\hat{n}_1 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \sin\theta_{n-2} \cos\phi, \tag{5.8a}$$

$$\hat{n}_2 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \sin\theta_{n-2} \sin\phi, \tag{5.8b}$$

$$\hat{n}_3 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \cos\theta_{n-2}, \tag{5.8c}$$

$$\hat{n}_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \cos\theta_{n-3}, \tag{5.8d}$$

$$\vdots \tag{5.8e}$$

$$\hat{n}_{n-2} = \sin\theta_1 \sin\theta_2 \cos\theta_3, \tag{5.8f}$$

$$\hat{n}_{n-1} = \sin\theta_1 \cos\theta_2, \tag{5.8g}$$

$$\hat{n}_n = \cos\theta_1. \tag{5.8h}$$

1. Show that in that frame,

$$\boldsymbol{a}\cdot\hat{\boldsymbol{n}} = \cos\theta_1. \tag{5.9}$$

2. Show that the integral reduces into

$$I_1 = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_{-1}^{1}(1-x_1^2)^{\frac{n-3}{2}}(ax_1)dx_1 = 0. \tag{5.10}$$

It is straightforward to find that

$$\begin{aligned} I_{2k+1} &= \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}(\boldsymbol{a}\cdot\hat{\boldsymbol{n}})^{2n+1} \\ &= a^{2k+1}\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_{-1}^{1}(1-x_1^2)^{\frac{n-3}{2}}x_1^{2k+1}dx_1 = 0, \quad n = 0,\, 1,\, 2,\, \cdots . \end{aligned} \tag{5.11}$$

3. In a similar way, show that

$$\begin{aligned} I_{2k} &= \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}(\boldsymbol{a}\cdot\hat{\boldsymbol{n}})^{2k} \\ &= a^{2k}\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_{-1}^{1}(1-x_1^2)^{\frac{n-3}{2}}x_1^{2k}dx_1 \\ &= a^{2[[k]]}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2}+k)}{\sqrt{\pi}\Gamma(\frac{n}{2}+k)}, \end{aligned} \tag{5.12}$$

where we have used

$$\int_{-1}^{1}(1-x^2)^a x^{2b}dx = \frac{1+(-1)^{2b}}{2}\frac{\Gamma(1+a)\Gamma(\frac{1}{2}+b)}{\Gamma(a+b+\frac{3}{2})}, \tag{5.13}$$

$$\int_{-1}^{1}(1-x_1^2)^{\frac{n-3}{2}}x_1^{2k}dx_1 = \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{n}{2}+k)}, \tag{5.14}$$

Therefore,

$$I_0 = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})} = 1, \tag{5.15a}$$

$$I_2 = a^2\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2}+1)}{\sqrt{\pi}\Gamma(\frac{n}{2}+1)} = a^2\frac{\Gamma(\frac{n}{2})\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\pi}\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{a^2}{n}, \tag{5.15b}$$

$$I_4 = a^4\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2}+2)}{\sqrt{\pi}\Gamma(\frac{n}{2}+2)} = a^4\frac{\Gamma(\frac{n}{2})\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\pi}\frac{n+2}{2}\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{3a^4}{n(n+2)}, \tag{5.15c}$$

$$I_6 = a^6\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2}+2)}{\sqrt{\pi}\Gamma(\frac{n}{2}+2)} = a^6\frac{\Gamma(\frac{n}{2})\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\pi}\frac{n+4}{2}\frac{n+2}{2}\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{5!!a^6}{n(n+2)(n+4)}, \tag{5.15d}$$

$$\vdots \tag{5.15e}$$

$$\begin{aligned} I_{2k} &= a^{2k}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2}+k)}{\sqrt{\pi}\Gamma(\frac{n}{2}+k)} = a^{2k}\frac{\Gamma(\frac{n}{2})\frac{2k-1}{2}\cdots\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{\sqrt{\pi}\frac{n+2k-2}{2}\cdots\frac{n+2}{2}\frac{n}{2}\Gamma(\frac{n}{2})} \\ &= \frac{(2k-1)(2k-3)\cdots 3\cdot 1\cdot a^{2k}}{n(n+2)(n+4)\cdots(n+2k-2)}, \\ &= \frac{(2k-1)!!(n-2)!!a^{2k}}{(n+2k-2)!!}, \end{aligned} \tag{5.15f}$$

for $n = 0, 1, 2, \cdots$. Here, $n!!$ is defined by

$$n!! = n(n-2)(n-4)\cdots, \tag{5.16a}$$

$$9!! = 9 \cdot 7 \cdot 5 \cdot \cdots \cdot 1, \tag{5.16b}$$

$$10!! = 10 \cdot 8 \cdot 6 \cdot \cdots \cdot 2. \tag{5.16c}$$

We can check these results by making use of the following MATHEMATICA code:

```
Table[{2 k,
   a^(2 k) Gamma[n/2] Gamma[1/2 + k]/Sqrt[Pi]/Gamma[n/2 + k] //
    FullSimplify}, {k, 0, 3}] // TableForm
```

Problem 5.3 Next we evaluate

$$J_{11} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \boldsymbol{a} \cdot \hat{\boldsymbol{n}}\, \boldsymbol{b} \cdot \hat{\boldsymbol{n}}, \tag{5.17}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

1. Show that we can choose the frame in which

$$\boldsymbol{a} = (0, 0, \cdots,\ 0,\ a), \tag{5.18a}$$

$$\boldsymbol{b} = (0, 0, \cdots, b\sin\theta, b\cos\theta), \tag{5.18b}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\hat{\boldsymbol{n}}$ is given by (5.8). Then we have

$$\begin{aligned} \boldsymbol{a} \cdot \hat{\boldsymbol{n}}\, \boldsymbol{b} \cdot \hat{\boldsymbol{n}} &= ab\cos\theta_1(\cos\theta_1\cos\theta + \sin\theta_1\cos\theta_2\sin\theta) \\ &= ab\, x_1(x_1\cos\theta + \sqrt{1-x_1^2}\, x_2\sin\theta), \end{aligned} \tag{5.19}$$

where we have set $x_i = \cos\theta_i$.

2. Show that the integral reduces into

$$\begin{aligned} I &= ab\frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n-2}{2})}\int_{-1}^{1} dx_1(1-x_1^2)^{\frac{n-3}{2}}\int_{-1}^{1} dx_2(1-x_2^2)^{\frac{n-4}{2}} x_1(x_1\cos\theta + \sqrt{1-x_1^2}x_2\sin\theta) \\ &= \boldsymbol{a}\cdot\boldsymbol{b}\frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n-2}{2})}\int_{-1}^{1} dx_1 x_1^2(1-x_1^2)^{\frac{n-3}{2}}\int_{-1}^{1} dx_2(1-x_2^2)^{\frac{n-4}{2}} \\ &= \boldsymbol{a}\cdot\boldsymbol{b}\frac{\Gamma(\frac{n}{2})}{2\Gamma(\frac{n}{2}+1)} \qquad \leftarrow \Gamma(\tfrac{n}{2}+1) = \tfrac{n}{2}\Gamma(\tfrac{n}{2}) \\ &= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{n}. \end{aligned} \tag{5.20}$$

where $\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta$ and the integrands that are odd functions of x_i are vanishing. The angular integrals are expressed as beta functions:

$$\int_{-1}^{1} dx_1 x_1^2 (1-x_1^2)^{\frac{n-3}{2}} = \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{n}{2}+1)}, \tag{5.21a}$$

$$\int_{-1}^{1} dx_2 (1-x_2^2)^{\frac{n-4}{2}} = \frac{\sqrt{\pi}\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n-1}{2})}. \tag{5.21b}$$

We can check the integral table by making use of the following MATHEMATICA code:

```
Integrate[x^2 (1 - x^2)^((n - 3)/2), {x, -1, 1}]
Integrate[(1 - x^2)^((n - 4)/2), {x, -1, 1}]
```

5.2 Rotationally Invariant Tensor Integrals

Problem 5.4 Consider a vector V^i and a rank-2 tensor W^{ij}. Under rotation, they transform like

$$V'^i = R^{ij} V^j, \tag{5.22a}$$

$$W'^{ij} = R^{ia} R^{ib} W^{ab}, \tag{5.22b}$$

where R is the matrix representing the rotation about $\hat{\boldsymbol{\theta}}$ by an angle $\theta = |\boldsymbol{\theta}|$.

1. Show that there is no vector V^i that is ***invariant under rotation***:

$$V'^i \equiv R^{ij} V^j = V^i \text{ for any } R. \tag{5.23}$$

2. Show that any rank-2 tensor W^{ij} can be decomposed into the form:

$$W^{ij} = S^{ij} + A^{[ij]}, \tag{5.24a}$$

$$S^{ij} = T^{ij} + D^{(ij)}, \tag{5.24b}$$

$$T^{ij} = \frac{1}{n}\delta^{ij} W^{kk}, \tag{5.24c}$$

$$D^{(ij)} = \frac{1}{2}(W^{ij} + W^{ji}) - \frac{1}{n}\delta^{ij} W^{kk}, \tag{5.24d}$$

$$A^{[ij]} = \frac{1}{2}(W^{ij} - W^{ji}) = \frac{1}{2}\epsilon_{ijk}\epsilon_{abk} W^{ab}, \tag{5.24e}$$

where the summation over a repeated index is assumed in $W^{kk} = \mathrm{Tr}(W^{ij})$. We shall find that $D^{(ij)}$ is symmetric traceless tensor and $A^{[ij]}$ is antisymmetric tensor.

3. Show that S^{ij} is symmetric and $A^{[ij]}$ is antisymmetric under exchange of $i \leftrightarrow j$.

4. Show that both $A^{[ij]}$ and $D^{(ij)}$ are traceless:

$$\mathrm{Tr}(D^{(ij)}) = D^{(kk)} = 0, \tag{5.25a}$$

$$\mathrm{Tr}(A^{[ij]}) = A^{[kk]} = 0. \tag{5.25b}$$

5. Show that T^{ij} is symmetric and traceful:

$$\mathrm{Tr}(T^{ij}) = \frac{1}{n}\delta^{\ell\ell} W^{kk} = W^{kk} = \mathrm{Tr}(W^{ij}). \tag{5.26}$$

6. Show that the ***rotationally symmetric*** (invariant) component of W^{ij} is T^{ij}:

$$T'^{ij} \equiv R^{ia} R^{ib}\, T^{ab} = T^{ij} \text{ for any } R. \tag{5.27}$$

Problem 5.5 The angular integrals of previous problems involve tensor integrals,

$$T^{i} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}\, \hat{\boldsymbol{n}}^i, \tag{5.28a}$$

$$T^{ij} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}\, \hat{\boldsymbol{n}}^i \hat{\boldsymbol{n}}^j, \tag{5.28b}$$

$$T^{ijk} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}\, \hat{\boldsymbol{n}}^i \hat{\boldsymbol{n}}^j \hat{\boldsymbol{n}}^k, \tag{5.28c}$$

$$T^{ijk\ell} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega}\, \hat{\boldsymbol{n}}^i \hat{\boldsymbol{n}}^j \hat{\boldsymbol{n}}^k \hat{\boldsymbol{n}}^\ell, \tag{5.28d}$$

$$\vdots$$

Now we make use of the rotational properties of vectors to simplify above tensor integrals without explicit angular integrations. T^i must satisfy the vector transformation under rotation:

$$T'^{i} = R^{ij}\, T^j, \tag{5.29}$$

where R is the matrix representing the rotation about $\hat{\boldsymbol{\theta}}$ by an angle $\theta = |\boldsymbol{\theta}|$. After the angular integration, T^i must be a linear combination of available vectors. However, because the only available vector $\hat{\boldsymbol{n}}$ is already integrated out, there is no available vector left. Therefore,

$$T'^{i} = 0. \tag{5.30}$$

1. Following the argument provided above, show that the integral T^{ij} must satisfy the following transformations,

$$T'^{ij} = R^{ia} R^{jb}\, T^{ab}, \tag{5.31a}$$

$$T'^{ijk} = R^{ia} R^{jb} R^{kc}\, T^{abc}, \tag{5.31b}$$

$$\vdots$$

2. Show that all of the tensors T^i, T^{ij}, T^{ijk}, $\cdots$ are completely symmetric under exchange of any two vector indices. Therefore, the tensor integrals T^i, T^{ij}, T^{ijk}, $\cdots$ are invariant under rotation.

3. Show that all of the rotationally invariant tensors that have odd numbers of vector indices must vanish.

4. Show that δ^{ij} is the only linearly independent tensor that is invariant under rotation:
$$A'^{ij} = R^{ia}R^{jb}A^{ab} = A^{ij}, \quad A^{ij} = c_2\delta^{ij}, \tag{5.32}$$
where c_2 is an arbitrary number.

5. Therefore, we can set
$$T^{ij} = c_2\delta^{ij}. \tag{5.33}$$
By multiplying δ^{ij} to both sides, we can determine the constant c_2. Show that
$$T^{ij} = \frac{1}{n}\delta^{ij}T^{kk}. \tag{5.34}$$

6. Show that the only linearly independent tensors that are invariant under rotation,
$$A'^{ijk\ell} = R^{ia}R^{jb}R^{kc}R^{\ell d}\,A^{abcd} = A^{ijk\ell}, \tag{5.35}$$
are $\delta^{ij}\delta^{k\ell}$, $\delta^{ik}\delta^{j\ell}$, and $\delta^{i\ell}\delta^{jk}$.

7. Therefore, we can set
$$T^{ijk\ell} = c_4(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}), \tag{5.36}$$
where we have used the rotational symmetry to set the common coefficient c_4 for the three contributions. By multiplying $\delta^{ij}\delta^{k\ell}$, or any other term on the right side, we can determine c_4. Show that
$$T^{ijk\ell} = \frac{1}{n(n+2)}(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})T^{ppqq}. \tag{5.37}$$

Problem 5.6 Consider a rank-6 tensor $T^{ijk\ell mn}$ that is invariant under rotation. Show that this tensor is decomposed into a linear combination of Kronecker deltas:
$$T^{ijk\ell mn} = (\delta^{ij}\Delta^{k\ell mn} + \delta^{ik}\Delta^{j\ell mn} + \delta^{i\ell}\Delta^{jkmn} + \delta^{im}\Delta^{jk\ell n} + \delta^{in}\Delta^{jk\ell m})T^{ppqqrr}, \tag{5.38a}$$
$$\Delta^{abcd} = \frac{1}{n(n+2)(n+4)}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}). \tag{5.38b}$$
The following REDUCE program verifies the above statement:

```
vecdim n;
operator dd;
vector p,q,r,s,u,v;
for all a,b,c,d let
dd(a,b,c,d)=1/(n*(n+2))*(a.b*c.d+a.c*b.d+a.d*b.c);
tt:=1/(n+4)*(p.q*dd(r,s,u,v)+p.r*dd(q,s,u,v)
+p.s*dd(r,q,u,v)+p.u*dd(r,s,q,v)+p.v*dd(r,s,u,q));
d0:=p.q*r.s*u.v;
d1:=p.s*r.q*u.v;
d2:=p.r*q.s*u.v;
index p,q,r,s,u,v;
xx1:=tt*d0;
xx2:=tt*d1;
xx3:=tt*d2;
```

5.3 Scalar Integrals

Problem 5.7 Consider an integral $I_1(\boldsymbol{a})$ in the 3-dimensional Euclidean space,

$$I_1(\boldsymbol{a}) = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}}, \tag{5.39}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$ is a constant vector defined in the 3-dimensional Euclidean space and $\hat{\boldsymbol{n}}$ is a unit vector. The magnitude of $\boldsymbol{a}$ is less than 1: $|\boldsymbol{a}| < 1$. The direction of $\hat{\boldsymbol{n}}$ varies and the integral $\int d\Omega_{\hat{\boldsymbol{n}}}$ is over the solid angle of $\hat{\boldsymbol{n}}$. Here, $\Omega = 4\pi$ is the solid angle. To evaluate this integral it is convenient to choose the x_3 axis along the constant vector $\boldsymbol{a}$ and employ the spherical polar coordinate system, in which

$$\boldsymbol{a} = (0, 0, |\boldsymbol{a}|), \tag{5.40a}$$

$$\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{5.40b}$$

1. Show that the integral $I_1(\boldsymbol{a})$ must be a scalar.

2. Show that the integrand of $I_1(\boldsymbol{a})$ is independent of the azimuthal angle ϕ and is dependent upon $|\boldsymbol{a}|$ and $\cos\theta$. Therefore, after integrating over the solid angle, $I_1(\boldsymbol{a})$ becomes a function of the only available scalar $\boldsymbol{a}^2$.

3. By integrating out the angles θ and ϕ, show that
$$I_1(\boldsymbol{a}) = \frac{1}{2}\int_{-1}^{1} dx \frac{1}{1+|\boldsymbol{a}|x} = \frac{1}{2|\boldsymbol{a}|}\int_{1-|\boldsymbol{a}|}^{1+|\boldsymbol{a}|} \frac{dt}{t} = \frac{\tanh^{-1}|\boldsymbol{a}|}{|\boldsymbol{a}|}, \tag{5.41}$$
where $x = \cos\theta$.

4. Show that the Taylor series expansion of $I_1(\boldsymbol{a})$ about $|\boldsymbol{a}| = 0$ is
$$I_1(\boldsymbol{a}) = \sum_{i=0}^{\infty} \frac{|\boldsymbol{a}|^{2n}}{2n+1} = 1 + \frac{|\boldsymbol{a}|^2}{3} + \frac{|\boldsymbol{a}|^4}{5} + \frac{|\boldsymbol{a}|^6}{7} + \frac{|\boldsymbol{a}|^8}{9} + \cdots . \tag{5.42}$$

Problem 5.8 In the 3-dimensional Euclidean space, we define
$$I_n(\boldsymbol{a}) = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{(1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}})^n} \quad \text{for} \quad n = 2,\, 3,\, 4,\, \cdots . \tag{5.43}$$
Show that
$$\begin{aligned} I_n(\boldsymbol{a}) &= \frac{1}{2}\int_{-1}^{1} dx \frac{1}{(1+|\boldsymbol{a}|x)^n}. \\ &= \frac{1}{2|\boldsymbol{a}|}\int_{1-|\boldsymbol{a}|}^{1+|\boldsymbol{a}|} \frac{dt}{t^n} \\ &= \frac{1}{2(n-1)|\boldsymbol{a}|}\left[\frac{1}{(1-|\boldsymbol{a}|)^{n-1}} - \frac{1}{(1+|\boldsymbol{a}|)^{n-1}}\right] \\ &= \frac{(1+|\boldsymbol{a}|)^{n-1} - (1-|\boldsymbol{a}|)^{n-1}}{2(n-1)|\boldsymbol{a}|(1-\boldsymbol{a}^2)^{n-1}}. \end{aligned} \tag{5.44}$$
First several values are
$$I_2(\boldsymbol{a}) = \frac{1}{1-\boldsymbol{a}^2}, \tag{5.45a}$$
$$I_3(\boldsymbol{a}) = \frac{1}{(1-\boldsymbol{a}^2)^2}, \tag{5.45b}$$
$$I_4(\boldsymbol{a}) = \frac{3+\boldsymbol{a}^2}{3(1-\boldsymbol{a}^2)^3}, \tag{5.45c}$$
$$I_5(\boldsymbol{a}) = \frac{1+\boldsymbol{a}^2}{(1-\boldsymbol{a}^2)^4}. \tag{5.45d}$$

Problem 5.9 By substituting $\boldsymbol{a} \to \boldsymbol{a}/\lambda$ into $I_1(\boldsymbol{a})$, we find that
$$\frac{I_1(\boldsymbol{a}/\lambda)}{\lambda} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{\lambda+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}} = \frac{1}{2|\boldsymbol{a}|}\log\frac{\lambda+|\boldsymbol{a}|}{\lambda-|\boldsymbol{a}|} = \frac{\tanh^{-1}\lambda|\boldsymbol{a}|}{\lambda|\boldsymbol{a}|}, \tag{5.46}$$
This can be the generating function for $I_n(\boldsymbol{a})$. By making use of the identity,
$$\frac{(-1)^{n-1}}{\Gamma(n)}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\frac{1}{\lambda} = \frac{1}{\lambda^n} \quad \text{for} \quad n \geq 1, \tag{5.47}$$
show that
$$I_n(\boldsymbol{a}) = \frac{(-1)^{n-1}}{2|\boldsymbol{a}|\Gamma(n)}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\log\frac{\lambda+|\boldsymbol{a}|}{\lambda-|\boldsymbol{a}|}\bigg|_{\lambda=1}. \tag{5.48}$$
The following MATHEMATICA code confirms the above derivation:

```
f[n_] := (-1)^(n - 1)/Factorial[n - 1]*D[1/x, {x, n - 1}];
g[n_] := (-1)^(n - 1)/Factorial[n - 1]*
   D[(Log[x + a] - Log[x - a])/(2 a), {x, n - 1}];
Do[Print[n, "=", FullSimplify[f[n]]], {n, 1, 10}]
Do[Print[n, "=", FullSimplify[g[n] /. x -> 1]], {n, 1, 10}]
```

Problem 5.10 Let us evaluate the integral

$$I_1(\boldsymbol{a}) = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}} \tag{5.49}$$

in n dimensions.

1. Show that, if we choose the x_n-axis along the constant vector $\boldsymbol{a}$, then we can simplify the angular integral as

$$I_1(\boldsymbol{a}) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_{-1}^{1} \frac{(1-x^2)^{\frac{n-3}{2}}}{1+|\boldsymbol{a}|x} dx, \tag{5.50}$$

   where $x = \hat{\boldsymbol{a}}\cdot\hat{\boldsymbol{n}}$. The integral can be evaluated by expanding the denominator about $x = 0$:

$$I_1(\boldsymbol{a}) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \sum_{k=0}^{\infty} (-1)^k |\boldsymbol{a}|^k \int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} x^k dx. \tag{5.51}$$

```
Expand[Series[1/(1 + x), {x, 0, 20}] - Sum[(-1)^{k} x^k, {k, 0, 20}]]
```

2. Show that the x integral becomes a beta function:

$$\int_{-1}^{1} (1-x^2)^{\frac{n-3}{2}} x^k dx = \frac{1+(-1)^k}{2} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+k}{2})}. \tag{5.52}$$

   This integral is non-vanishing only for even k.

```
Integrate[(1-x^2)^((n-3)/2)x^k ,{x,-1,1}]
((1 + (-1)^k) Gamma[(1 + k)/2] Gamma[1/2 (-1 + n)])/(2 Gamma[(k + n)/2])
```

3. Show that

$$I_1(\boldsymbol{a}) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})} \boldsymbol{a}^{2k}. \tag{5.53}$$

4. The series is expressed in terms of the hypergeometric function:

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})} \boldsymbol{a}^{2k} = \frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})} {}_2F_1[\tfrac{1}{2}, 1, \tfrac{n}{2}, \boldsymbol{a}^2]. \tag{5.54}$$

Show that

$$I_1(\boldsymbol{a}) = {}_2F_1[\tfrac{1}{2}, 1, \tfrac{n}{2}, \boldsymbol{a}^2], \tag{5.55}$$

where the hypergeometric function ${}_2F_1(a, b, c; z)$ is defined by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{5.56}$$

```
Sum[Gamma[k+1/2]/Gamma[k+n/2] x^(2k),{k,0,Infinity}]
(Sqrt[\[Pi]] Hypergeometric2F1[1/2, 1, n/2, x^2])/Gamma[n/2]
```

5. Show for $n = 3$ that

$$I_1(\boldsymbol{a}) = \frac{\tanh^{-1} |\boldsymbol{a}|}{|\boldsymbol{a}|} = \frac{1}{2|\boldsymbol{a}|} \log \frac{1 + |\boldsymbol{a}|}{1 - |\boldsymbol{a}|}. \tag{5.57}$$

This reproduces the previous result for $n = 3$.

Problem 5.11 Consider an integral

$$J_{11}(\boldsymbol{a}, \boldsymbol{b}) = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{(1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}})(1 + \boldsymbol{b} \cdot \hat{\boldsymbol{n}})}, \tag{5.58}$$

where the vectors $\boldsymbol{a} = (0, 0, \cdots, 0, a)$ and $\boldsymbol{b} = (0, 0, \cdots, b \sin\theta, b \cos\theta)$ are constant vectors defined in the n-dimensional Euclidean space and $\hat{\boldsymbol{n}}$ is a unit vector. The magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$ are less than 1: $|\boldsymbol{a}|$, $|\boldsymbol{b}| < 1$. The direction of $\hat{\boldsymbol{n}}$ varies and the integral $\int d\Omega_{\hat{\boldsymbol{n}}}$ is over the solid angle of $\hat{\boldsymbol{n}}$. Here, Ω is the solid angle in n dimensions. To evaluate this integral it is convenient to choose the x_n axis along $\boldsymbol{a}$, to choose the x_{n-1} axis so that $\boldsymbol{b}$ be on the plane spanned by x_n and x_{n-1} axes, and to employ the spherical polar coordinate system:

$$\boldsymbol{a} = (0, \cdots, 0, |\boldsymbol{a}|), \tag{5.59a}$$

$$\boldsymbol{b} = (0, \cdots, |\boldsymbol{b}| \sin\theta, |\boldsymbol{b}| \cos\theta), \tag{5.59b}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

1. Show that

$$\hat{\boldsymbol{n}} = (e_1, e_2, \cdots, e_n), \tag{5.60}$$

where

$$\begin{aligned}
e_1 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \sin\theta_{n-2} \cos\phi, \\
e_2 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \sin\theta_{n-2} \sin\phi, \\
e_3 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-3} \cos\theta_{n-2}, \\
e_4 &= \sin\theta_1 \sin\theta_2 \sin\theta_3 \cdots \cos\theta_{n-3}, \\
&\vdots \\
e_{n-2} &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
e_{n-1} &= \sin\theta_1 \cos\theta_2, \\
e_n &= \cos\theta_1.
\end{aligned}$$

2. Show that the integrand is independent of the polar angles θ_3, θ_4, $\cdots$, θ_{n-2} and the azimuthal angle ϕ.

3. Show that

$$\boldsymbol{a}\cdot\hat{\boldsymbol{n}} = a\cos\theta_1, \tag{5.61a}$$

$$\boldsymbol{b}\cdot\hat{\boldsymbol{n}} = b(\cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos\theta_2), \tag{5.61b}$$

$$\begin{aligned}
(1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}})(1+\boldsymbol{b}\cdot\hat{\boldsymbol{n}}) &= (1+a\cos\theta_1)\big[1+b(\cos\theta_1\cos\theta + \sin\theta_1\cos\theta_2\sin\theta)\big] \\
&= (1+a\,x)\big[1+b(cx+sy\sqrt{1-x^2})\big],
\end{aligned} \tag{5.61c}$$

where $c=\cos\theta$, $s=\sin\theta$, $x=\cos\theta_1$, and $y=\cos\theta_2$.

4. By integrating out these angles, show that

$$J_{11}(\boldsymbol{a},\boldsymbol{b}) = \frac{\Gamma(\frac{n}{2})}{\pi\Gamma(\frac{n-2}{2})}\int_{-1}^{1}dx\int_{-1}^{1}dy\frac{(1-x^2)^{\frac{n-3}{2}}(1-y^2)^{\frac{n-4}{2}}}{(1+a\,x)[1+b(cx+sy\sqrt{1-x^2})]}, \tag{5.62}$$

where $c=\cos\theta$, $s=\sin\theta$, $x=\cos\theta_1$, and $y=\cos\theta_2$. The evaluation of this integral is quite involved.

5. We introduce the Feynman parametrization

$$\frac{1}{AB} = \int_0^1 \frac{dt}{[(1-t)A+tB]^2}, \tag{5.63}$$

```
F=1+( (1-t)A+t B )^2
Integrate[1/F,{t,0,1}]
```

which can be verified in a straightforward way. By making use of the Feynman parametrization, show that

$$\begin{aligned}
J_{11}(\boldsymbol{a},\boldsymbol{b}) &= \int_0^1 dt \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{\left[1+\boldsymbol{c}(t)\cdot\hat{\boldsymbol{n}}\right]^2} \\
&= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^1 dt \int_{-1}^1 \frac{(1-x^2)^{\frac{n-3}{2}}}{\left[1+|\boldsymbol{c}(t)|x\right]^2} dx \\
&= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \int_0^1 dt \sum_{k=0}^{\infty} (k+1)(-1)^k |\boldsymbol{c}(t)|^k \int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} x^k dx \\
&= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{n}{2}+k)} \int_0^1 dt |\boldsymbol{c}(t)|^{2k} \\
&= \frac{\Gamma(\frac{n}{2})}{2\sqrt{\pi}|\boldsymbol{a}-\boldsymbol{b}|} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{n}{2}+k)} \left||\boldsymbol{a}|^{2k+1} - |\boldsymbol{b}|^{2k+1}\right| \\
&= \frac{1}{2|\boldsymbol{a}-\boldsymbol{b}|} \left||\boldsymbol{a}|{}_2F_1(\tfrac{1}{2},1,\tfrac{n}{2},\boldsymbol{a}^2) - |\boldsymbol{b}|{}_2F_1(\tfrac{1}{2},1,\tfrac{n}{2},\boldsymbol{b}^2)\right|, \qquad (5.64)
\end{aligned}$$

where $\boldsymbol{c}(t) = (1-t)\boldsymbol{a} + t\boldsymbol{b} = \boldsymbol{a} + (\boldsymbol{b}-\boldsymbol{a})t$ and we have used

$$\int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} x^k dx = \frac{1+(-1)^k}{2} \frac{\Gamma(\frac{1+k}{2})\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{n+k}{2})}, \qquad (5.65a)$$

$$\int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} x^{2k} dx = \frac{\Gamma(\frac{1}{2}+k)\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{n}{2}+k)}, \qquad (5.65b)$$

$$\int_0^1 dt |\boldsymbol{c}(t)|^{2k} = \frac{1}{(2k+1)|\boldsymbol{a}-\boldsymbol{b}|} \left||\boldsymbol{a}|^{2k+1} - |\boldsymbol{b}|^{2k+1}\right|, \qquad (5.65c)$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{n}{2}+k)} |\boldsymbol{a}|^{2k+1} = |\boldsymbol{a}|{}_2F_1(\tfrac{1}{2},1,\tfrac{n}{2},\boldsymbol{a}^2). \qquad (5.65d)$$

6. Note that

$$|\boldsymbol{a}|{}_2F_1(\tfrac{1}{2},1,\tfrac{n}{2},\boldsymbol{a}^2)\Big|_{n=3} = 2\tanh^{-1}|\boldsymbol{a}|. \qquad (5.66)$$

Show for $n=3$ that

$$\begin{aligned}
J_{11}(\boldsymbol{a},\boldsymbol{b}) &= \frac{1}{|\boldsymbol{b}-\boldsymbol{a}|} \int_{\mathrm{Min}[|\boldsymbol{a}|,|\boldsymbol{b}|]}^{\mathrm{Max}[|\boldsymbol{a}|,|\boldsymbol{b}|]} \frac{dt}{1-t^2} \\
&= \frac{1}{2|\boldsymbol{b}-\boldsymbol{a}|} \left|\log\frac{(1+|\boldsymbol{b}|)(1-|\boldsymbol{a}|)}{(1-|\boldsymbol{b}|)(1+|\boldsymbol{a}|)}\right| \\
&= \frac{\left|\tanh^{-1}|\boldsymbol{b}| - \tanh^{-1}|\boldsymbol{a}|\right|}{|\boldsymbol{b}-\boldsymbol{a}|}. \qquad (5.67)
\end{aligned}$$

By setting $\boldsymbol{b} = \boldsymbol{0}$, we find that the integral reproduces

$$I_1(|\boldsymbol{a}|) = \frac{\tanh^{-1}|\boldsymbol{a}|}{|\boldsymbol{a}|}. \qquad (5.68)$$

Problem 5.12 Let us make use of the generating function approach to evaluate the integral

$$J_{mn}(\boldsymbol{a}, \boldsymbol{b}) = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{(1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}})^m(1+\boldsymbol{b}\cdot\hat{\boldsymbol{n}})^n}. \tag{5.69}$$

For simplicity, let $n = 3$.

1. By making an appropriate rescaling of $J_{11}(\boldsymbol{a}, \boldsymbol{b})$, show that

$$\frac{J_{11}(\boldsymbol{a}/\alpha, \boldsymbol{b}/\beta)}{\alpha\beta} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{(\alpha+\boldsymbol{a}\cdot\hat{\boldsymbol{n}})(\beta+\boldsymbol{b}\cdot\hat{\boldsymbol{n}})} = \frac{1}{2\,|\alpha\boldsymbol{b}-\beta\boldsymbol{a}|}\left|\log\frac{(\beta+|\boldsymbol{b}|)(\alpha-|\boldsymbol{a}|)}{(\beta-|\boldsymbol{b}|)(\alpha+|\boldsymbol{a}|)}\right| \tag{5.70}$$

2. Show that

$$J_{mn}(\boldsymbol{a}, \boldsymbol{b}) = \frac{(-1)^{m+n}}{2\Gamma(m)\Gamma(n)} \frac{\partial^{m-1}}{\partial\alpha^{n-1}} \frac{\partial^{n-1}}{\partial\beta^{n-1}} \frac{1}{|\alpha\boldsymbol{b}-\beta\boldsymbol{a}|}\left|\log\frac{(\beta+|\boldsymbol{b}|)(\alpha-|\boldsymbol{a}|)}{(\beta-|\boldsymbol{b}|)(\alpha+|\boldsymbol{a}|)}\right|\Bigg|_{\alpha=\beta=1}. \tag{5.71}$$

5.4 Reduction of Tensor Integrals

Problem 5.13 We consider a vector integral

$$T^i = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\hat{n}^i}{1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}}, \tag{5.72}$$

where $\boldsymbol{a} = (a^1, a^2, \cdots, a^n)$ is a constant vector defined in the n-dimensional Euclidean space and the integral is over the solid angle of the unit vector $\hat{\boldsymbol{n}}$.

1. Show that T^i must transform like a vector under rotation and it must be expressed as

$$T^i = ca^i. \tag{5.73}$$

2. By multiplying the constant vector a^i to both sides, determine the coefficient c to find that

$$T^i = \frac{a^i}{\boldsymbol{a}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a}\cdot\hat{\boldsymbol{n}}}{1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}}. \tag{5.74}$$

3. Show that the vector integral is completely determined in terms of a single scalar integral:

$$T^i = \frac{a^i}{\boldsymbol{a}^2}\left[1 - \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}}\right]. \tag{5.75}$$

Problem 5.14 We consider a vector integral

$$T^i = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\hat{n}^i}{(1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}})(1+\boldsymbol{b}\cdot\hat{\boldsymbol{n}})}, \tag{5.76}$$

where $\boldsymbol{a} = (a^1, a^2, \cdots, a^n)$ and $\boldsymbol{b} = (b^1, b^2, \cdots, b^n)$ are constant vectors defined in the n-dimensional Euclidean space and the integral is over the solid angle of the unit vector $\hat{\boldsymbol{n}}$. In order to simplify our analysis, we assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0. \tag{5.77}$$

1. Show that T^i must transform like a vector under rotation and it must be expressed as

$$T^i = c_1 a^i + c_2 b^i. \tag{5.78}$$

2. By multiplying the constant vectors a^i and b^i to both sides, we obtain two simultaneous linear equations for c_1 and c_2. Determine the coefficients c_1 and c_2 to find that

$$T^i = \frac{a^i}{\boldsymbol{a}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a} \cdot \hat{\boldsymbol{n}}}{(1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}})(1 + \boldsymbol{b} \cdot \hat{\boldsymbol{n}})} + \frac{b^i}{\boldsymbol{b}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{b} \cdot \hat{\boldsymbol{n}}}{(1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}})(1 + \boldsymbol{b} \cdot \hat{\boldsymbol{n}})}. \tag{5.79}$$

3. Show that the vector integral reduces into the form

$$\begin{aligned} T^i = {} & \frac{a^i}{\boldsymbol{a}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1 + \boldsymbol{b} \cdot \hat{\boldsymbol{n}}} + \frac{b^i}{\boldsymbol{b}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} \\ & - \left(\frac{a^i}{\boldsymbol{a}^2} + \frac{b^i}{\boldsymbol{b}^2} \right) \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{(1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}})(1 + \boldsymbol{b} \cdot \hat{\boldsymbol{n}})}. \end{aligned} \tag{5.80}$$

The following REDUCE program is useful to check the above derivation:

```
vector p,q,u;
t1:=p.u;
t2:=q.u;
LHS:=n.u;
RHS:=c1*t1+c2*t2;
index u;
ss:=solve({LHS*t1=RHS*t1,LHS*t2=RHS*t2},{c1,c2});
remind u;
ans:=sub(ss,RHS);
let p.q=0;
ans;
ansf:= n.p*p.u/p.p + n.q*q.u/q.q;
ans-ansf;
```

Problem 5.15 We consider a tensor integral

$$T^{ij} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\hat{n}^i \hat{n}^j}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}}, \tag{5.81}$$

where $\boldsymbol{a} = (a^1, a^2, \cdots, a^n)$ is a constant vector defined in the n-dimensional Euclidean space and the integral is over the solid angle of the unit vector $\hat{\boldsymbol{n}}$.

1. Show that T^{ij} is symmetric under exchange of the two vector indices and the tensor must be expressed as a linear combination

$$T^{ij} = c_1 \delta^{ij} + c_2 a^i a^j, \tag{5.82}$$

where δ^{ij} is invariant under rotation. Notice that $a^i a^j$ is not invariant under rotation, while $a^i a^j$ is symmetric.

2. By multiplying the constant vector δ^{ij} and $a^i a^j$, we find two linear equations for c_i. Show that the solution is

$$T^{ij} = \frac{\delta^{ij}}{(n-1)\boldsymbol{a}^2} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a}^2 - (\boldsymbol{a} \cdot \hat{\boldsymbol{n}})^2}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} - \frac{a^i a^j}{(n-1)\boldsymbol{a}^4} \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a}^2 - n(\boldsymbol{a} \cdot \hat{\boldsymbol{n}})^2}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}}. \tag{5.83}$$

3. Show that

$$\begin{aligned}
\int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{(\boldsymbol{a} \cdot \hat{\boldsymbol{n}})^2}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} &= \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1 - 1 + (\boldsymbol{a} \cdot \hat{\boldsymbol{n}})^2}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} \\
&= \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} + \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} (\boldsymbol{a} \cdot \hat{\boldsymbol{n}} - 1) \\
&= -1 + \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{1}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}}.
\end{aligned} \tag{5.84}$$

4. By making use of the reduction formula (5.84), show that

$$T^{ij} = \frac{\delta^{ij}}{(n-1)\boldsymbol{a}^2} \left[1 + \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a}^2 - 1}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} \right] - \frac{a^i a^j}{(n-1)\boldsymbol{a}^4} \left[n + \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\boldsymbol{a}^2 - n}{1 + \boldsymbol{a} \cdot \hat{\boldsymbol{n}}} \right]. \tag{5.85}$$

The following REDUCE program confirms the above derivation:

```
vecdim d;

vector q,u,v;

t1:=u.v;

t2:=q.u*q.v;

LHS:=n.u*n.v;

RHS:=c1*t1+c2*t2;
```

```
index u,v;
ss:=solve({LHS*t1=RHS*t1,LHS*t2=RHS*t2},{c1,c2});
remind u,v;
let n.n=1;
ans:=sub(ss,RHS);
ansf:=
(q.q- n.q^2)*u.v/(d-1)/q.q
-(q.q-n.q^2*d)*q.u*q.v/(d-1)/q.q^2;
ans-ansf;
```

Problem 5.16 Let us consider the rank-3 tensor integral T^{ijk} defined in the n-dimensional Euclidean space:

$$T^{ijk} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{\Omega} \frac{\hat{n}^i \hat{n}^j \hat{n}^k}{1+\boldsymbol{a}\cdot\hat{\boldsymbol{n}}}, \tag{5.86}$$

where $\boldsymbol{a}$ is a constant vector and the integral is over the solid angle of the unit vector $\hat{\boldsymbol{n}}$ and Ω is the solid angle in n dimensions.

1. Show that T^{ijk} is symmetric under exchange of any two indices and T^{ijk} must be expressed as the following linear combination:

   $$T^{ijk} = c_1(\delta^{ij}a^k + \delta^{jk}a^i + \delta^{ki}a^j) + c_2 a^i a^j a^k, \tag{5.87}$$

   where c_1 and c_2 are scalars.

2. Find the values for c_i.

3. Find the most compact expression for T^{ijk} that contains the minimum number of the scalar integrals.

Problem 5.17 Let us evaluate the integrals I, I^i, and I^{ij} that are defined by

$$I = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}}{4\pi(1+\delta\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{q}})}, \tag{5.88a}$$

$$I^i = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}\hat{n}^i}{4\pi(1+\delta\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{q}})}, \tag{5.88b}$$

$$I^{ij} = \int \frac{d\Omega_{\hat{\boldsymbol{n}}}\hat{n}^i\hat{n}^j}{4\pi(1+\delta\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{q}})}, \tag{5.88c}$$

where $\hat{\boldsymbol{q}}$ is a unit constant vector, δ is a positive real number ($0<\delta<1$), and the integral is over the direction of the unit vector $\hat{\boldsymbol{n}}$. All of the Euclidean vectors are defined in 3 dimensions. In this problem, we demonstrate how to evaluate tensor integrals component by component.

1. It is convenient to choose a frame of reference in which $\hat{\boldsymbol{q}}$ is along the z axis so that

$$\hat{\boldsymbol{q}} = (0,0,1), \tag{5.89a}$$

$$\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{5.89b}$$

   Show in that frame that

$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{q}} = \cos\theta = x, \tag{5.90a}$$

$$d\Omega_{\hat{\boldsymbol{n}}} = d(\cos\theta)d\phi = dxd\phi, \tag{5.90b}$$

   where $-1 \le x = \cos\theta \le 1$.

2. Show that I, I^i, and I^{ij} reduce into

$$I = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} \frac{d\phi}{1+\delta\cos\theta}, \tag{5.91a}$$

$$I^i = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} \frac{d\phi}{1+\delta\cos\theta}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^i, \tag{5.91b}$$

$$I^{ij} = \frac{1}{4\pi}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} \frac{d\phi}{1+\delta\cos\theta}\begin{pmatrix} \sin^2\theta\cos^2\phi & \sin^2\theta\sin\phi\cos\phi & \sin\theta\cos\theta\cos\phi \\ \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi & \sin\theta\cos\theta\sin\phi \\ \sin\theta\cos\theta\cos\phi & \sin\theta\cos\theta\sin\phi & \cos^2\theta \end{pmatrix}^{ij}. \tag{5.91c}$$

3. Confirm the following integral table:

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi = 1, \tag{5.92a}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\cos\phi = \frac{1}{2\pi}\int_0^{2\pi} d\phi\sin\phi = \frac{1}{2\pi}\int_0^{2\pi} d\phi\sin\phi\cos\phi = 0, \tag{5.92b}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\cos^2\phi = \frac{1}{2\pi}\int_0^{2\pi} d\phi\sin^2\phi = \frac{1}{2}. \tag{5.92c}$$

4. Show that

$$I = \frac{1}{2}\int_{-1}^{1} dx\frac{1}{1+\delta x}, \tag{5.93a}$$

$$I^i = \frac{\hat{q}^i}{2}\int_{-1}^{1} dx\frac{x}{1+\delta x}, \tag{5.93b}$$

$$\begin{aligned} I^{ij} &= \frac{1}{2}\int_{-1}^{1} d\cos\theta\frac{1}{1+\delta\cos\theta}\begin{pmatrix} \frac{1}{2}\sin^2\theta & 0 & 0 \\ 0 & \frac{1}{2}\sin^2\theta & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix}^{ij} \\ &= \frac{\delta_\perp^{ij}}{4}\int_{-1}^{1} dx\frac{1-x^2}{1+\delta x} + \frac{\hat{q}^i\hat{q}^j}{2}\int_{-1}^{1} dx\frac{x^2}{1+\delta x}, \end{aligned} \tag{5.93c}$$

where

$$\delta_\perp^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{ij} \quad \text{and} \quad \hat{q}^i \hat{q}^j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{ij} . \tag{5.94}$$

5. Verify the following integral table:

$$\frac{1}{2}\int_{-1}^{1} dx \frac{1}{1+\delta x} = \frac{\tanh^{-1}\delta}{\delta}, \tag{5.95a}$$

$$\frac{1}{2}\int_{-1}^{1} dx \frac{x}{1+\delta x} = \frac{\delta - \tanh^{-1}\delta}{\delta^2}, \tag{5.95b}$$

$$\frac{1}{4}\int_{-1}^{1} dx \frac{1-x^2}{1+\delta x} = \frac{\delta - (1-\delta^2)\tanh^{-1}\delta}{2\delta^3}, \tag{5.95c}$$

$$\frac{1}{2}\int_{-1}^{1} dx \frac{x^2}{1+\delta x} = \frac{-\delta + \tanh^{-1}\delta}{\delta^3}. \tag{5.95d}$$

```
F1=Simplify[Integrate[1/(2(1+d x)),{x,-1,1}]]
F2=Simplify[Integrate[x/(2(1+d x)),{x,-1,1}]]
F3=Simplify[Integrate[(1-x^2)/(4(1+d x)),{x,-1,1}]]
F4=Simplify[Integrate[x^2/(2(1+d x)),{x,-1,1}]]
Simplify[   ArcTanh[d] /d              -F1]
Simplify[(d-ArcTanh[d])/d^2           -F2]
Simplify[(d-(1-d^2)ArcTanh[d])/(2d^3) -F3]
Simplify[(-d+ArcTanh[d])/d^3 -F4]
```

6. As the last step, verify the following integral table for I, I^i, and I^{ij}.

$$I = \frac{\tanh^{-1}\delta}{\delta}, \tag{5.96a}$$

$$I^i = \hat{q}^i \frac{1}{\delta^2}(\delta - \tanh^{-1}\delta), \tag{5.96b}$$

$$I^{ij} = \frac{1}{\delta^3}\left\{\frac{1}{2}\delta_\perp^{ij}\Big[\delta - (1-\delta^2)\tanh^{-1}\delta\Big] + \hat{q}^i\hat{q}^j\Big[-\delta + \tanh^{-1}\delta\Big]\right\}. \tag{5.96c}$$

6. Radial Integrals

Problem 6.1 We introduce an intgral

$$I(m^2) = \int \frac{d^n \boldsymbol{p}}{(2\pi)^n} \frac{1}{\boldsymbol{p}^2 + m^2}, \tag{6.1}$$

where $\boldsymbol{p} = (p^1, p^2, \cdots, p^n)$ is a vector in the n-dimensional Euclidean space and the range of the integration is given by

$$\int \frac{d^n\boldsymbol{p}}{(2\pi)^n} = \prod_{k=1}^{n} \int_{-\infty}^{\infty} \frac{dp^k}{2\pi}. \tag{6.2}$$

1. By rescaling the integral

$$1 = \int_0^\infty e^{-t} dt, \tag{6.3}$$

   show that

$$\frac{1}{p^2 + m^2} = \int_0^\infty e^{-(p^2+m^2)t} dt. \tag{6.4}$$

2. Show that

$$I(m^2) = \int_0^\infty dt e^{-m^2 t} \int \frac{d^n\boldsymbol{p}}{(2\pi)^n} e^{-t\boldsymbol{p}^2}, \tag{6.5}$$

   and that $\boldsymbol{p}$ integral is a product of n gaussian integrals:

$$\int \frac{d^n\boldsymbol{p}}{(2\pi)^n} e^{-t\boldsymbol{p}^2} = \frac{1}{(2\pi)^n} \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{-tx_k^2} dx_k. \tag{6.6}$$

3. By making use of the previous result, show that

$$I(m^2) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty dt\, t^{-\frac{n}{2}} e^{-m^2 t}. \tag{6.7}$$

4. By rescaling the integral over t with $m^2 t \to u$, show that

$$\int_0^\infty dt\, t^{-\frac{n}{2}} e^{-m^2 t} = m^{n-2} \Gamma\left(1 - \frac{n}{2}\right). \tag{6.8}$$

   Therefore, $I(m^2)$ reduces into the form:

$$I(m^2) = \frac{(m^2)^{\frac{n}{2}-1}}{(4\pi)^{n/2}} \Gamma\left(1 - \frac{n}{2}\right). \tag{6.9}$$

5. Note that the $\Gamma(x)$ diverges for $x = 0, -1, -2, \cdots$. Show that the integral $I(m^2)$ is convergent only for $n = 1$ and diverges for all integers $n \geq 2$. For example, if we put $n = 2$, then the integral is divergent logarithmically $\propto \log \Lambda$ as the cutoff (upper bound) $p_{\max} = \Lambda \to \infty$. This is called the logarithmic ultraviolet (UV) divergence. If we put $n = 3$, then the integral is divergent linearly as the cutoff Λ. For $n = 3$, $I(m^2)$ diverges quadratically : $I(m^2) \propto \Lambda^2$. These divergences are called the power UV divergences.

Problem 6.2 We evaluate the integral $I(m^2)$ in an alternative way. We notice that the integrand of the integral (6.1) is independent of the direction of $\boldsymbol{p}$. Therefore, we can integrate over the whole solid angle easily.

1. By integrating over the angles first, show that

$$I(m^2) = \int \frac{d^n\boldsymbol{p}}{(2\pi)^n} \frac{1}{\boldsymbol{p}^2 + m^2} = \frac{2}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty dp \frac{p^{n-1}}{p^2 + m^2}, \tag{6.10}$$

   where

$$p \equiv \sqrt{\boldsymbol{p}^2}. \tag{6.11}$$

2. By rescaling the integral with $p = mt$, show that

$$I(m^2) = \frac{2m^{n-2}}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty dt \frac{t^{n-1}}{t^2 + 1}. \tag{6.12}$$

3. By changing the variable $t^2 = u$, show that

$$I(m^2) = \frac{m^{n-2}}{\Gamma(n/2)(4\pi)^{n/2}} \int_0^\infty du \frac{u^{\frac{n}{2}-1}}{1 + u}. \tag{6.13}$$

4. By changing the variable

$$\frac{1}{1+u} = 1 - t, \tag{6.14}$$

   show that

$$\int_0^\infty du \frac{u^{a-1}}{1+u} = \int_0^1 dt\, t^{a-1}(1-t)^{-a} = B(a, 1-a) = \Gamma(a)\Gamma(1-a). \tag{6.15}$$

   Therefore,

$$I(m^2) = \frac{(m^2)^{\frac{n}{2}-1}}{(4\pi)^{n/2}} \Gamma\left(1 - \frac{n}{2}\right), \tag{6.16}$$

   This reproduces the previous result in Eq. (6.9).

Problem 6.3 $I(m^2)$ can be used as a generating function of the following radial integrals:

$$I_\alpha(m^2) = \int \frac{d^n\boldsymbol{p}}{(2\pi)^n} \frac{1}{(\boldsymbol{p}^2 + m^2)^\alpha}. \tag{6.17}$$

1. Show that

$$I_\alpha(m^2) = \frac{1}{\Gamma(\alpha)} \left(-\frac{\partial}{\partial m^2}\right)^{\alpha - 1} I_\alpha(m^2). \tag{6.18}$$

```
Do[Print[n,"=",D[(-1)^(n-1)/x/Gamma[n],{x,n-1}]],{n,1,20}]
```

2. Show that

$$I_1(m^2) = \frac{m^{n-2}}{(4\pi)^{n/2}}\Gamma\left(1-\frac{n}{2}\right), \tag{6.19a}$$

$$I_2(m^2) = \frac{m^{n-4}}{(4\pi)^{n/2}}\frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}, \tag{6.19b}$$

$$I_3(m^2) = \frac{m^{n-6}}{(4\pi)^{n/2}}\frac{\Gamma\left(3-\frac{n}{2}\right)}{\Gamma(3)}, \tag{6.19c}$$

$$\vdots$$

3. Show that

$$I_\alpha(m^2) = \int \frac{d^n\boldsymbol{p}}{(2\pi)^n}\frac{1}{(\boldsymbol{p}^2+m^2)^\alpha} = \frac{m^{n-2\alpha}}{(4\pi)^{n/2}}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}. \tag{6.20}$$

```
Do[Print[n, "=",
  FullSimplify[(m^2)^(n/2 - a) Gamma[a - n/2]/Gamma[a] -
     Gamma[1 - n/2]*
      D[(-1)^(a - 1)/Gamma[a] x^(n/2 - 1), {x, a - 1}] /.
    x -> m^2]], {a, 1, 20}]
```

Problem 6.4 Let us consider an example of a parametrization scheme called the Feynman parametrizations:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A+(B-A)x]^2}. \tag{6.21}$$

Note that the two factors in the denominator, A and B, are merged into a square of a singe variable at the expense of the introduction of an integration over a new parameter x. This method is particularly useful in evaluating loop integrals in perturbation theory.

1. Verify the partial fraction identity,

$$\frac{1}{AB} = \frac{1}{B-A}\left(\frac{1}{A}-\frac{1}{B}\right). \tag{6.22}$$

2. Show that

$$\int_A^B \frac{dt}{t^2} = \frac{1}{A}-\frac{1}{B}. \tag{6.23}$$

3. From the previous problems, we obtain

$$\frac{1}{AB} = \frac{1}{B-A}\int_A^B \frac{dt}{t^2}. \tag{6.24}$$

By making use of this result, verify the Feynman parametrization formula in Eq. (6.21).

4. It is trivial to show that the parametrization in Eq. (6.21) is symmetric under exchange of the coefficients $x \leftrightarrow (1-x)$:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[(1-x)A + xB]^2} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}. \tag{6.25}$$

We notice that the sum of the coefficients for A and B is always unity. By introducing the integral of a Dirac delta function, show that the Feynman parametrization in Eq. (6.25) can be written in a symmetric form:

$$\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \, \frac{\delta(1-x-y)}{(xA+yB)^2}. \tag{6.26}$$

Problem 6.5 Let us generalize the result in Eq. (6.26) into the case of three factors in the denominator:

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \, \frac{\delta(1-x-y-z)}{(xA+yB+zC)^3}. \tag{6.27}$$

1. By making use of the partial fraction for $1/(BC)$, show that

$$\frac{1}{ABC} = \frac{1}{C-B}\left(\frac{1}{AB} - \frac{1}{CA}\right). \tag{6.28}$$

2. By making use of Eq. (6.25), show that

$$\begin{aligned}\frac{1}{AB} - \frac{1}{CA} &= \int_0^1 \frac{dx}{[xA+(1-x)B]^2} - \int_0^1 \frac{dx}{[xA+(1-x)C]^2} \\ &= \int_0^1 dx \left\{\frac{1}{F^2} - \frac{1}{[F+(1-x)(C-B)]^2}\right\},\end{aligned} \tag{6.29}$$

where $F \equiv xA + (1-x)B$.

3. Verify the following definite integral:

$$\int_\alpha^\beta \frac{dt}{t^{n+1}} = \frac{1}{n}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right) \quad \text{for} \quad n > 0. \tag{6.30}$$

Therefore,

$$\frac{1}{F^2} - \frac{1}{[F+(1-x)(C-B)]^2} = 2\int_0^{(1-x)(C-B)} \frac{dt}{(F+t)^3}. \tag{6.31}$$

4. Changing the variable $t = (C - B)y$, show that

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} \frac{dy}{[F + (C - B)y]^3}. \tag{6.32}$$

5. Substituting $F = xA + (1 - x)B$, we find that

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[xA + (1 - x - y)B + yC]^3}. \tag{6.33}$$

Verify the following symmetric version of the Feynman parametrization for $1/(ABC)$:

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1 - x - y - z)}{(xA + yB + zC)^3}. \tag{6.34}$$

Problem 6.6 There are quite a few modified versions of the above Feynman parametrization. Show that all of the following parametrizations are equivalent among one another.

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\delta(1 - x - y - z)}{(xA + yB + zC)^3}, \tag{6.35a}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^{1-x} dy \frac{1}{[xA + yB + (1 - x - y)C]^3}, \tag{6.35b}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dt \frac{1 - x}{\{xA + (1 - x)[tB + (1 - t)C]\}^3}, \tag{6.35c}$$

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^1 dt \frac{x}{\{(1 - x)A + x[(1 - t)B + tC]\}^3}. \tag{6.35d}$$

Problem 6.7 By applying mathematical induction, show that

$$\frac{1}{A_1 A_2 \cdots A_n} = (n - 1)! \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \frac{\delta(1 - x_1 - x_2 - \cdots - x_n)}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n}. \tag{6.36}$$

Problem 6.8 Consider the following integral

$$I = \int \frac{d^n \boldsymbol{p}}{(2\pi)^n} \frac{1}{(\boldsymbol{p}^2 + m^2)[(\boldsymbol{p} - \boldsymbol{q})^2 + m^2]}, \tag{6.37}$$

where $\boldsymbol{q}$ is a constant vector in the n-dimensional Euclidean space.

1. By making use of the Feynman parametrization, we can combine the two denominator factors as

$$I = \int_0^1 dt \int \frac{d^n \boldsymbol{p}}{(2\pi)^n} \frac{1}{\left\{t(\boldsymbol{p}^2 + m^2) + (1 - t)[(\boldsymbol{p} - \boldsymbol{q})^2 + m^2]\right\}^2}. \tag{6.38}$$

Show that

$$I = \int_0^1 dt \int \frac{d^n \boldsymbol{p}}{(2\pi)^n} \frac{1}{\left\{\left[\boldsymbol{p} - (1 - t)\boldsymbol{q}\right]^2 + t(1 - t)\boldsymbol{q}^2 + m^2\right\}^2}. \tag{6.39}$$

2. Show that the integral is invariant under translation of the integral variable

$$\boldsymbol{p} \to \boldsymbol{k} = \boldsymbol{p} - (1-t)\boldsymbol{q}, \tag{6.40}$$

where $\boldsymbol{k}$ is a new integral variable. Then we have

$$I = \int_0^1 dt \int \frac{d^n\boldsymbol{k}}{(2\pi)^n} \frac{1}{\left[\boldsymbol{k}^2 + t(1-t)\boldsymbol{q}^2 + m^2\right]^2}. \tag{6.41}$$

3. The integrand is now independent of the direction of $\boldsymbol{k}$ and we can integrate over the angles easily. The radial integral can be computed by making use of the integral table (6.20). Show that

$$I = \frac{m^{n-4}\Gamma\left(2-\frac{n}{2}\right)}{(4\pi)^{n/2}} \int_0^1 dt \left[t(1-t)\boldsymbol{q}^2 + m^2\right]^{\frac{n}{2}-2}. \tag{6.42}$$

```
Integrate[(t (1 - t) - 1/4)^(n/2 - 2), {t, 0, 1}]
```

```
ConditionalExpression[(I^n 2^(4 - n))/(-3 + n), Re[n] > 3]
```

4. Show that the integral becomes UV divergent at $n = 4$.

III. Group Theory

7. Group

Definition 7.1 A ***group*** G is a set of elements g_i that satisfies the following conditions:

1. For all g_1, $g_2 \in G$ the product $g_1 g_2$ is also an element of G.

2. There exists an element $e \in G$ called the ***identity*** such that for any $g \in G$,

$$ge = eg = g. \tag{7.1}$$

3. The multiplication of three elements satisfies the ***associative law***:

$$g_1(g_2 g_3) = (g_1 g_2) g_3, \tag{7.2}$$

for any $g_i \in G$.

4. For any $g \in G$, there exists an element g^{-1} called the ***inverse*** of g such that

$$g g^{-1} = g^{-1} g = e. \tag{7.3}$$

Exercise 7.2 Verify the following statements.

1. The identity e of a group G is uniquely defined.

2. The inverse g^{-1} of an element $g \in G$ is uniquely defined.

Exercise 7.3 Show that the following sets of numbers satisfy the group requirements.

1. $\mathbb{R} - \{0\} \equiv \{x | x \text{ is a real number and } x \neq 0\}$.

2. $\mathbb{C} - \{0\} \equiv \{z | z \text{ is a complex number and } z \neq 0\}$.

3. The general linear group $\mathrm{GL}(N, \mathbb{R})$ is a set of $N \times N$ real matrices A with $\mathfrak{Det}[A] \neq 0$.

4. The special linear group $\mathrm{SL}(N, \mathbb{C})$ is a set of $N \times N$ complex matrices A with $\mathfrak{Det}[A] = 1$.

5. The orthogonal group $\mathrm{O}(N, \mathbb{R})$ is a set of $N \times N$ real matrices A defined by

$$\mathrm{O}(N, \mathbb{R}) = \{A \in \mathrm{GL}(N, R) | A^T A = AA^T = \mathbb{1}\}, \tag{7.4}$$

where $\mathbb{1}$ is the $N \times N$ identity matrix.

6. The special orthogonal group $\mathrm{SO}(N, \mathbb{R})$ is a set of $N \times N$ real matrices A defined by

$$\mathrm{SO}(N, \mathbb{R}) = \{A \in \mathrm{GL}(N, R) | A^T A = AA^T = \mathbb{1} \text{ and } \mathfrak{Det}[A] = +1\}. \tag{7.5}$$

This group is also called the rotation group.

7. The unitary group $\mathrm{U}(N)$ is a set of $N \times N$ complex matrices defined by

$$\mathrm{U}(N) = \{A \in \mathrm{GL}(N, \mathbb{C}) | A^\dagger A = AA^\dagger = \mathbb{1}\}. \tag{7.6}$$

8. The special unitary group $\mathrm{SU}(N)$ is a set of $N \times N$ complex matrices defined by

$$\mathrm{SU}(N) = \{A \in \mathrm{GL}(n, \mathbb{C}) | A^\dagger A = AA^\dagger = \mathbb{1} \text{ and } \mathfrak{Det}[A] = +1\}. \tag{7.7}$$

8. SO(N)

Exercise 8.1 In general we need N^2 real parameters to represent an arbitrary real $N \times N$ matrix. Suppose $\boldsymbol{e}_i$ is the ith row vector of a matrix $R \in \mathrm{O}(N)$.

1. Show that the condition $R^T R = \mathbb{1}$ is equivalent to $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ for i, $j = 1, 2, \cdots, N$.

2. The number of constraints is

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}, \tag{8.1}$$

where N constraints are for $i = j$ and $\frac{1}{2}N(N-1)$ constraints are for $i \neq j$. Show that we need $\frac{1}{2}N(N-1)$ real parameters to represent an arbitrary matrix in $\mathrm{O}(N)$.

3. Show that $\mathfrak{Det}[R^T] = \mathfrak{Det}[R]$ and $\mathfrak{Det}[R] = \pm 1$.

4. $\mathrm{SO}(N)$ is a subset of $\mathrm{O}(N)$ that satisfies $\mathfrak{Det}[R] = 1$, for any $R \in \mathrm{SO}(N)$. Show that $\mathrm{SO}(N)$ is a continuous subgroup of $\mathrm{O}(N)$.

8.1 SO(2)

Exercise 8.2 Therefore, we need only a single real parameter to represent matrices in SO(2).

1. By making use of this result, show that the following matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{8.2}$$

   represents any element of SO(2) where θ is a real number.

2. Show that any element $A \in \mathrm{O}(2)$ with $\mathfrak{Det}[A] = -1$ can be parametrized by

$$A(\theta) = R(\theta)\mathbb{P}_1, \tag{8.3}$$

   where $\mathbb{P}_1$ represents the reflection, $x \to -x$, whose matrix representation is given by

$$\mathbb{P}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.4}$$

   Show also that the $\{A(\theta)|A(\theta) = R(\theta)\mathbb{P}_1,\ \theta \in \mathbb{R}\}$ is not a group. Therefore, every element of O(2) can be parameterized by

$$O(2) = \{M|M = R(\theta) \text{ or } R(\theta)\mathbb{P}_1,\ \theta \in \mathbb{R}\}. \tag{8.5}$$

3. Let us consider the parity transformation $\mathbb{P} = -\mathbb{1}$. Show that $\mathbb{P}$ is an element of O(2) by finding the parameter θ to satisfy $R(\theta)$ or $R(\theta)\mathbb{P}_1$.

4. Show that $\{\mathbb{1}, \mathbb{P}\}$ is a subgroup of $\mathrm{O}(N)$.

5. Show that $\{\mathbb{1}, \mathbb{P}_1, \mathbb{P}_2, \mathbb{P}\}$ is a subgroup of $\mathrm{O}(N)$, where $\mathbb{P}_i$ represents the reflection of $x_i \to -x_i$ and $\mathbb{P}$ is the parity.

$$\mathbb{P}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{P} = -\mathbb{1}. \tag{8.6}$$

Problem 8.3 We observe that every element of the matrix representation $R(\theta)$ for $\mathrm{SO}(N)$ is analytic: $R(\theta)^{ij}$ is differentiable to any order for any value of the parameter θ.

1. Show that $R(0) = \mathbb{1}$.

2. Show that for any integer $n > 0$,

$$R(\theta) = [R(\theta/n)]^n. \tag{8.7}$$

3. Show that

$$R(\theta/n) = \mathbb{1} + \frac{\theta}{n}G + O\big[\,(\theta/n)^2\,\big], \tag{8.8}$$

where G is a traceless anti-hermitian matrix which is defined by

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{8.9}$$

It is trivial to show that $G^{2n} = (-1)^n\mathbb{1}$ and $G^{2n+1} = (-1)^n G$.

4. Show that

$$R(\theta) = \lim_{n\to\infty}\left(\mathbb{1} + \frac{\theta}{n}G\right)^n = e^{\theta G}, \tag{8.10}$$

where the exponential of an $N \times N$ matrix A is defined by

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!}. \tag{8.11}$$

5. Provide the reason why the terms in Eq. (8.8) of order $(\theta/n)^2$ or higher are consistently negligible without introducing any errors to Eq. (8.10).

6. By an explicit calculation of the matrix exponential, show that

$$e^{\theta G} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{8.12}$$

7. Because $R(0) = \mathbb{1}$ is well defined and $R(\theta)$ is analytic for any θ, we can make a Taylor series expansion about $\theta = 0$. Show that

$$\frac{d^{2n}}{d\theta^n}R(\theta)\Big|_{\theta=0} = (-1)^n, \tag{8.13a}$$

$$\frac{d^{2n+1}}{d\theta^n}R(\theta)\Big|_{\theta=0} = (-1)^n G. \tag{8.13b}$$

8. It is now straightforward to show that the Taylor series expansion of $e^{\theta G}$ about $\theta = 0$ reproduces the right side of Eq. (8.12). We define the generator $L = iG$:

$$L = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{8.14}$$

Show that L is a traceless hermitian matrix and

$$R(\theta) = e^{-i\theta L}. \tag{8.15}$$

8.2 SO(3)

Problem 8.4 We consider SO(3). We need three real parameters θ^1, θ^2, and θ^3 to represent a matrix $R(\boldsymbol{\theta})$ in SO(3), where $\boldsymbol{\theta} \equiv (\theta^1, \theta^2, \theta^3)$. Evidently the identity matrix $\mathbb{1}$ is an element of SO(3). We set $R(\mathbf{0}) = \mathbb{1}$, where $\mathbf{0} \equiv (0, 0, 0)$. We assume that $R(\boldsymbol{\theta})$ is analytic with respect to every component of $\boldsymbol{\theta}$. We define

$$L^k = i\frac{\partial}{\partial\theta^k} R(\boldsymbol{\theta})\big|_{\boldsymbol{\theta}=\mathbf{0}}. \tag{8.16}$$

$$R(\boldsymbol{\theta}) = 1 - iL^k\theta^k + O(\boldsymbol{\theta}^2), \tag{8.17}$$

where we use the Einstein convention for summation of repeated indices.

1. We choose the parameters θ^i to be the angle of rotation about the axis x^i. Show in this case that

$$R(\theta^1, 0, 0) = e^{-iL^1\theta^1}, \tag{8.18a}$$

$$R(0, \theta^2, 0) = e^{-iL^2\theta^2}, \tag{8.18b}$$

$$R(0, 0, \theta^3) = e^{-iL^3\theta^3}, \tag{8.18c}$$

   where

$$L^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad L^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8.19}$$

   These three rotation matrices can be parametrized by

$$R(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{L}}. \tag{8.20}$$

   We would like to show that the set $\mathrm{SO}(3) = \{R(\boldsymbol{\theta}) | \theta^i \in \mathbb{R}\}$ is a group.

2. Show that $\mathbb{1} \in \mathrm{SO}(3)$.

3. Show that the inverse of $R(\boldsymbol{\theta})$ is

$$R(\boldsymbol{\theta})^{-1} = [R(\boldsymbol{\theta})]^T = R(-\boldsymbol{\theta}). \tag{8.21}$$

4. As the last step, we need to show that there exists a three-vector $\boldsymbol{\phi}$ such that

$$R(\boldsymbol{\theta}_1)R(\boldsymbol{\theta}_2) = R(\boldsymbol{\phi}). \tag{8.22}$$

Show that if $\boldsymbol{\theta}_1$ is parallel or anti-parallel to $\boldsymbol{\theta}_2$, then

$$R(\boldsymbol{\theta}_1)R(\boldsymbol{\theta}_2) = R(\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2). \tag{8.23}$$

However, this is not true if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ is neither parallel nor anti-parallel.

Problem 8.5 Now we know that

$$R(0,0,\theta) = e^{-i\theta L^3} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.24}$$

represents the matrix that rotates a vector about $\hat{\boldsymbol{z}}$ axis by an angle θ. We introduce a rotation matrix O that transforms $\hat{\boldsymbol{z}}$ to $\hat{\boldsymbol{n}} = (\hat{n}^1, \hat{n}^2, \hat{n}^3)$:

$$O \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \hat{n}^1 \\ \hat{n}^2 \\ \hat{n}^3 \end{pmatrix}. \tag{8.25}$$

Let $R(\boldsymbol{\theta})$ be the rotation matrix that rotates a vector about an axis parallel to a unit vector $\hat{\boldsymbol{n}}$ by an angle θ.

1. Show that

$$R(\boldsymbol{\theta}) = OR(0,0,\theta)O^T, \tag{8.26}$$

   where $\boldsymbol{\theta} = \theta\hat{\boldsymbol{n}}$.

2. We set $\hat{\boldsymbol{n}} = (0,-1,0)$ that can be obtained by rotating $\hat{\boldsymbol{z}}$ about $\hat{\boldsymbol{x}}$ by $\pi/2$. Show that

$$O = R(\tfrac{1}{2}\pi, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{8.27}$$

3. Show that

$$R(\boldsymbol{\theta}) = OR(0,0,\theta)O^T = R(-\theta\hat{\boldsymbol{y}}). \tag{8.28}$$

```
r1:=mat((1,0,0),(0,c,-s),(0,s,c));
r2:=mat((c,0,s),(0,1,0),(-s,0,c));
r3:=mat((c,-s,0),(s,c,0),(0,0,1));
```

```
oh:=mat((1,0,0),(0,0,-1),(0,1,0));

sub({c=0,s=1},r1)-oh;

oht:=tp(oh);

x:=mat((1),(0),(0));

y:=mat((0),(1),(0));

z:=mat((0),(0),(1));

oh*x-x;

oh*y-z;

oh*z+y;

oh*r3*oht-tp(r2);
```

Problem 8.6 Let us continue to show the closure property of SO(3).

1. As a simple case, we consider $R(\theta^1\hat{\boldsymbol{x}}^1)$ and $R(\theta^2\hat{\boldsymbol{x}}^2)$. Suppose $[L^1, L^2]$ is a linear combination of the generators L^i:

$$[L^1, L^2] = c_1 L^1 + c_2 L^2 + c_3 L^3, \tag{8.29}$$

where c_i is a number. Show that, if the condition (8.29) is satisfied, then there exists a vector $\boldsymbol{\phi} = \phi\hat{\boldsymbol{n}}$ such that

$$R(\theta^1\hat{\boldsymbol{x}}^1)R(\theta^2\hat{\boldsymbol{x}}^2) = R(\boldsymbol{\phi}). \tag{8.30}$$

2. By making use of the matrix representation (8.19) for L^i, show that

$$[L^1, L^2] = iL^3. \tag{8.31}$$

3. By making use of the matrix representation (8.19) for L^i, show that

$$[L^i, L^j] = i\epsilon^{ijk} L^k, \tag{8.32}$$

where summation over $k = 1, 2, 3$ is assumed and ϵ^{ijk} is a totally anisymmetric tensor. The antisymmetric tensor ϵ^{ijk} is the ***structure constant*** of SO(3).

```
L1:=mat((0,0,0),(0,0,-i),(0,i,0));

L2:=mat((0,0,i),(0,0,0),(-i,0,0));

L3:=mat((0,-i,0),(i,0,0),(0,0,0));

L1*L2-L2*L1-(i*L3);

L2*L3-L3*L2-(i*L1);

L3*L1-L1*L3-(i*L2);
```

4. Generalizing the results shown above, show that SO(3) is closed under multiplication. This verifies that SO(3) is a group. Therefore, any multiple rotations can be expressed in terms of a finite rotation about a fixed angle:

$$R(\boldsymbol{\theta}_1)R(\boldsymbol{\theta}_2)\cdots R(\boldsymbol{\theta}_n) = R(\boldsymbol{\phi}). \tag{8.33}$$

8.3 Baker-Campbell-Hausdorff formula

Exercise 8.7 Consider two matrices A and $B \in \mathrm{GL}(N, \mathbb{R})$. We would like to find a matrix C such that $e^A e^B = e^C$.

1. Show that

$$e^A e^B = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{A^a B^b}{a!b!} \tag{8.34}$$

2. The logarithmic function is known to be analytic for any complex number z such that $z \notin \mathbb{C} - [1, \infty)$ and $\log 1 = 0$. Show that for any $z \notin \mathbb{C} - [1, \infty)$, $\log z$ can be expanded in a Taylor series expansion about $z = 1$ as

$$\log z = \log[1 + (z-1)] = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(z-1)^k. \tag{8.35}$$

```
Normal[Series[Log[1+x],{x,0,10}]]
x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + x^7/7 - x^8/8 + x^9/9
```

In a similar manner, we can define the logarithm of a matrix $C \in \mathrm{GL}(N, \mathbb{R})$ as

$$\log C = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(C - \mathbb{1})^k. \tag{8.36}$$

3. Show that

$$\log(e^A e^B) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(e^A e^B - \mathbb{1})^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \widetilde{\sum} \frac{A^{a_1}B^{b_1}\cdots A^{a_k}B^{b_k}}{a_1!b_1!\cdots a_k!b_k!}, \tag{8.37}$$

where $\widetilde{\sum}$ denotes the summation over all of the indices a_i and b_i except for a single case $a_1 = \cdots = a_k = b_1 = \cdots = b_k = 0$.

4. We would like to reorganize the summation as

$$\log(e^A e^B) = \sum_{n=1}^{\infty} P_n(A, B), \tag{8.38}$$

where $P_n(A,B)$ is the matrix version of a ***homogeneous polynomial*** of degree n. For example, $x^2+2xy+y^2$ is a homogeneous polynomial of degree 2. Let us compute $P_n(A,B)$ order by order. We define F_k such that

$$\log(e^A e^B) = \sum_{k=1}^{\infty} F_k, \quad F_k = \frac{(-1)^{k-1}}{k}(e^A e^B - 1)^k. \tag{8.39}$$

Show that

$$\begin{aligned} F_1 &= (e^A e^B - \mathbb{1}) = (\mathbb{1} + A + \tfrac{1}{2!}A^2 + \tfrac{1}{3!}A^3 + \cdots)(\mathbb{1} + B + \tfrac{1}{2!}B^2 + \tfrac{1}{3!}B^3 + \cdots) - \mathbb{1} \\ &= A + B + \tfrac{1}{2!}(A^2+B^2) + AB + \tfrac{1}{3!}(A^3+B^3) + \tfrac{1}{2!}(A^2B + AB^2) \\ &\quad + \tfrac{1}{4!}(A^4+B^4) + \tfrac{1}{3!}(A^3B + AB^3) + \tfrac{1}{2!2!}A^2B^2 + \cdots . \qquad (8.40a) \\ F_2 &= -\frac{1}{2}(e^A e^B - \mathbb{1})^2, \qquad (8.40b) \\ F_3 &= \frac{1}{3}(e^A e^B - \mathbb{1})^3. \qquad (8.40c) \end{aligned}$$

5. We also define a projection operator Π_n that projects out a homogeneous polynomial of A and B of degree n. Show that

$$\Pi_1(F_1) = A + B, \tag{8.41a}$$

$$\Pi_1(F_k) = 0, \quad k \geq 2. \tag{8.41b}$$

Therefore, we have shown that

$$P_1(A,B) = A + B. \tag{8.42}$$

6. Show that

$$\Pi_2(F_1) = \tfrac{1}{2}(A^2+B^2) + AB, \tag{8.43a}$$

$$\Pi_2(F_2) = -\tfrac{1}{2}(A+B)^2, \tag{8.43b}$$

$$\Pi_2(F_k) = 0, \quad k \geq 3. \tag{8.43c}$$

It is straightforward to show that

$$P_2(A,B) = \sum_{k=1}^{2} \Pi_2(F_k) = \tfrac{1}{2}[A,B]. \tag{8.44}$$

7. Show that

$$\Pi_3(F_1) = \tfrac{1}{6}(A^3+B^3) + \tfrac{1}{2}(A^2B + AB^2), \tag{8.45a}$$

$$\Pi_3(F_2) = -(A^2+B^2)(A+B) - (A+B)(A^2+B^2), \tag{8.45b}$$

$$\Pi_3(F_3) = \tfrac{1}{3}(A+B)^3, \tag{8.45c}$$

$$\Pi_3(F_k) = 0, \quad k \geq 4. \tag{8.45d}$$

It is straightforward to show that

$$P_3(A,B) = \sum_{k=1}^{3} \Pi_3(F_k) = \frac{1}{12}\Big([A,[A,B]] + [[A,B],B]\Big). \tag{8.46}$$

The general expression for $P_n(A,B)$ was first computed by Dynkin [Dynkin, E. B., Evaluation of the coefficients of the Campbell-Hausdorff formula, Dokl. Akad. Nauk SSSR 57, 323 (1947).]

$$\log(e^A e^B) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{a_i+b_i>0 \\ 1\le i\le n}} \frac{\big[A^{a_1}B^{b_1}A^{a_2}B^{b_2}\ldots A^{a_n}B^{b_n}\big]}{a_1!\,b_1!\,a_2!\,b_2!\,\cdots a_n!\,b_n!\sum_{i=1}^{n}(a_i+b_i)}, \tag{8.47}$$

where the nested commutator is defined by

$$\big[A^{a_1}B^{b_1}A^{a_2}B^{b_2}\ldots X^{a_n}Y^{b_n}\big] = [\underbrace{A,[A,\ldots[A,}_{a_1}[\underbrace{B,[B,\ldots[B,}_{b_1}\ \ldots\ [\underbrace{A,[A,\ldots[A,}_{a_n}[\underbrace{B,[B,\ldots B}_{b_n}]]\ldots]]. \tag{8.48}$$

In summary,

$$e^A e^B = \exp\left[A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}\big([A,[A,B]] + [[A,B],B]\big) + \cdots\right]. \tag{8.49}$$

Problem 8.8 If we restrict ourselves to SO(N) and SU(N), then an element g of a group is always parametrized by

$$\mathfrak{G} = \{g | g = e^{-i\alpha^k G^k}, \alpha^k \in \mathbb{R}\}, \tag{8.50}$$

where the generator G^k, which is traceless and hermitian, satisfies the commutation relations:

$$[G^i, G^j] = i f^{ijk} G^k. \tag{8.51}$$

Here, f^{ijk} is the structure constant of the group which is real and totally antisymmetric. By making use of the Baker-Campbell-Hausdorff formula in Eq. (8.49), prove that the product of two elements g_1 and g_2

$$g_1 = e^{-i\alpha^k G^k}, \quad g_2 = e^{-i\beta^k G^k}, \tag{8.52}$$

is also an element of the group by finding a set of real numbers γ^k such that

$$g_1 g_2 = e^{-i\gamma^k G^k}. \tag{8.53}$$

Problem 8.9 Show that, if $[A,[A,B]] = [[A,B],B] = 0$, then

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} = e^{\frac{1}{2}[A,B]} e^{A+B}, \tag{8.54a}$$

$$e^B e^A = e^{A+B} e^{-\frac{1}{2}[A,B]} = e^{-\frac{1}{2}[A,B]} e^{A+B}, \tag{8.54b}$$

$$\begin{aligned} e^{A+B} &= e^{-\frac{1}{2}[A,B]} e^A e^B = e^A e^B e^{-\frac{1}{2}[A,B]} \\ &= e^{\frac{1}{2}[A,B]} e^B e^A = e^B e^A e^{\frac{1}{2}[A,B]}, \end{aligned} \tag{8.54c}$$

$$e^A e^B = e^B e^A e^{[A,B]}, \tag{8.54d}$$

$$e^A e^B e^{-A} = e^B e^{[A,B]}. \tag{8.54e}$$

Problem 8.10 Consider two matrices A and $B \in \mathrm{GL}(N,\mathbb{R})$. We define a matrix $M(\lambda)$ by

$$M(\lambda) = e^{\lambda A} B e^{-\lambda A} = \sum_k \frac{\lambda^k}{k!} C_k, \tag{8.55}$$

where λ is a complex number and $C_k \in \mathrm{GL}(N,\mathbb{R})$ is independent of λ.

$$M(\lambda) = e^{\lambda A} C_0 e^{-\lambda A}, \quad \leftarrow \quad C_0 = B \tag{8.56a}$$

$$\frac{\partial M(\lambda)}{\partial \lambda} = e^{\lambda A} C_1 e^{-\lambda A} = e^{\lambda A}[A,B] e^{-\lambda A}, \quad \leftarrow \quad C_1 = [A,B] \tag{8.56b}$$

$$\frac{\partial^2 M(\lambda)}{\partial \lambda^2} = e^{\lambda A} C_2 e^{-\lambda A} = e^{\lambda A}[A,[A,B]] e^{-\lambda A}, \quad \leftarrow \quad C_2 = [A,[A,B]] \tag{8.56c}$$

$$\vdots \tag{8.56d}$$

$$\frac{\partial^k M(\lambda)}{\partial \lambda^k} = e^{\lambda A} C_k e^{-\lambda A}, \tag{8.56e}$$

$$\frac{\partial^{k+1} M(\lambda)}{\partial \lambda^{k+1}} = e^{\lambda A} C_{k+1} e^{-\lambda A} = e^{\lambda A}[A,C_k] e^{-\lambda A}, \quad \leftarrow \quad C_{k+1} = [A,C_k] \tag{8.56f}$$

$$\vdots$$

Therefore, the Taylor series expansion of $M(\lambda)$ about $\lambda = 0$ is then

$$M(\lambda) = e^{\lambda A} B e^{-\lambda A} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\partial^k M(\lambda)}{\partial \lambda^k}\Big|_{\lambda=0} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} C_k, \tag{8.57}$$

where

$$C_0 = B, \tag{8.58a}$$

$$C_{k+1} = [A, C_k], \tag{8.58b}$$

for any $k \geq 0$.

1. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots . \tag{8.59}$$

2. By making use of Eq. (8.59), show that

$$e^A e^B e^{-A} = \exp\left(A + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots\right). \tag{8.60}$$

3. Show that

$$e^A e^{-A} = \mathbb{1}. \tag{8.61}$$

This confirms that

$$(e^A)^{-1} = e^{-A}. \tag{8.62}$$

Problem 8.11 Suppose $R(\boldsymbol{\theta})$ is the rotation matrix that rotates a 3-dimensional Euclidean vector about an axis $\hat{\boldsymbol{\theta}}$ by an angle $\theta = |\boldsymbol{\theta}|$:

$$R(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{L}}. \tag{8.63}$$

1. Show that

$$[R(\boldsymbol{\theta})]^{-1} = R(-\boldsymbol{\theta}) = e^{+i\boldsymbol{\theta}\cdot\boldsymbol{L}}. \tag{8.64}$$

2. By making use of the fact that

$$[R(\boldsymbol{\theta})]^{-1} = [R(\boldsymbol{\theta})]^T, \tag{8.65}$$

show that

$$\boldsymbol{L}^\dagger = (\boldsymbol{L}^T)^* = \boldsymbol{L}. \tag{8.66}$$

Problem 8.12 The angular momentum generator is given by

$$[J^i, J^j] = i\epsilon^{ijk} J^k. \tag{8.67}$$

Let us consider the transformation of the operator $O = J^3$ under rotation about the y axis by an angle θ. The operator in the new coordinate system must be expressed as

$$O' = \mathfrak{D}\,(\theta\hat{\boldsymbol{y}})\, O\, [\mathfrak{D}\,(\theta\hat{\boldsymbol{y}})]^{-1}, \tag{8.68}$$

where

$$\mathfrak{D}\,(\theta\hat{\boldsymbol{y}}) = e^{-i\theta J^2}. \tag{8.69}$$

1. Show that

$$\begin{aligned} e^{-i\theta J^2} J^3 e^{+i\theta J^2} &= J^3 + (-i\theta)\left[J^2, J^3\right] + \frac{(-i\theta)^2}{2!}[J^2,[J^2,J^3]] + \frac{(-i\theta)^3}{3!}[J^2,[J^2,[J^2,J^3]]] \\ &\quad + \frac{(-i\theta)^4}{4!}[J^2,[J^2,[J^2,[J^2,J^3]]]] + \frac{(-i\theta)^5}{5!}[J^2,[J^2,[J^2,[J^2,[J^2,J^3]]]]] + \cdots \\ &= J^3 + \theta J^1 - \frac{\theta^2}{2!}J^3 - \frac{\theta^3}{3!}J^1 + \frac{\theta^4}{4!}J^3 - \frac{\theta^5}{5!}J^1 + \cdots . \end{aligned} \tag{8.70}$$

2. Show that

$$e^{-i\theta J^2} J^3 e^{+i\theta J^2} = J^3 \cos\theta + J^1 \sin\theta. \tag{8.71}$$

Explain the reason why the right-hand side does not have the contribution proportional J^2.

3. If we set $\theta \to \pi/2$, then we find that

$$e^{-i\theta J^2} J^3 e^{+i\theta J^2} = J^1. \tag{8.72}$$

Interpret the physical meaning of this identity based on rotation.

Problem 8.13 Show that

$$e^{-\frac{i}{2}\theta\sigma^2} \sigma^3 e^{+\frac{i}{2}\theta\sigma^2} = \sigma^3 \cos\theta + \sigma^1 \sin\theta. \tag{8.73}$$

9. SU(N)

Definition 9.1 The ***special unitary group*** SU(N) is the set of $N \times N$ complex matrices U that satisfies the following conditions:

$$U^\dagger U = 1, \tag{9.1a}$$

$$\mathfrak{Det}[U] = 1. \tag{9.1b}$$

Exercise 9.2 Show that the set of matrices that satisfies the conditions (9.1) is a group.

Exercise 9.3 In general we need N^2 real parameters to represent an arbitrary $N \times N$ matrix whose elements are all real.

1. Show that we need $2N^2$ real parameters to represent an arbitrary $N \times N$ matrix.

2. Show that we need $N^2 - 1$ real parameters to represent an arbitrary matrix in an SU(N) group.

Problem 9.4 Consider a group SU(2).

1. Show that any matrix U of the SU(2) group can be expressed as

$$U = \boldsymbol{a} \cdot \boldsymbol{\sigma} = a^i \sigma^i, \tag{9.2}$$

where $\boldsymbol{a} = (a^1, a^2, a^3)$ is a three-dimensional vector whose components a^i's are real and σ^i's are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.3}$$

2. Show that any 2×2 hermitian matrix H can be expressed as

$$H = a^0 \mathbb{1} + \boldsymbol{a} \cdot \boldsymbol{\sigma}, \tag{9.4}$$

where $\mathbb{1}$ is the 2×2 identity matrix and a^i is real for $i = 0$, 1, 2, and 3.

3. Show that σ^i is traceless and hermitian.

9.1 Generators

We expand an element $A(\boldsymbol{\alpha}) \in \mathrm{SU}(N) \subset \mathrm{GL}(N, \mathbb{C})$ near $A(\boldsymbol{\alpha} = \boldsymbol{0}) = \mathbb{1}$, where $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \cdots, \alpha^{N^2-1})$ is a set of $N^2 - 1$ real variables α^i. $A(\boldsymbol{\alpha})$ is assumed to be analytic with respect to every variable α^i. Then for any finite α^a, the Taylor series expansion of $A(\boldsymbol{\alpha})$ about $\boldsymbol{\alpha} = \boldsymbol{0}$ is

$$\begin{aligned} A(\boldsymbol{\alpha}) &= \exp\big[\boldsymbol{\alpha} \cdot \boldsymbol{\nabla}_{\boldsymbol{z}}\big] A(\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}} = \mathbb{1} + \alpha^a \frac{\partial A(\boldsymbol{\alpha})}{\partial \alpha^a}\bigg|_{\boldsymbol{\alpha}=\boldsymbol{0}} + \frac{\alpha^a \alpha^b}{2!} \frac{\partial^2 A(\boldsymbol{\alpha})}{\partial \alpha^a \partial \alpha^b}\bigg|_{\boldsymbol{\alpha}=\boldsymbol{0}} + \cdots \\ &= \exp\big[-i\boldsymbol{\alpha} \cdot \big(i\boldsymbol{\nabla}_{\boldsymbol{z}}\big)\big] A(\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \end{aligned} \tag{9.5}$$

where

$$\nabla_{\boldsymbol{z}} \equiv \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^{N^2-1}} \right). \tag{9.6}$$

We define the generator T^a for $a = 1$, 2, $\cdots$, $N^2 - 1$ for the SU(N):

$$\boldsymbol{T} = \big(T^1, T^2, \cdots T^{N^2-1}\big) = i\boldsymbol{\nabla}_{\boldsymbol{z}} A(\boldsymbol{z})\Big|_{\boldsymbol{z}=\boldsymbol{0}}. \tag{9.7}$$

Problem 9.5 1. Show that for any finite $\alpha^i \in \mathbb{R}$,

$$A(\boldsymbol{\alpha}) = \lim_{n \to \infty} \left(\mathbb{1} - \frac{i}{n} \boldsymbol{\alpha} \cdot \boldsymbol{T} \right)^n = e^{-i\boldsymbol{\alpha} \cdot \boldsymbol{T}}, \tag{9.8}$$

where $\boldsymbol{\alpha} \cdot \boldsymbol{T} = \alpha^a T^a$ and the summation over $a = 1$, 2, $\cdots$, $N^2 - 1$ is assumed.

2. By making use of Baker-Campbell-Hausdorff formula, show that

$$[A(\boldsymbol{\alpha})]^{-1} = e^{+i\boldsymbol{\alpha}\cdot\boldsymbol{T}} = A(-\boldsymbol{\alpha}). \tag{9.9}$$

3. By making use of the property of $A(\boldsymbol{\alpha}) \in \mathrm{SU}(N)$, $[A(\boldsymbol{\alpha})]^{-1} = [A(\boldsymbol{\alpha})]^{\dagger}$, show that T^a is hermitian:

$$\boldsymbol{T}^{\dagger} = \boldsymbol{T}, \quad \boldsymbol{\alpha}\cdot\boldsymbol{T}^{\dagger} = \boldsymbol{\alpha}\cdot\boldsymbol{T}. \tag{9.10}$$

Therefore, there exists a unitary matrix

$$U = \begin{pmatrix} \psi_1^{\dagger} \\ \psi_2^{\dagger} \\ \vdots \end{pmatrix}, \tag{9.11}$$

where ψ_i is an eigenvector of $\boldsymbol{\alpha}\cdot\boldsymbol{T}$:

$$\boldsymbol{\alpha}\cdot\boldsymbol{T}\psi_i = \lambda_i\psi_i. \tag{9.12}$$

4. It is straightforward to show that $U\boldsymbol{\alpha}\cdot\boldsymbol{T}U^{\dagger}$ is diagonal. Show that

$$UA(\boldsymbol{\alpha})U^{\dagger} = \begin{pmatrix} e^{-i\lambda_1} & 0 & 0 & \cdots \\ 0 & e^{-i\lambda_2} & 0 & \cdots \\ 0 & 0 & e^{-i\lambda_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9.13}$$

5. Show that

$$\mathfrak{Det}[A(\boldsymbol{\alpha})] = e^{-i\mathrm{Tr}[\boldsymbol{\alpha}\cdot\boldsymbol{T}]} = +1. \tag{9.14}$$

The condition of the determinant requires that the $\boldsymbol{\alpha}\cdot\boldsymbol{T}$ is traceless for any $\boldsymbol{\alpha}$. Therefore, the generator of $\mathrm{SU}(N)$ is traceless:

$$\mathrm{Tr}[T^a] = 0. \tag{9.15}$$

Problem 9.6 Suppose $H \in \mathrm{SL}(N, \mathbb{C})$ is hermitian:

$$H^{\dagger} = H. \tag{9.16a}$$

Assume that there exists eigenvalues λ_i and corresponding eigenvector ψ_i, an N-dimensional column vector whose elements are complex numbers, that satisfy

$$H\psi_i = \lambda_i\psi_i. \tag{9.16b}$$

1. Show that eigenvectors of distinct eigenvalues are orthogonal to each other:

$$\psi_i^\dagger \psi_j = 0 \quad \text{if} \quad \lambda_i \neq \lambda_j. \tag{9.16c}$$

2. Show that eigenvalues are real.

3. Provide a way to construct an orthonormal set of eigenvectors. Describe a way how to construct an orthonormal set of eigenvectors if any two eigenvalues are identical.

4. We have constructed an orthonormal set of eigenvectors that satisfies

$$\psi_i^\dagger \psi_j = \delta_{ij} \quad \text{and} \quad \psi_i^\dagger H \psi_j = \lambda_i \delta_{ij}. \quad \leftarrow \text{no sum over } i \tag{9.16d}$$

   Let us construct a matrix A such that

$$A = \begin{pmatrix} \psi_1^\dagger \\ \psi_2^\dagger \\ \vdots \end{pmatrix} \quad \text{and} \quad A^\dagger = \begin{pmatrix} \psi_1 \, \psi_2 \, \cdots \end{pmatrix}. \tag{9.16e}$$

   Show that A is unitary:

$$A^\dagger A = \mathbb{1}. \tag{9.16f}$$

5. Show that $AHA^\dagger$ is diagonalized:

$$AHA^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9.16g}$$

   We have shown that any hermitian matrix has real eigenvalues and can be diagonalized.

9.2 Structure constant f^{abc} of SU(N)

The structure constant f^{abc} of SU(N) is defined by the commutator of two generators:

$$[T^a, T^b] = i f^{abc} T^c. \tag{9.17}$$

Problem 9.7 We would like to show that f^{abc} is antisymmetric under exchange of any two adjacent indices.

1. By makig use of the relation $(T^a)^\dagger = T^a$, show that f^{abc} is real.

2. By making use of the definition (9.17), show that

$$f^{abc} = -f^{bac}. \tag{9.18}$$

3. Show that

$$f^{abc} = -\frac{i}{T_F}\mathrm{Tr}\big([T^a, T^b]T^c\big) \quad \text{if} \quad \mathrm{Tr}(T^aT^b) = T_F\delta^{ab}. \tag{9.19}$$

4. By making use of Tr(AB)=Tr(BA), show that

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB). \tag{9.20}$$

5. Show that

$$f^{abc} = f^{bca} = f^{cab} = -f^{acb} = -f^{bac} = -f^{cba}. \tag{9.21}$$

   Therefore, we have shown that f^{abc} is totally antisymmetric under exchange of any two adjacent indices.

6. By multiplying T^e to the following Jacobi identity,

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0, \tag{9.22}$$

   and taking the trace, show that

$$f^{abd}f^{cde} + f^{bcd}f^{ade} + f^{cad}f^{bde} = 0. \tag{9.23}$$

Problem 9.8 Consider an arbitray hermitian matrix $H \in \mathrm{GL}(N, \mathbb{C})$: $H^\dagger = H$.

1. Show that the number of free real parameters that determines H is N^2.

2. We have shown that there are $N^2 - 1$ traceless hermitian matrices $T^a \in \mathrm{GL}(N, \mathbb{C})$ that are the generators of SU(N). In addition, $\mathbb{1}$ is a real diagonal matrix whose trace is $\mathrm{Tr}(\mathbb{1}) = N$. Show that H is completely determined by

$$H = \alpha^0\mathbb{1} + \boldsymbol{\alpha}\cdot\boldsymbol{T}, \tag{9.24}$$

   where $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \cdots, \alpha^{N^2-1})$ and $\alpha^i \in \mathbb{R}$ for $i = 0,\ 1,\ 2,\ \cdots,\ N^2 - 1$.

3. By making use of the commutation relations $[T^a, T^b] = if^{abc}T^c$, show that the condition $\boldsymbol{\alpha} = \mathbf{0}$ must be satisfied if $[H, T^a] = 0$ for any a. Thus any hermitian matrix that commutes with all of the generators T^a of SU(N) must be proportional to $\mathbb{1}$.

Problem 9.9 Let T^a be a generator of the fundamental representation for SU(N) whose commutation relations satisfies Lie algebra:

$$[T^a, T^b] = if^{abc}T^c. \tag{9.25}$$

The ***Casimir operator*** $\mathbb{C}_F$ for the fundamental representation of SU(N) is defined by

$$\mathbb{C}_F = T^a T^a, \tag{9.26}$$

where a is summed over.

1. Show that $\mathbb{C}_F$ is hermitian.

2. Show that $\mathbb{C}_F$ commutes with any of the generators T^b.

$$[\mathbb{C}_F, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a = 0. \tag{9.27}$$

   In summary, $\mathbb{C}_F$ is proportional to the identity matrix:

$$\mathbb{C}_F = C_F \mathbb{1}, \tag{9.28}$$

   where C_F is a real number that depends on the normalization of T^a.

Problem 9.10 The generator T^a of the fundamental representation for SU(N) is traceless, hermitian, and satisfies the commutation relation

$$[T^a, T^b] = if^{abc}T^c. \tag{9.29}$$

1. Show that, for any $a \neq b$,

$$\mathrm{Tr}[T^a T^b] = 0 \quad \text{for} \quad a \neq b. \tag{9.30}$$

2. By making use of the relation $\mathbb{C}_F = C_F\mathbb{1}$, show that

$$\mathrm{Tr}(T^a T^a) = C_F N, \tag{9.31}$$

   where a is summed over. We can always change the normalization of each T^a. The conventional choice of the normalization for T^a is

$$\mathrm{Tr}(T^a T^b) = T_F \delta^{ab}, \quad \text{where} \quad T_F = \frac{1}{2}. \tag{9.32}$$

3. Show that

$$C_F = \frac{T_F(N^2-1)}{N} = \frac{N^2-1}{2N}. \tag{9.33}$$

4. Show that any hermitian matrix $H \in \mathrm{GL}(N, \mathbb{C})$ can be expressed as

$$H = \frac{\mathbb{1}}{N}\mathrm{Tr}(H) + \frac{T^a}{T_F}\mathrm{Tr}(HT^a). \tag{9.34}$$

Problem 9.11 The Pauli matrices σ^i's are the generators of the fundamental representation for SU(2). They are traceless and hermitian, and satisfy the commutation relation:

$$[\sigma^i, \sigma^j] = i f^{ijk}\sigma^k. \tag{9.35}$$

1. Show that

$$\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\epsilon^{ijk}\sigma^k. \tag{9.36}$$

2. Find the structure constant f^{ijk}.

3. By making use of the relation $\mathbb{C}_F = C_F\mathbb{1}$, show that

$$\mathrm{Tr}(\sigma^i\sigma^i) = 2C_F \quad \text{and} \quad \mathrm{Tr}(\sigma^i\sigma^j) = T_F\delta^{ij}. \tag{9.37}$$

4. Show that

$$T_F = 2 \quad \text{and} \quad C_F = 3. \tag{9.38}$$

5. Show that any hermitian matrix $H \in \mathrm{GL}(2, \mathbb{C})$ can be expressed as

$$H = \frac{\mathbb{1}}{2}\mathrm{Tr}(H) + \frac{\sigma^i}{2}\mathrm{Tr}(H\sigma^i). \tag{9.39}$$

Problem 9.12 Let us consider SU(3).

$$[T^a, T^b] = i f^{abc}T^c. \tag{9.40}$$

1. By making use of the relation $\mathbb{C}_F = C_F\mathbb{1}$, show that

$$\mathrm{Tr}(T^aT^a) = 3C_F \quad \text{and} \quad \mathrm{Tr}(T^aT^b) = T_F\delta^{ab}. \tag{9.41}$$

2. Show that

$$T_F = \frac{1}{2} \quad \text{and} \quad C_F = \frac{4}{3}. \tag{9.42}$$

3. Show that any hermitian matrix $H \in \mathrm{GL}(3, \mathbb{C})$ can be expressed as

$$H = \frac{\mathbb{1}}{3}\mathrm{Tr}(H) + 2T^a\mathrm{Tr}(HT^a). \tag{9.43}$$

Problem 9.13 The product of any two hermitian matrices A and B can be expressed as

$$AB = \frac{1}{2}\{A, B\} + \frac{1}{2}[A, B]. \tag{9.44}$$

It is trivial to show that the anticommutator $\{A, B\} = AB + BA$ is hermitian and the commutator $[A, B] = AB - BA$ is antihermitian:

$$\{A, B\}^\dagger = \{A, B\}, \quad [A, B]^\dagger = -[A, B]. \tag{9.45}$$

1. Show that

$$T^aT^b = \frac{1}{2}\{T^a, T^b\} + \frac{i}{2}f^{abc}T^c, \tag{9.46}$$

where T^a is the generator for the fundamental representation of SU(N).

2. Because $\{T^a, T^b\}$ is hermitian, we can parametrize $\{T^a, T^b\}$ as

$$\{T^a, T^b\} = \frac{2T_F}{N}\delta^{ab}\mathbb{1} + d^{abc}T^c. \tag{9.47}$$

Show that

$$T^aT^b = \frac{T_F}{N}\delta^{ab}\mathbb{1} + \frac{1}{2}(d^{abc} + if^{abc})T^c. \tag{9.48}$$

3. By making use of Eq. (9.48), show that

$$\mathbb{C}_F = T^aT^a = C_F\mathbb{1}, \quad C_F = \frac{T_F(N^2 - 1)}{N}. \tag{9.49}$$

This result reproduces Eq. (9.33).

Problem 9.14 Let us find properties of d^{abc}.

1. By making use of Eqs. (9.32) and (9.47), show that

$$d^{abc} = \frac{1}{T_F}\mathrm{Tr}\big(\{T^a, T^b\}T^c\big). \tag{9.50}$$

2. Show that d^{abc} is totally symmetric:

$$d^{abc} = d^{bca} = d^{cab} = d^{acb} = d^{bac} = d^{cba}. \tag{9.51}$$

Problem 9.15 Show that $d^{ijk} = 0$ for all $i,\ j,\ k = 1,\ 2,\ 3$ in SU(2).

9.3 Completeness relation

We have shown that any hermitian matrix $H \in \mathrm{GL}(N, \mathbb{C})$ can be expressed as

$$H = \frac{\mathbb{1}}{N}\mathrm{Tr}(H) + \frac{T^a}{T_F}\mathrm{Tr}(HT^a). \tag{9.52}$$

The ij element of the matrix H is

$$H_{ij} = \frac{1}{N}\delta_{ij}H_{kk} + \frac{1}{T_F}T^a_{ij}H_{\ell k}T^a_{k\ell}. \tag{9.53}$$

Problem 9.16 H_{ij} can be expressed as

$$H_{ij} = \delta_{il}\delta_{jk}H_{lk}. \tag{9.54}$$

By making use of this trick, solve the following problems.

1. By comparing the coefficients of H_{lk} on both sides of Eq. (9.53), show that

$$\delta_{i\ell}\delta_{jk} = \frac{1}{N}\delta_{ij}\delta_{k\ell} + \frac{1}{T_F}T^a_{ij}T^a_{k\ell}. \tag{9.55}$$

   Therefore,

$$T^a_{ij}T^a_{k\ell} = T_F\left(-\frac{1}{N}\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}\right). \tag{9.56}$$

   This identity is called the ***completeness relation***.

2. By multiplying δ_{jk} to both sides, show that

$$\mathbb{C}_F = \frac{T_F(N^2-1)}{N}\mathbb{1}. \tag{9.57}$$

3. By making use of the completeness relation, show that

$$(T^aT^bT^a)_{i\ell} = T^a_{ij}T^b_{jk}T^a_{k\ell} = T^b_{jk}T_F\left(-\frac{1}{N}\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}\right) = -\frac{T_F}{N}T^b_{i\ell}. \tag{9.58}$$

   Therefore,

$$T^aT^bT^a = -\frac{T_F}{N}T^b. \tag{9.59}$$

4. By making use of Eqs. (9.57) and (9.59), show that

$$T^aT^aT^bT^b = T^aT^bT^bT^a = C_F^2\mathbb{1} = \frac{T_F^2(N^2-1)^2}{N^2}\mathbb{1}, \tag{9.60a}$$

$$T^aT^bT^aT^c = -\frac{T_F}{N}T^bT^c, \tag{9.60b}$$

$$T^aT^bT^aT^b = -\frac{T_FC_F}{N}\mathbb{1} = -\frac{T_F^2(N^2-1)}{N^2}\mathbb{1}. \tag{9.60c}$$

5. Show that

$$\mathrm{Tr}(T^a) = 0, \tag{9.61a}$$

$$\mathrm{Tr}(T^aT^b) = T_F\delta^{ab}, \tag{9.61b}$$

$$\mathrm{Tr}(T^aT^a) = C_FN = T_F(N^2-1), \tag{9.61c}$$

$$\mathrm{Tr}(T^aT^bT^a) = 0. \tag{9.61d}$$

6. Show that

$$\mathrm{Tr}(T^aT^aT^bT^b) = \mathrm{Tr}(T^aT^bT^bT^a) = C_F^2N = \frac{T_F^2(N^2-1)^2}{N}, \tag{9.62a}$$

$$\mathrm{Tr}(T^aT^bT^aT^b) = -T_FC_F = -\frac{T_F^2(N^2-1)}{N}, \tag{9.62b}$$

$$\mathrm{Tr}(T^cT^cT^aT^b) = \mathrm{Tr}(T^cT^aT^bT^c) = C_F\mathrm{Tr}(T^aT^b) = C_FT_F\delta^{ab} = \frac{T_F^2(N^2-1)}{N}\delta^{ab}, \tag{9.62c}$$

$$\mathrm{Tr}(T^cT^aT^cT^b) = \mathrm{Tr}(T^aT^cT^bT^c) = -\frac{T_F}{N}\mathrm{Tr}(T^aT^b) = -\frac{T_F^2}{N}\delta^{ab}. \tag{9.62d}$$

Problem 9.17 Let us consider a hermitian matrix $H \in \mathrm{GL}(2,\mathbb{C})$. We have shown that

$$H = \frac{\mathbb{1}}{2}\mathrm{Tr}(H) + \frac{\sigma^a}{2}\mathrm{Tr}(H\sigma^a), \tag{9.63}$$

and the ij element of the matrix H is

$$H_{ij} = \frac{1}{2}\delta_{ij}H_{kk} + \frac{1}{2}\sigma^a_{ij}H_{\ell k}\sigma^a_{k\ell}. \tag{9.64}$$

1. By comparing the coefficients of H_{lk} on both sides of Eq. (9.64), show that

$$\delta_{i\ell}\delta_{jk} = \frac{1}{2}\delta_{ij}\delta_{k\ell} + \frac{1}{2}\sigma^a_{ij}\sigma^a_{k\ell}. \tag{9.65}$$

   Therefore, the completeness relation is

$$\sigma^a_{ij}\sigma^a_{k\ell} = -\delta_{ij}\delta_{k\ell} + 2\delta_{i\ell}\delta_{jk}. \tag{9.66}$$

2. By multiplying δ_{jk} to both sides, show that

$$\mathbb{C}_F = 3 \times \mathbb{1}. \tag{9.67}$$

3. By making use of the completeness relation, show that

$$(\sigma^a\sigma^b\sigma^a)_{i\ell} = \sigma^a_{ij}\sigma^b_{jk}\sigma^a_{k\ell} = \sigma^b_{jk}\left(-\delta_{ij}\delta_{k\ell} + 2\delta_{i\ell}\delta_{jk}\right) = -\sigma^b_{i\ell}. \tag{9.68}$$

   Therefore,

$$\sigma^a\sigma^b\sigma^a = -\sigma^b. \tag{9.69}$$

4. By making use of Eqs. (9.67) and (9.69), show that

$$\sigma^a\sigma^a\sigma^b\sigma^b = \sigma^a\sigma^b\sigma^b\sigma^a = 9 \times \mathbb{1}, \tag{9.70a}$$

$$\sigma^a\sigma^b\sigma^a\sigma^c = -\sigma^b\sigma^c, \tag{9.70b}$$

$$\sigma^a\sigma^b\sigma^a\sigma^b = -3 \times \mathbb{1}. \tag{9.70c}$$

5. Show that

$$\mathrm{Tr}(\sigma^a) = 0, \tag{9.71a}$$

$$\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}, \tag{9.71b}$$

$$\mathrm{Tr}(\sigma^a\sigma^a) = 6, \tag{9.71c}$$

$$\mathrm{Tr}(\sigma^a\sigma^b\sigma^a) = 0. \tag{9.71d}$$

6. Show that

$$\mathrm{Tr}(\sigma^a\sigma^a\sigma^b\sigma^b) = \mathrm{Tr}(\sigma^a\sigma^b\sigma^b\sigma^a) = 18, \tag{9.72a}$$

$$\mathrm{Tr}(\sigma^a\sigma^b\sigma^a\sigma^b) = -6, \tag{9.72b}$$

$$\mathrm{Tr}(\sigma^c\sigma^c\sigma^a\sigma^b) = \mathrm{Tr}(\sigma^c\sigma^a\sigma^b\sigma^c) = 3\,\mathrm{Tr}(\sigma^a\sigma^b) = 6\,\delta^{ab}, \tag{9.72c}$$

$$\mathrm{Tr}(\sigma^c\sigma^a\sigma^c\sigma^b) = \mathrm{Tr}(\sigma^a\sigma^c\sigma^b\sigma^c) = -\mathrm{Tr}(\sigma^a\sigma^b) = -2\,\delta^{ab}. \tag{9.72d}$$

Problem 9.18 According to Eq. (9.36),

$$\sigma^i\sigma^j = \delta^{ij}\,\mathbb{1} + i\epsilon^{ijk}\sigma^k. \tag{9.73}$$

Carry out the following calculation by applying this relation without relying on the completeness relation.

1. Show that

$$\sigma^a\sigma^b\sigma^a = -\sigma^b. \tag{9.74}$$

2. Show that

$$\sigma^a\sigma^a\sigma^b\sigma^b = \sigma^a\sigma^b\sigma^b\sigma^a = 9 \times \mathbb{1}, \tag{9.75a}$$

$$\sigma^a\sigma^b\sigma^a\sigma^c = -\sigma^b\sigma^c, \tag{9.75b}$$

$$\sigma^a\sigma^b\sigma^a\sigma^b = -3 \times \mathbb{1}. \tag{9.75c}$$

3. Show that

$$\mathrm{Tr}(\sigma^a) = 0, \tag{9.76a}$$

$$\mathrm{Tr}(\sigma^a\sigma^b) = 2\delta^{ab}, \tag{9.76b}$$

$$\mathrm{Tr}(\sigma^a\sigma^a) = 6, \tag{9.76c}$$

$$\mathrm{Tr}(\sigma^a\sigma^b\sigma^a) = 0. \tag{9.76d}$$

4. Show that

$$\mathrm{Tr}(\sigma^a\sigma^a\sigma^b\sigma^b) = \mathrm{Tr}(\sigma^a\sigma^b\sigma^b\sigma^a) = 18, \tag{9.77a}$$

$$\mathrm{Tr}(\sigma^a\sigma^b\sigma^a\sigma^b) = -6, \tag{9.77b}$$

$$\mathrm{Tr}(\sigma^c\sigma^c\sigma^a\sigma^b) = \mathrm{Tr}(\sigma^c\sigma^a\sigma^b\sigma^c) = 3\,\mathrm{Tr}(\sigma^a\sigma^b) = 6\,\delta^{ab}, \tag{9.77c}$$

$$\mathrm{Tr}(\sigma^c\sigma^a\sigma^c\sigma^b) = \mathrm{Tr}(\sigma^a\sigma^c\sigma^b\sigma^c) = -\mathrm{Tr}(\sigma^a\sigma^b) = -2\,\delta^{ab}. \tag{9.77d}$$

9.4 Adjoint representation

Problem 9.19 We recall that

$$f^{abc} = -\frac{i}{T_F}\mathrm{Tr}(T^c[T^a, T^b]). \tag{9.78}$$

We define the generator $\mathbb{t}^c$ of the adjoint representation of SU(N):

$$\mathbb{t}^c_{ab} \equiv -if^{abc} = -\frac{1}{T_F}\mathrm{Tr}(T^a[T^b, T^c]) = -\frac{1}{T_F}\mathrm{Tr}(T^b[T^c, T^a]) = -\frac{1}{T_F}\mathrm{Tr}(T^c[T^a, T^b]), \tag{9.79}$$

where a, b, $c = 1, 2, \cdots, N^2-1$. Note that the number of generators $\mathbb{t}^c$ is N^2-1 and each generator is an $(N^2-1)\times(N^2-1)$ matrix. According to Eq. (9.23), we have

$$f^{abe}f^{cde} + f^{bce}f^{ade} + f^{cae}f^{bde} = 0. \tag{9.80}$$

1. We define sets of permutations of (a, b, c) as $\sigma(a, b, c)$:

$$\sigma(a, b, c) \equiv \sigma^+(a, b, c) \cup \sigma^-(a, b, c), \tag{9.81a}$$

$$\sigma^+(a, b, c) = \{(a, b, c), (b, c, a), (c, a, b)\}, \tag{9.81b}$$

$$\sigma^-(a, b, c) = \{(a, c, b), (b, a, c), (c, b, a)\}. \tag{9.81c}$$

Let (x, y, z) be a permutation of (a, b, c). We define a sign function ϵ for a permutation:

$$\epsilon(x, y, z) = +1, \quad \text{if} \quad (x, y, z) \in \sigma^+(a, b, c), \tag{9.82a}$$

$$\epsilon(x, y, z) = -1, \quad \text{if} \quad (x, y, z) \in \sigma^-(a, b, c), \tag{9.82b}$$

$$\epsilon(x, y, z) = 0, \quad \text{if} \quad (x, y, z) \notin \sigma(a, b, c). \tag{9.82c}$$

Show that

$$f^{abc} = -\frac{2i}{3!T_F} \sum_{(a',b',c')\in\sigma(a,b,c)} \epsilon(a',b',c')\mathrm{Tr}(T^{a'}T^{b'}T^{c'}), \tag{9.83}$$

where $\epsilon(a,b,c)$ is totally antisymmetric.

2. Show for any $a = 1, 2, \cdots, N^2 - 1$ that $\mathbb{t}^a$ is traceless and hermitian:

$$\mathrm{Tr}(\mathbb{t}^a) = 0 \quad \text{and} \quad (\mathbb{t}^a)^\dagger = \mathbb{t}^a. \tag{9.84}$$

3. Show that

$$\begin{aligned}
f^{abe}f^{cde} &= i f^{abe}\mathbb{t}^e_{cd}, &(9.85\text{a})\\
(\mathbb{t}^a\mathbb{t}^b)_{cd} &= \mathbb{t}^a_{ce}\mathbb{t}^b_{ed} = -f^{cea}f^{edb} = -f^{cae}f^{bde}, &(9.85\text{b})\\
(\mathbb{t}^b\mathbb{t}^a)_{cd} &= \mathbb{t}^b_{ce}\mathbb{t}^a_{ed} = -f^{ceb}f^{eda} = +f^{bce}f^{ade}, &(9.85\text{c})\\
[\mathbb{t}^a,\mathbb{t}^b]_{cd} &= i f^{abe}\mathbb{t}^e_{cd}. &(9.85\text{d})
\end{aligned}$$

In summary, the structure constant of the adjoint representation is the same as that of the fundamental representation of SU(N):

$$[\mathbb{t}^a,\mathbb{t}^b] = i f^{abe}\mathbb{t}^e. \tag{9.86}$$

4. The Casimir operator for the adjoint representation of SU(N) can be defined by

$$\mathbb{C}_A \equiv \mathbb{t}^a\mathbb{t}^a. \tag{9.87}$$

Show that

$$[\mathbb{C}_A,\mathbb{t}^a] = \mathbb{t}^a[\mathbb{t}^a,\mathbb{t}^b] + [\mathbb{t}^a,\mathbb{t}^b]\mathbb{t}^a = 0, \tag{9.88}$$

for all $a = 1, 2, \cdots, N^2 - 1$.

5. According to Eq. (9.19),

$$f^{abc} = -\frac{1}{T_F}\mathrm{Tr}([T^a,T^b]T^c). \tag{9.89}$$

Show that

$$\begin{aligned}
(\mathbb{C}_A)_{ab} &= (\mathbb{t}^x)_{ay}(\mathbb{t}^x)_{yb} = f^{xya}f^{xyb}\\
&= -\frac{1}{T_F^2}\mathrm{Tr}([T^x,T^y]T^a)\mathrm{Tr}([T^x,T^y]T^b).\\
&= \frac{2}{T_F^2}\Big[\mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^yT^xT^b) - \mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^xT^yT^b)\Big]. \qquad (9.90)
\end{aligned}$$

6. By making use of the completeness relation,

$$T^x_{ij}T^x_{k\ell} = T_F\left(-\frac{1}{N}\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}\right), \tag{9.91}$$

show that

$$\mathrm{Tr}(T^x A)\mathrm{Tr}(T^x B) = (T^x_{ij}A_{ji})(T^x_{k\ell}B_{\ell k}) = T_F\left[-\frac{1}{N}\mathrm{Tr}(A)\mathrm{Tr}(B) + \mathrm{Tr}(AB)\right]. \tag{9.92}$$

7. By making use of the identity (9.92), show that

$$\mathrm{Tr}(T^a)\mathrm{Tr}(T^a) = 0, \tag{9.93a}$$

$$\mathrm{Tr}(T^xT^a)\mathrm{Tr}(T^xT^b) = T_F^2\delta^{ab}. \tag{9.93b}$$

8. Show that

$$\begin{aligned}
\mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^xT^yT^b) &= T_F\left[-\frac{1}{N}\mathrm{Tr}(T^yT^a)\mathrm{Tr}(T^yT^b) + \mathrm{Tr}(T^yT^aT^yT^b)\right] \\
&= T_F\left[-\frac{T_F^2}{N}\delta^{ab} - \frac{T_F^2}{N}\delta^{ab}\right] = -\frac{2T_F^3}{N}\delta^{ab}, \qquad (9.94)\\
\mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^yT^xT^b) &= \mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^xT^bT^y) \\
&= T_F\left[-\frac{1}{N}\mathrm{Tr}(T^yT^a)\mathrm{Tr}(T^bT^y) + \mathrm{Tr}(T^yT^aT^bT^y)\right] \\
&= T_F\left[-\frac{T_F^2}{N}\delta^{ab} + \frac{T_F^2(N^2-1)}{N}\delta^{ab}\right] = \frac{T_F^3(N^2-2)}{N}\delta^{ab}. \qquad (9.95)
\end{aligned}$$

It is straightforward to show that

$$(\mathbb{C}_A)_{ab} = 2T_F N\delta^{ab}. \tag{9.96}$$

Therefore,

$$\mathbb{C}_A = C_A\mathbb{1} = 2T_F N\mathbb{1}. \tag{9.97}$$

9. Show that

$$f^{abp}f^{abq} = C_A\delta^{pq}, \tag{9.98a}$$

$$f^{abc}f^{abc} = C_A(N^2-1) = 2T_F N(N^2-1). \tag{9.98b}$$

Problem 9.20 According to Eq. (9.50),

$$d^{abc} = \frac{1}{T_F}\mathrm{Tr}(\{T^a, T^b\}T^c). \tag{9.99}$$

1. Show that

$$d^{abp}f^{abq} = 0. \tag{9.100}$$

2. Show that

$$d^{xya}d^{xyb} = \frac{2}{T_F^2}\Big[\mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^xT^yT^b) + \mathrm{Tr}(T^xT^yT^a)\mathrm{Tr}(T^yT^xT^b)\Big]. \tag{9.101}$$

3. Show that

$$d^{abc}d^{abd} = \frac{2T_F(N^2-4)}{N}\delta^{cd}, \tag{9.102a}$$

$$d^{abc}d^{abc} = \frac{2T_F(N^2-4)(N^2-1)}{N}. \tag{9.102b}$$

Problem 9.21 According to Eq. (9.47),

$$\{T^a, T^b\} = \frac{2T_F}{N}\delta^{ab}\mathbb{1} + d^{abc}T^c. \tag{9.103}$$

By multiplying δ^{ab} to the above identity, show that

$$d^{aab} = 0, \tag{9.104}$$

where a is summed over $a = 1, 2, \cdots, N^2-1$. Show also that

$$d^{aab} = d^{aba} = d^{baa} = 0. \tag{9.105}$$

[[NEEDTOBEEDITED, Adjoint representation of SU(2) is the rotational generator of SO(3)]]

9.5 Gell-Mann matrices

Exercise 9.22 We have shown that the number of generators for the SU(N) is N^2-1. Therefore, SU(3) has 8 generators. We also have shown that the generators for SU(N) are traceless hermitian. Conventional choice of the generators is

$$T^a = \frac{1}{2}\lambda_a, \quad a = 1, 2, \cdots, 8, \tag{9.106}$$

where λ_a's are called the ***Gell-Mann matrices***. It is convenient to construct 3×3 traceless hermitain matrices by making use of the 2×2 Pauli matrices that are also traceless hermitian. Note that σ_1 and σ_2 have vanishing diagonal elements. The only Pauli matrix that has non-vanishing diagonal elements is σ_3.

1. Show that only two elements are independent among the diagonal elements of Gell-Mann matrices.

2. The first three entries of λ_a are chosen so that the ij element is identical to σ_a for i, $j = 1$, 2:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{9.107}$$

3. The next two entries λ_a are chosen so that the ij element is identical to σ_1 and σ_2 for i, $j = 1$, 3:

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{9.108}$$

4. Two more entries are chosen so that the ij element is identical to σ_1 and σ_2 for i, $j = 2$, 3:

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{9.109}$$

5. Now we determine the last entry. Because off-diagonal elements are already fixed completely, we have to find an entry that has non-vanishing diagonal elements. Because $\lambda_3 = \mathrm{diag}[1, -1, 0]$, we can choose a diagonal matrix whose diagonal elements construct a 3-dimensional vector that is orthogonal to $(1, -1, 0)$. A simple choice is

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{9.110}$$

6. Show that

$$\mathrm{Tr}(\lambda_a) = 0, \tag{9.111a}$$

$$\lambda_a^\dagger = \lambda_a, \tag{9.111b}$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}. \tag{9.111c}$$

Problem 9.23 According to Eq. (9.19) and Eq. (9.106),

$$f^{abc} = -\frac{i}{4}\mathrm{Tr}\big([\lambda^a, \lambda^b]\lambda^c\big). \tag{9.112}$$

1. Show that nonvanishing structure constants f^{abc} are given by

$$f^{123} = 1, \tag{9.113a}$$

$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \tag{9.113b}$$

$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}. \tag{9.113c}$$

We can use the antisymmetric properties to find other permuations:

$$f^{abc} = f^{bca} = f^{cab} = -f^{bac} = -f^{acb} = -f^{cba}. \tag{9.114}$$

For example,

$$f^{123} = f^{231} = f^{312} = +1, \tag{9.115a}$$

$$f^{213} = f^{132} = f^{321} = -1. \tag{9.115b}$$

2. Show that

$$f^{abc} f^{abc} = 24, \tag{9.116}$$

where repeated indicies are summed over. The following REDUCE program can be used as an independent check of above derivations.

```
procedure ta(n);
begin scalar m;
if n=1 then m:=mat((0,1,0),(1,0,0),(0,0,0));
if n=2 then m:=mat((0,-i,0),(i,0,0),(0,0,0));
if n=3 then m:=mat((1,0,0),(0,-1,0),(0,0,0));
if n=4 then m:=mat((0,0,1),(0,0,0),(1,0,0));
if n=5 then m:=mat((0,0,-i),(0,0,0),(i,0,0));
if n=6 then m:=mat((0,0,0),(0,0,1),(0,1,0));
if n=7 then m:=mat((0,0,0),(0,0,-i),(0,i,0));
if n=8 then m:=mat((1,0,0),(0,1,0),(0,0,-2))/sqrt(3);
return m/2;
end;

procedure f(a,b,c);
```

```
begin scalar aa,bb,cc,xx,ans;
aa:=ta(a);
bb:=ta(b);
cc:=ta(c);
xx:=aa*bb-bb*aa;
xx:=2*xx*cc;
ans:=trace(xx)/i;
return ans;
end;
cas:=mat((0,0,0),(0,0,0),(0,0,0));
id:=mat((1,0,0),(0,1,0),(0,0,1));
for a:=1:8 do <<x:=ta(a);cas:=cas+x*x>>;cas-4/3*id;
for a:=1:8 do <<x:=ta(a);write a,trace(x*x);>>;
f(1,2,3)-1;
f(1,4,7)-1/2;
f(1,6,5)-1/2;
f(2,4,6)-1/2;
f(2,5,7)-1/2;
f(3,4,5)-1/2;
f(3,7,6)-1/2;
f(4,5,8)-sqrt(3)/2;
f(6,7,8)-sqrt(3)/2;
24-for a:=1:8 sum for b:=1:8 sum for c:=1:8 sum f(a,b,c)^2;
```

IV. Minkowski Space

10. Minkowski space

10.1 Metric tensor

Exercise 10.1 In the n-dimensional Euclidean space, the distance $d(\boldsymbol{x},\boldsymbol{y})$ between two points $\boldsymbol{x}=(x^1,x^2,\cdots,x^n)$ and $\boldsymbol{y}=(y^1,y^2,\cdots,y^n)$ is defined by

$$d(\boldsymbol{x},\boldsymbol{y})=\sqrt{(\boldsymbol{x}-\boldsymbol{y})^2}=\sqrt{(\boldsymbol{x}-\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}, \tag{10.1}$$

where the scalar product of the Euclidean vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\boldsymbol{x}\cdot\boldsymbol{y}=x^iy^i. \tag{10.2}$$

This can be expressed in terms of the ***metric tensor*** δ^{ij} in the n-dimensional Euclidean space:

$$\boldsymbol{x}\cdot\boldsymbol{y}=x^i\delta^{ij}y^j. \tag{10.3}$$

Show that the matrix representation of the metric tensor of the n-dimensional Euclidean space is

$$\delta^{ij}=(\mathbb{1})^{ij}=\begin{pmatrix}1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\\\vdots&\vdots&&\ddots\\&&&\end{pmatrix}. \tag{10.4}$$

Problem 10.2 1. Show that the metric tensor δ^{ij} of the n-dimensional Eucliean space is invariant under rotation:

$$\delta^{ij}=R^{ia}(\boldsymbol{\theta})R^{jb}(\boldsymbol{\theta})\delta^{ab}, \tag{10.5}$$

where

$$R(\boldsymbol{\theta})=R(\theta\hat{\boldsymbol{n}})=\exp\big[-i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\big] \tag{10.6}$$

is the rotation matrix about an axis $\hat{\boldsymbol{n}}$ by an angle θ. For example,

$$R(\theta\hat{\boldsymbol{z}})\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}, \tag{10.7}$$

in three dimensions. Therefore, the metric tensor is the same in any frame of references that are related by rotation.

2. Show for $n=3$ that the matrix representation of the generator $\boldsymbol{J}_{ij}=(J^1_{ij},J^2_{ij},J^3_{ij})$ is

$$J^k_{ij}=-i\epsilon^{ijk}. \tag{10.8}$$

Problem 10.3 Show that the scalar product is invariant under rotation:

$$\boldsymbol{x}'\cdot\boldsymbol{y}'=\boldsymbol{x}\cdot\boldsymbol{y}, \tag{10.9}$$

where the primed vector is obtained by rotation:

$$x'^i=R^{ij}(\boldsymbol{\theta})x^j. \tag{10.10}$$

Exercise 10.4 The $(n+1)$-dimensional Minkowski space consists of a single time component

$$x^0=ct, \tag{10.11}$$

where c is the speed of light and t is the time, and n spatial components defined in the n-dimensional Euclidean space. An element x of that space is called a four-vector:

$$x=(x^0,x^1,\cdots,x^n)=(x^0,\boldsymbol{x}). \tag{10.12}$$

We use a Greek letter to represent an index for the four-vector. For example, the μth component is x^μ. In the $(n+1)$-dimensional Minkowski space, the distance $d(x,y)$ between two points $x=(x^0,\boldsymbol{x})$ and $y=(y^0,\boldsymbol{y})$ is defined by

$$d(x,y)=\sqrt{(x-y)^2}=\sqrt{(x-y)\cdot(x-y)}=\sqrt{(x^0-y^0)^2-(\boldsymbol{x}-\boldsymbol{y})^2}, \tag{10.13}$$

where the scalar product of two four-vectors x and y is

$$x\cdot y=x^0y^0-\boldsymbol{x}\cdot\boldsymbol{y}=x^0y^0-x^iy^i. \tag{10.14}$$

We introduce two ways to express components of a four-vector. One way is the ***contravariant*** form x^μ and the other way is the ***covariant*** form x_μ:

$$x^\mu = (x^0, +\boldsymbol{x}), \tag{10.15a}$$

$$x_\mu = (x^0, -\boldsymbol{x}). \tag{10.15b}$$

Note that

$$x^0 = x_0 = ct, \tag{10.16a}$$

$$x_1 = -x^1 = -x, \tag{10.16b}$$

$$x_2 = -x^2 = -y, \tag{10.16c}$$

$$x_3 = -x^3 = -z, \tag{10.16d}$$

in $3+1$ dimensions.

1. Show that the scalar product of two four-vectors x and y can be expressed as

$$x \cdot y = x_\mu y^\mu = x^\mu y_\mu. \tag{10.17}$$

2. Show that

$$x \cdot y \neq x^\mu y^\mu,\ x_\mu y_\mu. \tag{10.18}$$

   Therefore, in any pair of repeated four-vector indices, one must be covariant and the other must be contravariant.

3. Show that the scalar product $x \cdot y$ can be expressed as

$$x \cdot y = x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu}, \tag{10.19}$$

   where

$$g^{\mu\nu} = g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1,\, 2,\, 3, \cdots, n, \\ 0, & \mu \neq \nu. \end{cases} \tag{10.20}$$

4. The tensor $g^{\mu\nu}$ is called the metric tensor of the Minkowski space. Show that its matrix representation is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \cdots \\ 0 & -1 & 0 \cdots \\ 0 & 0 & -1 \\ \vdots & \vdots & & \ddots \end{pmatrix}^{\mu\nu}, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \cdots \\ 0 & -1 & 0 \cdots \\ 0 & 0 & -1 \\ \vdots & \vdots & & \ddots \end{pmatrix}_{\mu\nu}. \tag{10.21}$$

5. Show that the matrix representations of $g^{\mu}{}_{\nu}$ and $g_{\mu}{}^{\nu}$ are given by

$$g^{\mu}{}_{\nu} = g^{\mu\alpha} g_{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 \cdots \\ 0 & 1 & 0 \cdots \\ 0 & 0 & 1 \\ \vdots & \vdots & & \ddots \end{pmatrix}^{\mu}_{\nu}, \tag{10.22a}$$

$$g_{\mu}{}^{\nu} = g_{\mu\alpha} g^{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 \cdots \\ 0 & 1 & 0 \cdots \\ 0 & 0 & 1 \\ \vdots & \vdots & & \ddots \end{pmatrix}^{\nu}_{\mu}. \tag{10.22b}$$

Note that we must not use the form g_{μ}^{ν} which is ambiguous.

11. Lorentz transformation

11.1 Definition

Problem 11.1 ***Lorentz transformation*** represents the rules of the coordinate transformations of physical quantities X in an inertial reference frame S to the corresponding quantities X' in another inertial reference frame S'. If there is a physical quantity s defined in S that is invariant under Lorentz transformation, $s = s'$, then we call s a ***Lorentz scalar***. Under Lorentz transformation, a four-vector x^{ν} defined in S transforms into x'^{μ} as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}, \tag{11.1}$$

where the summation over the repeated index ν is assumed for $\mu = 0$, 1, 2, and 3. We restrict ourselves for $n+1$ Minkowski space. Because we have required that the scalar product of two four-vectors is invariant in any inertial reference frame, the scalar product must be a Lorentz scalar:

$$x' \cdot y' = x \cdot y. \tag{11.2}$$

An implicit way defining Lorentz transformation is to require the transformation matrix Λ in Eq. (14.1) to respect the invariance of the scalar product.

1. Show that the metric tensor $g^{\mu\nu}$ is invariant under Lorentz transformation:

$$g^{\mu\nu} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g^{\alpha\beta}. \tag{11.3}$$

2. ***Parity transformation*** $\mathbb{P}$ flips the sign of each spatial component and keeps the time component of a four-vector:

$$x'^{\mu} = \mathbb{P}^{\mu}{}_{\nu} x^{\nu}, \quad \mathbb{P}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{\mu}_{\;\;\nu}. \tag{11.4}$$

Show that $\mathbb{P}^2 = \mathbb{1}$ guarantees the invariance of the scalar product, where

$$\mathbb{1}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}_{\;\;\nu}. \tag{11.5}$$

Therefore, the parity transformation (11.4) is a Lorentz transformation. $\{\mathbb{1}, \mathbb{P}\}$ forms a discrete group.

3. ***Time reversal transformation*** $\mathbb{T}$ flips the sign of the time component and keeps the spatial components of a four-vector:

$$x'^{\mu} = \mathbb{T}^{\mu}{}_{\nu} x^{\nu}, \quad \mathbb{T}^{\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}_{\;\;\nu}. \tag{11.6}$$

Show that $\mathbb{T}^2 = \mathbb{1}$ guarantees the invariance of the scalar product. Therefore, the time reversal transformation (11.6) is a Lorentz transformation. $\{\mathbb{1}, \mathbb{T}\}$ forms a discrete group.

4. ***Pure rotational transformation*** $\mathbb{R}$ keeps the time component the same and transforms the spatial components according to rotation. Show that

$$\mathbb{R}^{0}{}_{0}(\theta\hat{\boldsymbol{n}}) = 1, \tag{11.7a}$$

$$\mathbb{R}^{0}{}_{i}(\theta\hat{\boldsymbol{n}}) = \mathbb{R}^{i}{}_{0}(\theta\hat{\boldsymbol{n}}) = 0, \quad i = 1,\, 2,\, 3, \tag{11.7b}$$

$$\mathbb{R}^{i}{}_{j}(\theta\hat{\boldsymbol{n}}) = \exp\big[-i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\big]^{\,i}{}_{j}, \quad i,\, j = 1,\, 2,\, 3. \tag{11.7c}$$

where

$$(J^1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (J^2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (J^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11.8}$$

Show that $\mathbb{R}^2 = \mathbb{1}$ guarantees the invariance of the scalar product. Therefore, the pure rotation is a Lorentz transformation. Show also that pure rotation forms a continuous group.

5. Show that there exist three more generators to represent complete set of Lorentz transformation of a four-vector. These generators represents Lorentz boosts.

Problem 11.2 Under a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{11.9}$$

where the summation over the repeated index ν is assumed for $\mu = 0$, 1, 2, and 3.

1. Show that

$$\frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu}. \tag{11.10}$$

2. Show that

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu}. \tag{11.11}$$

   Therefore, we write

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{11.12}$$

3. Show that

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \quad \partial_\mu[[=]] \left(\frac{1}{c}\frac{\partial}{\partial t}, +\boldsymbol{\nabla}\right). \tag{11.13}$$

11.2 Rotation Generators $\boldsymbol{J}$

Problem 11.3 We would like to find the rotation matrix $\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{n}})$ for a rotation about an axis parallel to a unit vector $\hat{\boldsymbol{n}} = (\hat{n}^1, \hat{n}^2, \hat{n}^3)$ by an angle θ. Under the transformation, the 3-dimensional position vector transforms like

$$x'^i = \mathbb{R}^i{}_j(\theta\hat{\boldsymbol{n}}) x^j. \tag{11.14}$$

1. Show that

$$\boldsymbol{x}' = \boldsymbol{x}[[\cos\theta]] + \hat{\boldsymbol{n}} \times \boldsymbol{x} \sin\theta + \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{x})(1 - \cos\theta). \tag{11.15}$$

2. By making use of the above relation, read off the matrix element $\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{n}})$ to find that

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{n}}) = \delta^{ij}\cos\theta + \hat{\boldsymbol{n}}^i\hat{\boldsymbol{n}}^j(1 - \cos\theta) + \epsilon^{ikj}\hat{\boldsymbol{n}}^k \sin\theta. \tag{11.16}$$

3. Check this relation for special cases:

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}^i\hat{\boldsymbol{x}}^j + (\hat{\boldsymbol{y}}^i\hat{\boldsymbol{y}}^j + \hat{\boldsymbol{z}}^i\hat{\boldsymbol{z}}^j)\cos\theta + (-\hat{\boldsymbol{y}}^i\hat{\boldsymbol{z}}^j + \hat{\boldsymbol{z}}^i\hat{\boldsymbol{y}}^j)\sin\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}^{i}{}_{j} \tag{11,17a}$$

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{y}}) = \hat{\boldsymbol{y}}^i\hat{\boldsymbol{y}}^j + (\hat{\boldsymbol{x}}^i\hat{\boldsymbol{x}}^j + \hat{\boldsymbol{z}}^i\hat{\boldsymbol{z}}^j)\cos\theta + (-\hat{\boldsymbol{z}}^i\hat{\boldsymbol{x}}^j + \hat{\boldsymbol{x}}^i\hat{\boldsymbol{z}}^j)\sin\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}^{i}{}_{j} \tag{11,17b}$$

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{z}}) = \hat{\boldsymbol{z}}^i\hat{\boldsymbol{z}}^j + (\hat{\boldsymbol{x}}^i\hat{\boldsymbol{x}}^j + \hat{\boldsymbol{y}}^i\hat{\boldsymbol{y}}^j)\cos\theta + (-\hat{\boldsymbol{x}}^i\hat{\boldsymbol{y}}^j + \hat{\boldsymbol{y}}^i\hat{\boldsymbol{x}}^j)\sin\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{i}{}_{j} \tag{11.17c}$$

Problem 11.4 We recall the matrix representations for the generators for the rotation.

$$(J^1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}^{\mu}{}_{\nu}, \quad (J^2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}, \quad (J^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}. \tag{11.18}$$

1. Show that

$$(J^1)^{2n\,\mu}{}_{\nu} = (J^1)^{\mu}{}_{\alpha_1}(J^1)^{\alpha_1}{}_{\alpha_2}(J^1)^{\alpha_2}{}_{\alpha_3}\cdots(J^1)^{\alpha_{2n}}{}_{\nu} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}^{\mu}{}_{\nu}, \tag{11.19}$$

$$(J^2)^{2n\,\mu}{}_{\nu} = (J^2)^{\mu}{}_{\alpha_1}(J^2)^{\alpha_1}{}_{\alpha_2}(J^2)^{\alpha_2}{}_{\alpha_3}\cdots(J^2)^{\alpha_{2n}}{}_{\nu} = \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}^{\mu}{}_{\nu}, \tag{11.20}$$

$$(J^3)^{2n\,\mu}{}_{\nu} = (J^3)^{\mu}{}_{\alpha_1}(J^3)^{\alpha_1}{}_{\alpha_2}(J^3)^{\alpha_2}{}_{\alpha_3}\cdots(J^3)^{\alpha_{2n}}{}_{\nu} = \begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{pmatrix}^{\mu}{}_{\nu}. \tag{11.21}$$

2. Show that

$$(J^1)^{2n+1\,\mu}{}_{\nu} = (J^1)^{\mu}{}_{\alpha_1}(J^1)^{\alpha_1}{}_{\alpha_2}(J^1)^{\alpha_2}{}_{\alpha_3}\cdots(J^1)^{\alpha_{2n+1}}{}_{\nu} = (J^1)^{\mu}{}_{\nu}, \tag{11.22}$$

$$(J^2)^{2n+1\,\mu}{}_{\nu} = (J^2)^{\mu}{}_{\alpha_1}(J^2)^{\alpha_1}{}_{\alpha_2}(J^2)^{\alpha_2}{}_{\alpha_3}\cdots(J^2)^{\alpha_{2n+1}}{}_{\nu} = (J^2)^{\mu}{}_{\nu}, \tag{11.23}$$

$$(J^3)^{2n+1\,\mu}{}_{\nu} = (J^3)^{\mu}{}_{\alpha_1}(J^3)^{\alpha_1}{}_{\alpha_2}(J^3)^{\alpha_2}{}_{\alpha_3}\cdots(J^3)^{\alpha_{2n+1}}{}_{\nu} = (J^3)^{\mu}{}_{\nu}. \tag{11.24}$$

3. Show that

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{x}}) = \sum_{n=0}^{\infty}\frac{(-i\theta)^n}{n!}(J^1)^{n\,i}{}_j = \begin{pmatrix} 1&0&0\\ 0&\cos\theta&-\sin\theta\\ 0&\sin\theta&\cos\theta \end{pmatrix}^{i}{}_{j}, \tag{11.25}$$

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{y}}) = \sum_{n=0}^{\infty}\frac{(-i\theta)^n}{n!}(J^2)^{n\,i}{}_j = \begin{pmatrix} \cos\theta&0&\sin\theta\\ 0&1&0\\ -\sin\theta&0&\cos\theta \end{pmatrix}^{i}{}_{j}, \tag{11.26}$$

$$\mathbb{R}^i{}_j(\theta\hat{\boldsymbol{z}}) = \sum_{n=0}^{\infty}\frac{(-i\theta)^n}{n!}(J^3)^{n\,i}{}_j = \begin{pmatrix} \cos\theta&-\sin\theta&0\\ \sin\theta&\cos\theta&0\\ 0&0&1 \end{pmatrix}^{i}{}_{j}. \tag{11.27}$$

11.3 Boost Generators $\boldsymbol{K}$

Problem 11.5 The Lorentz boost,

$$x'^{\mu} = \mathbb{B}^{\mu}{}_{\nu}(\beta\hat{\boldsymbol{x}}_1)x^{\nu}, \tag{11.28}$$

along the $\hat{\boldsymbol{x}}$ axis transforms the four-momentum of a rest particle from p to p'

$$p^{\mu} = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \longrightarrow \quad p'^{\mu} = \begin{pmatrix} E/c \\ p \\ 0 \\ 0 \end{pmatrix}, \quad E = \sqrt{(mc^2)^2 + (pc)^2}. \tag{11.29}$$

According to the special theory of relativity, the mass m' of a moving particle with velocity $v = \beta c$ increases by

$$m' = \gamma m, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{11.30}$$

where m is the rest mass. Therefore, the energy E and momentum p of that moving particle become

$$E = m'c^2 = \gamma mc^2, \quad p = m'v = m\gamma v. \tag{11.31}$$

1. Show that

$$\beta = \frac{pc}{E}, \quad \gamma = \frac{E}{mc^2}. \tag{11.32}$$

2. Show that

$$\mathbb{B}^{\mu}{}_{\nu}(\beta\hat{\boldsymbol{x}}_1) = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}{}_{\nu} \tag{11.33}$$

3. Generalizing the results to the boosts along $\hat{\boldsymbol{x}}_2$ and $\hat{\boldsymbol{x}}_3$, show that

$$\mathbb{B}^{\mu}{}_{\nu}(\beta\hat{\boldsymbol{x}}_2) = \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}{}_{\nu}, \quad \mathbb{B}^{\mu}{}_{\nu}(\beta\hat{\boldsymbol{x}}_3) = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}^{\mu}{}_{\nu} \tag{11.34}$$

4. Show that $\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_1)$, $\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_2)$, and $\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_3)$ satisfy the condition $x'\cdot y' = x\cdot y$ of the Lorentz transformation.

5. Show that $\det[\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_i)] = +1$ for $i = 1, 2, 3$.

Problem 11.6 We notice that the spatial components that are perpendicular to the axis of boost remain unchanged in the Lorentz boost. Let us consider a boost by $\boldsymbol{\beta} = \beta\hat{\boldsymbol{\beta}}$ along a unit vector in a frame, where a particle is at rest. We define

$$\boldsymbol{x}_\parallel = \frac{\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{x})}{\beta^2}, \tag{11.35a}$$

$$\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel = \boldsymbol{x} - \frac{\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{x})}{\beta^2}. \tag{11.35b}$$

1. Show that

$$x'^0 = \gamma\left(x^0 + \boldsymbol{\beta}\cdot\boldsymbol{x}\right). \tag{11.36}$$

This condition determines the first row of the boost matrix as

$$\mathbb{B}^0{}_\nu(\boldsymbol{\beta}) = \begin{pmatrix}\gamma & \gamma\beta^1 & \gamma\beta^2 & \gamma\beta^3\end{pmatrix}_\nu. \tag{11.37}$$

2. Show that

$$\begin{aligned}\boldsymbol{x}' &= \gamma\hat{\boldsymbol{\beta}}\left(\beta x^0 + \hat{\boldsymbol{\beta}}\cdot\boldsymbol{x}\right) + \boldsymbol{x} - \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{x})\\ &= \gamma\boldsymbol{\beta}x^0 + \boldsymbol{x} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\boldsymbol{x}).\end{aligned} \tag{11.38}$$

Therefore,

$$x'^i = \gamma\beta^i x^0 + x^i + \frac{\gamma-1}{\beta^2}\beta^i\beta^j x^j. \tag{11.39}$$

3. Show that the first column of the boost matrix is determined as

$$\mathbb{B}^\mu{}_0(\boldsymbol{\beta}) = \begin{pmatrix}\gamma\\ \gamma\beta^1\\ \gamma\beta^2\\ \gamma\beta^3\end{pmatrix}^\mu. \tag{11.40}$$

4. Show that the ij element of the boost matrix is determined as

$$\mathbb{B}^i{}_j(\boldsymbol{\beta}) = \delta^{ij} + \frac{\gamma-1}{\beta^2}\beta^i\beta^j = \left(\mathbb{1}_{3\times3} + \frac{\gamma-1}{\beta^2}\boldsymbol{\beta}\otimes\boldsymbol{\beta}^T\right)^i{}_j, \tag{11.41}$$

where

$$\mathbb{1}_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{pmatrix}, \quad \boldsymbol{\beta}^T = \begin{pmatrix} \beta^1 & \beta^2 & \beta^3 \end{pmatrix}, \quad (\boldsymbol{\beta} \otimes \boldsymbol{\beta}^T)^i{}_j = \beta^i \beta^j. \tag{11.42}$$

5. In summary, the boost matrix is completely determined as

$$\mathbb{B}^\mu{}_\nu(\boldsymbol{\beta}) = \begin{pmatrix} \gamma & \gamma\beta^1 & \gamma\beta^2 & \gamma\beta^3 \\ \gamma\beta^1 & 1 + (\gamma-1)(\hat{\beta}^1)^2 & (\gamma-1)\hat{\beta}^1\hat{\beta}^2 & (\gamma-1)\hat{\beta}^1\hat{\beta}^3 \\ \gamma\beta^2 & (\gamma-1)\hat{\beta}^2\hat{\beta}^1 & 1 + (\gamma-1)(\hat{\beta}^2)^2 & [[(\gamma-1)]]\hat{\beta}^2\hat{\beta}^3 \\ \gamma\beta^3 & (\gamma-1)\hat{\beta}^3\hat{\beta}^1 & (\gamma-1)\hat{\beta}^3\hat{\beta}^2 & 1 + (\gamma-1)(\hat{\beta}^3)^2 \end{pmatrix}^\mu{}_\nu, \tag{11.43}$$

where

$$\hat{\beta}^i = \frac{\beta^i}{\beta}. \tag{11.44}$$

6. Show that $\det[\mathbb{B}^\mu{}_\nu(\boldsymbol{\beta})] = +1$ for $i = 1, 2, 3$.

Problem 11.7 Let us consider $[\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_1)]$.

1. By making use of the relation,

$$\gamma^2(1-\beta^2) = 1, \tag{11.45}$$

show that $[\mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_1)]$ can be expressed of the form

$$(\mathbb{B}^1)^\mu{}_\nu(\phi) \equiv \mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_1) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu, \tag{11.46a}$$

$$(\mathbb{B}^2)^\mu{}_\nu(\phi) \equiv \mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_2) = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu, \tag{11.46b}$$

$$(\mathbb{B}^3)^\mu{}_\nu(\phi) \equiv \mathbb{B}^\mu{}_\nu(\beta\hat{\boldsymbol{x}}_3) = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}^\mu{}_\nu, \tag{11.46c}$$

where

$$\gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi. \tag{11.47}$$

2. For an infinitesimal parameter ϕ, show that each boost matrix becomes

$$(\mathbb{B}^i)^\mu{}_\nu(\phi) = \mathbb{1} - i\phi K^i + O(\phi^2), \tag{11.48}$$

where

$$(K^1)^\mu{}_\nu = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}, \quad (K^2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}, \quad (K^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}. \tag{11.49}$$

3. Show that for any finite ϕ,

$$(\mathbb{B}^i)^\mu{}_\nu(\phi) = [[\lim_{n\to\infty}]] \left(\mathbb{1} - i\frac{\phi}{n}K^i\right)^{n\,\mu}{}_{\nu} = \exp\left[-i\phi K^i\right]^\mu{}_\nu. \tag{11.50}$$

4. Show that

$$(K^1)^{2n\,\mu}{}_\nu = (K^1)^\mu{}_{\alpha_1}(K^1)^{\alpha_1}{}_{\alpha_2}(K^1)^{\alpha_2}{}_{\alpha_3}\cdots(K^1)^{\alpha_{2n}}{}_\nu = (-1)^n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}, \tag{11.51a}$$

$$(K^2)^{2n\,\mu}{}_\nu = (K^2)^\mu{}_{\alpha_1}(K^2)^{\alpha_1}{}_{\alpha_2}(K^2)^{\alpha_2}{}_{\alpha_3}\cdots(K^2)^{\alpha_{2n}}{}_\nu = (-1)^n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}{}_{\nu}, \tag{11.51b}$$

$$(K^3)^{2n\,\mu}{}_\nu = (K^3)^\mu{}_{\alpha_1}(K^3)^{\alpha_1}{}_{\alpha_2}(K^3)^{\alpha_2}{}_{\alpha_3}\cdots(K^3)^{\alpha_{2n}}{}_\nu = (-1)^n \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}{}_{\nu}. \tag{11.51c}$$

5. Show that

$$(K^1)^{2n+1\,\mu}{}_\nu = (K^1)^\mu{}_{\alpha_1}(K^1)^{\alpha_1}{}_{\alpha_2}(K^1)^{\alpha_2}{}_{\alpha_3}\cdots(K^1)^{\alpha_{2n+1}}{}_\nu = (-1)^n(K^1)^\mu{}_\nu, \tag{11.52a}$$

$$(K^2)^{2n+1\,\mu}{}_\nu = (K^2)^\mu{}_{\alpha_1}(K^2)^{\alpha_1}{}_{\alpha_2}(K^2)^{\alpha_2}{}_{\alpha_3}\cdots(K^2)^{\alpha_{2n+1}}{}_\nu = (-1)^n(K^2)^\mu{}_\nu, \tag{11.52b}$$

$$(K^3)^{2n+1\,\mu}{}_\nu = (K^3)^\mu{}_{\alpha_1}(K^3)^{\alpha_1}{}_{\alpha_2}(K^3)^{\alpha_2}{}_{\alpha_3}\cdots(K^3)^{\alpha_{2n+1}}{}_\nu = (-1)^n(K^3)^\mu{}_\nu. \tag{11.52c}$$

6. Show that

$$(\mathbb{B}^1)^\mu{}_\nu(\phi) = \sum_{n=0}^{\infty} \frac{(-i\phi)^n}{n!}(K^1)^{n\,\mu}{}_\nu = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu , \tag{11.53}$$

$$(\mathbb{B}^2)^\mu{}_\nu(\phi) = \sum_{n=0}^{\infty} \frac{(-i\phi)^n}{n!}(K^2)^{n\,\mu}{}_\nu = \begin{pmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^\mu{}_\nu , \tag{11.54}$$

$$(\mathbb{B}^3)^\mu{}_\nu(\phi) = \sum_{n=0}^{\infty} \frac{(-i\phi)^n}{n!}(K^3)^{n\,\mu}{}_\nu = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix}^\mu{}_\nu . \tag{11.55}$$

11.4 Commutation Relations for $\boldsymbol{J}$ and $\boldsymbol{K}$

Problem 11.8 Show that the Lorentz transformation with determinant +1 must be expressed as

$$\Lambda^\mu{}_\nu(\boldsymbol{\theta}, \boldsymbol{\phi}) = \exp\left[-i\theta^i J^i - i\phi^i K^i\right]^\mu{}_\nu \tag{11.56}$$

and is a group of SO(4).

Problem 11.9 Let $\boldsymbol{V}$ be an arbitray three-vector operator. Under rotation, this operator must transform like

$$\boldsymbol{V'} = [[R^{-1}(\theta\hat{\boldsymbol{n}})\boldsymbol{V}R(\theta\boldsymbol{n}) = R(\theta\hat{\boldsymbol{n}})^T\boldsymbol{V}R(\theta\boldsymbol{n}) = R(-\theta\hat{\boldsymbol{n}})\boldsymbol{V}R(\theta\boldsymbol{n}),]] \tag{11.57}$$

where

$$R(\theta\hat{\boldsymbol{n}}) = \exp\left[-i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\right]. \tag{11.58}$$

1. For an infinitesimal angle θ and an arbitrary unit vector $\hat{\boldsymbol{n}}$, show that

$$\boldsymbol{V'} = \boldsymbol{V} + \theta\hat{\boldsymbol{n}}\times\boldsymbol{V} + O(\theta^2). \tag{11.59}$$

2. For an infinitestimal transformation ($\theta\to 0$), show that

$$\boldsymbol{V'} = \boldsymbol{V}[[+]]i\theta[\hat{\boldsymbol{n}}\cdot\boldsymbol{J}, \boldsymbol{V}] + O(\theta^2). \tag{11.60}$$

3. Based on these results, show that

$$(\mathbb{1}[[+]]i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J})\boldsymbol{V}(\mathbb{1}[[-]]i\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}) = \boldsymbol{V} + \theta\hat{\boldsymbol{n}}\times\boldsymbol{V}. \tag{11.61}$$

   We can read off the coefficient of $\hat{n}^i$ to find that

$$[J^i, V^j] = i\epsilon^{ijk}V^k. \tag{11.62}$$

4. Based on the fact that the angular momentum must be a vector operator, the following commutation relation must hold.

$$[J^i, J^j] = i\epsilon^{ijk}J^k. \tag{11.63}$$

Problem 11.10 We generalize the previous result for the three-vector operator to the four-vector operator V^μ. We consider an infinitesimal rotation operator $\mathbb{R}(\theta\hat{\boldsymbol{n}})$ with $\theta\to 0$.

1. For an infinitesimal angle θ and an arbitrary unit vector $\hat{\boldsymbol{n}}$, show that

$$V' = (V'^0, \boldsymbol{V}') = (V^0, \boldsymbol{V} + \theta\hat{\boldsymbol{n}}\times\boldsymbol{V}) + O(\theta^2) = V + (0, \theta\hat{\boldsymbol{n}}\times\boldsymbol{V}) + O(\theta^2). \tag{11.64}$$

2. For an infinitestimal transformation ($\theta\to 0$), show that

$$V'^0 = V^0, \tag{11.65a}$$

$$\boldsymbol{V}' = \boldsymbol{V}[[+]]i\theta[\hat{\boldsymbol{n}}\cdot\boldsymbol{J}, \boldsymbol{V}] + O(\theta^2). \tag{11.65b}$$

3. Based on these results, show that

$$[J^i, V^0] = 0, \tag{11.66a}$$

$$[J^i, V^j] = i\epsilon^{ijk}V^k. \tag{11.66b}$$

4. We define antisymmetric tensor operator $M^{\alpha\beta}$ such that

$$M^{12} \equiv -M^{21} = J^3, \quad M^{23} \equiv -M^{32} = J^1, \quad M^{31} \equiv -M^{13} = J^2. \tag{11.67}$$

   Based on the previous results, show that

$$[M^{ij}, V^0] = 0, \tag{11.68a}$$

$$[M^{12}, V^1] = [J^3, V^1] = iV^2 = -iV^2 g^{11}, \tag{11.68b}$$

$$[M^{12}, V^2] = [J^3, V^2] = -iV^1 = iV^1 g^{22}, \tag{11.68c}$$

$$[M^{12}, V^3] = [J^3, V^3] = 0, \tag{11.68d}$$

$$[M^{12}, V^k] = i(V^1 g^{2k} - V^2 g^{1k}), \tag{11.68e}$$

$$[M^{ij}, V^k] = i(V^i g^{jk} - V^j g^{ik}). \tag{11.68f}$$

Problem 11.11 We consider an arbitrary boost by $\phi = \tanh^{-1}\beta$ along an arbitray axis.

$$\Lambda^{\mu}{}_{\nu} = \exp\left[-i\boldsymbol{\phi}\cdot\boldsymbol{K}\right]^{\mu}{}_{\nu} \tag{11.69}$$

1. For any parameter $\phi = \tanh^{-1}\beta$ and an arbitrary boost along a unit vector $\hat{\boldsymbol{n}}$, show that

$$(V'^0, \boldsymbol{V}') = [[\Lambda^{-1}(\boldsymbol{\phi})(V^0, \boldsymbol{V})\Lambda(\boldsymbol{\phi}) = \Lambda(-\boldsymbol{\phi})(V^0, \boldsymbol{V})\Lambda(\boldsymbol{\phi}).]] \tag{11.70}$$

2. For an infinitestimal transformation ($[[\phi]] \to 0$), show that four-momentum transforms like

$$V'^0 = V^0 + \boldsymbol{\beta}\cdot\boldsymbol{V} + O(\beta^2) = V^0 + \phi\hat{\boldsymbol{n}}\cdot\boldsymbol{V} + O(\phi^2), \tag{11.71a}$$

$$\boldsymbol{V}' = \boldsymbol{\beta}V^0 + \boldsymbol{V} + O(\beta^2) = \phi\hat{\boldsymbol{n}}V^0 + [[\boldsymbol{V}]] + O(\phi^2). \tag{11.71b}$$

3. Based on these results, show that

$$(\mathbb{1}[[+]]i\phi\hat{\boldsymbol{n}}\cdot\boldsymbol{K})V^0(\mathbb{1}[[-]]i\phi\hat{\boldsymbol{n}}\cdot\boldsymbol{K}) = V^0 + \phi\hat{\boldsymbol{n}}\cdot\boldsymbol{V}, \tag{11.72a}$$

$$(\mathbb{1}[[+]]i\phi\hat{\boldsymbol{n}}\cdot\boldsymbol{K})\boldsymbol{V}(\mathbb{1}[[-]]i\phi\hat{\boldsymbol{n}}\cdot\boldsymbol{K}) = \boldsymbol{V} + \phi\hat{\boldsymbol{n}}V^0. \tag{11.72b}$$

   We can read off the coefficient of $\hat{n}^i$ to find that

$$[K^i, V^0] = [[-]]iV^i, \tag{11.73a}$$

$$[K^i, V^j] = [[-]]i\delta^{ij}V^0 = [[]]iV^0 g^{ij}, \tag{11.73b}$$

4. We define antisymmetric tensor operator $M^{\alpha\beta}$ such that

$$M^{[[01]]} \equiv -M^{[[10]]} = K^1, \quad M^{[[02]]} \equiv -M^{[[20]]} = K^2, \quad M^{[[03]]} \equiv -M^{[[30]]} = K^3. \tag{11.74}$$

5. Based on these results, show that

$$[M^{i0}, V^0] = [[-]][K^i, V^0] = iV^i g^{00} = i(V^i g^{00} - V^0 g^{i0}), \tag{11.75a}$$

$$[M^{0i}, V^0] = [[]][K^i, V^0] = -iV^i g^{00} = i(V^0 g^{i0} - V^i g^{00}), \tag{11.75b}$$

$$[M^{i0}, V^j] = [[-]][K^i, V^j] = -iV^0 g^{ij} = i(V^i g^{0j} - V^0 g^{ij}), \tag{11.75c}$$

$$[M^{0i}, V^j] = [[]]][K^i, V^j] = iV^0 g^{ij} = i(V^0 g^{ij} - V^i g^{0j}). \tag{11.75d}$$

Problem 11.12 We have shown that

$$[M^{ij}, V^0] = i(V^i g^{j0} - V^j g^{i0}), \tag{11.76a}$$

$$[M^{ij}, V^k] = i(V^i g^{jk} - V^j g^{ik}), \tag{11.76b}$$

$$[M^{i0}, V^0] = i(V^i g^{00} - V^0 g^{i0}), \tag{11.76c}$$

$$[M^{0i}, V^0] = i(V^0 g^{i0} - V^i g^{00}), \tag{11.76d}$$

$$[M^{i0}, V^j] = i(V^i g^{0j} - V^0 g^{ij}), \tag{11.76e}$$

$$[M^{0i}, V^j] = i(V^0 g^{ij} - V^i g^{0j}). \tag{11.76f}$$

Show that this result is equivalent to

$$[M^{\mu\nu}, V^\alpha] = i(V^\mu g^{\nu\alpha} - V^\nu g^{\mu\alpha}). \tag{11.77}$$

Problem 11.13 We investigate the commutation relations among the 6 generators of the Lorentz transformation:

$$[P, Q]^\mu{}_\nu = P^\mu{}_\alpha Q^\alpha{}_\nu - Q^\mu{}_\alpha P^\alpha{}_\nu. \tag{11.78}$$

1. Show that

$$[J^i, J^j]^\mu{}_\nu = i\epsilon^{ijk}(J^k)^\mu{}_\nu, \tag{11.79a}$$

$$[K^i, K^j]^\mu{}_\nu = -i\epsilon^{ijk}(J^k)^\mu{}_\nu, \tag{11.79b}$$

$$[J^i, K^j]^\mu{}_\nu = i\epsilon^{ijk}(K^k)^\mu{}_\nu. \tag{11.79c}$$

2. Based on the commutation relations

$$[J^i, J^j]^\mu{}_\nu = i\epsilon^{ijk}(J^k)^\mu{}_\nu, \tag{11.80a}$$

$$[J^i, K^j]^\mu{}_\nu = i\epsilon^{ijk}(K^k)^\mu{}_\nu, \tag{11.80b}$$

   show that both $\boldsymbol{J}$ and $\boldsymbol{K}$ are vector operators.

3. Based on the commutation relation

$$[K^i, K^j]^\mu{}_\nu = -i\epsilon^{ijk}(J^k)^\mu{}_\nu \tag{11.81}$$

   confirm that a set of successsive boosts along $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ results in a rotation about the axis along $\hat{\boldsymbol{z}}$.

```
l1:=mat((0,0,0,0),(0,0,0,0),(0,0,0,-i),(0,0,i,0));
l2:=mat((0,0,0,0),(0,0,0,i),(0,0,0,0),(0,-i,0,0));
l3:=mat((0,0,0,0),(0,0,-i,0),(0,i,0,0),(0,0,0,0));
k1:=mat((0,i,0,0),(i,0,0,0),(0,0,0,0),(0,0,0,0));
k2:=mat((0,0,i,0),(0,0,0,0),(i,0,0,0),(0,0,0,0));
k3:=mat((0,0,0,i),(0,0,0,0),(0,0,0,0),(i,0,0,0));

l1*l2-l2*l1-(i*l3);
l2*l3-l3*l2-(i*l1);
l3*l1-l1*l3-(i*l2);

k1*k2-k2*k1-(-i*l3);
k2*k3-k3*k2-(-i*l1);
k3*k1-k1*k3-(-i*l2);

l1*k2-k2*l1-(i*k3);
l2*k3-k3*l2-(i*k1);
l3*k1-k1*l3-(i*k2);
matrix a1,a2,a3,b1,b2,b3;
a1:=(l1+i*k1)/2;
a2:=(l2+i*k2)/2;
a3:=(l3+i*k3)/2;

b1:=(l1-i*k1)/2;
b2:=(l2-i*k2)/2;
b3:=(l3-i*k3)/2;
a1*b1-b1*a1;
a1*b2-b2*a1;
a1*b3-b3*a1;
a2*b1-b1*a2;
a2*b2-b2*a2;
a2*b3-b3*a2;
a3*b1-b1*a3;
```

```
a3*b2-b2*a3;
a3*b3-b3*a3;

a1*a2-a2*a1-(i*a3);
a2*a3-a3*a2-(i*a1);
a3*a1-a1*a3-(i*a2);
b1*b2-b2*b1-(i*b3);
b2*b3-b3*b2-(i*b1);
b3*b1-b1*b3-(i*b2);
```

Problem 11.14

We define

$$(A^i)^\mu{}_\nu \equiv \frac{1}{2}(J^i + iK^i)^\mu{}_\nu, \tag{11.82a}$$

$$(B^i)^\mu{}_\nu \equiv \frac{1}{2}(J^i - iK^i)^\mu{}_\nu. \tag{11.82b}$$

1. Show that

$$[A^i, B^j]^\mu{}_\nu = 0, \tag{11.83a}$$

$$[A^i, A^j]^\mu{}_\nu = i\epsilon^{ijk}(A^k)^\mu{}_\nu, \tag{11.83b}$$

$$[B^i, B^j]^\mu{}_\nu = i\epsilon^{ijk}(B^k)^\mu{}_\nu. \tag{11.83c}$$

2. Show that SO(4) is equivalent to the direct product of two SU(2) groups:

$$\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2). \tag{11.84}$$

11.5 Commutation Relations for $M^{\mu\nu}$

Problem 11.15 We recall the 6 generators of the Lorentz transformation $\Lambda^\mu{}_\nu = \exp\left[-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\phi}\cdot\boldsymbol{K}\right]^\mu{}_\nu$ with $\det(\Lambda^\mu{}_\nu) = +1$:

$$(J^1)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad (J^2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad (J^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11.85a}$$

$$(K^1)^\mu{}_\nu = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K^2)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K^3)^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{11.85b}$$

The argument of the exponential function in the transformation matrix can be expressed as

$$\Lambda^\mu{}_\nu = \exp\left[-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\phi}\cdot\boldsymbol{K}\right]^\mu{}_\nu = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}M^{\alpha\beta}\right]^\mu{}_\nu, \tag{11.86}$$

where the matrices $(M^{\alpha\beta})^\mu{}_\nu$ are defined by

$$(M^{00})^\mu{}_\nu \equiv 0, \tag{11.87a}$$

$$(M^{[[0i]]})^\mu{}_\nu \equiv -(M^{[[i0]]})^\mu{}_\nu = (K^i)^\mu{}_\nu, \tag{11.87b}$$

$$(M^{ij})^\mu{}_\nu \equiv -(M^{ji})^\mu{}_\nu = \epsilon^{ijk}(J^k)^\mu{}_\nu. \tag{11.87c}$$

1. Show that

$$(J^1)^\mu{}_\nu = (M^{12})^\mu{}_\nu, \quad (J^2)^\mu{}_\nu = (M^{23})^\mu{}_\nu, \quad (J^3)^\mu{}_\nu = (M^{31})^\mu{}_\nu. \tag{11.88}$$

2. Show that

$$(K^1)^\mu{}_\nu = (M^{[[01]]})^\mu{}_\nu, \quad (K^2)^\mu{}_\nu = (M^{[[02]]})^\mu{}_\nu, \quad (K^3)^\mu{}_\nu = (M^{[[03]]})^\mu{}_\nu. \tag{11.89}$$

3. Show that

$$(\theta^1, \theta^2, \theta^3) = (\omega_{23}, \omega_{31}, \omega_{12}) = -(\omega_{32}, \omega_{13}, \omega_{21}). \tag{11.90a}$$

$$(\phi^1, \phi^2, \phi^3) = [[(\omega_{01}, \omega_{02}, \omega_{03}) = -(\omega_{10}, \omega_{20}, \omega_{30}).]] \tag{11.90b}$$

4. Show that both $(J^i)^{\mu\nu}$ and $(K^i)^{\mu\nu}$ are antisymmetric:

$$(J^i)^{\mu\nu} = -(J^i)^{\nu\mu}, \tag{11.91}$$

$$(K^i)^{\mu\nu} = -(K^i)^{\nu\mu}. \tag{11.92}$$

5. Show that the commutation relations

$$[J^i, J^j]^\mu{}_\nu = i\epsilon^{ijk}(J^k)^\mu{}_\nu \tag{11.93}$$

is equivalent to

$$[M^{23}, M^{23}]^\mu{}_\nu = [J^1, J^1]^\mu{}_\nu = 0, \tag{11.94a}$$

$$[M^{23}, M^{31}]^\mu{}_\nu = [J^1, J^2]^\mu{}_\nu = i(J^3)^\mu{}_\nu = i(M^{12})^\mu{}_\nu, \tag{11.94b}$$

$$[M^{31}, M^{23}]^\mu{}_\nu = [J^2, J^1]^\mu{}_\nu = -i(J^1)^\mu{}_\nu = -i(M^{12})^\mu{}_\nu, \tag{11.94c}$$

and similar relations that can be obtained by the replacements: $(1,2,3) \to (2,3,1)$ and $(1,2,3) \to (3,1,2)$.

6. Show that the commutation relations

$$[K^i, K^j]^\mu{}_\nu = -i\epsilon^{ijk}(J^k)^\mu{}_\nu \tag{11.95}$$

is equivalent to

$$[M^{[[01]]}, M^{[[01]]}]^\mu{}_\nu = [K^1, K^1]^\mu{}_\nu = 0, \tag{11.96a}$$

$$[M^{[[01]]}, M^{[[02]]}]^\mu{}_\nu = [K^1, K^2]^\mu{}_\nu = -i(J^3)^\mu{}_\nu = -i(M^{12})^\mu{}_\nu, \tag{11.96b}$$

$$[M^{[[02]]}, M^{[[01]]}]^\mu{}_\nu = [K^2, K^1]^\mu{}_\nu = i(J^3)^\mu{}_\nu = i(M^{12})^\mu{}_\nu, \tag{11.96c}$$

and similar relations that can be obtained by the replacements: $(1,2,3) \to (2,3,1)$ and $(1,2,3) \to (3,1,2)$.

7. Show that the commutation relations

$$[J^i, K^j]^\mu{}_\nu = i\epsilon^{ijk}(K^k)^\mu{}_\nu \tag{11.97}$$

is equivalent to

$$[M^{23}, M^{[[01]]}]^\mu{}_\nu = [J^1, K^1]^\mu{}_\nu = 0, \tag{11.98a}$$

$$[M^{23}, M^{[[02]]}]^\mu{}_\nu = [J^1, K^2]^\mu{}_\nu = i(K^3)^\mu{}_\nu = i(M^{[[03]]})^\mu{}_\nu, \tag{11.98b}$$

$$[M^{[[02]]}, M^{23}]^\mu{}_\nu = [K^2, J^1]^\mu{}_\nu = -i(K^3)^\mu{}_\nu = -i(M^{[[03]]})^\mu{}_\nu, \tag{11.98c}$$

and similar relations that can be obtained by the replacements: $(1,2,3) \to (2,3,1)$ and $(1,2,3) \to (3,1,2)$.

8. Show that the previous relations are equivalent to

$$[M^{23}, M^{23}]^\mu{}_\nu = 0, \tag{11.99a}$$

$$[M^{23}, M^{31}]^\mu{}_\nu = i(M^{12})^\mu{}_\nu = -i(M^{21})^\mu{}_\nu = i(M^{21}g^{33})^\mu{}_\nu, \tag{11.99b}$$

$$[M^{31}, M^{23}]^\mu{}_\nu = -i(M^{12})^\mu{}_\nu = i(M^{12}g^{33})^\mu{}_\nu, \tag{11.99c}$$

$$[M^{[[01]]}, M^{[[01]]}]^\mu{}_\nu = 0, \tag{11.99d}$$

$$[M^{[[01]]}, M^{[[02]]}]^\mu{}_\nu = -i(M^{12})^\mu{}_\nu = -i(M^{12}g^{00})^\mu{}_\nu, \tag{11.99e}$$

$$[M^{[[02]]}, M^{[[01]]}]^\mu{}_\nu = i(M^{12})^\mu{}_\nu = -i(M^{21})^\mu{}_\nu = -i(M^{21}g^{00})^\mu{}_\nu, \tag{11.99f}$$

$$[M^{23}, M^{[[01]]}]^\mu{}_\nu = 0, \tag{11.99g}$$

$$[M^{23}, M^{[[02]]}]^\mu{}_\nu = i(M^{[[03]]})^\mu{}_\nu = -i(M^{30}g^{22})^\mu{}_\nu, \tag{11.99h}$$

$$[M^{[[02]]}, M^{23}]^\mu{}_\nu = -i(M^{[[03]]})^\mu{}_\nu = i(M^{[[30]]})^\mu{}_\nu = -i(M^{03}g^{22})^\mu{}_\nu. \tag{11.99i}$$

9. Show that all of the above relations are completely obtained from

$$\begin{aligned}[M^{\mu\nu}, M^{\alpha\beta}]^\rho{}_\sigma &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} - M^{\nu\beta}g^{\mu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} - M^{\nu\alpha}g^{\mu\beta}\Big)\Big]^\rho{}_\sigma \\ &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} + g^{\mu\beta}M^{\nu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}M^{\nu\beta}\Big)\Big]^\rho{}_\sigma .\end{aligned} \tag{11.100a}$$

Problem 11.16 We recall that

$$(J^1)^\mu{}_\nu = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}, \quad (J^2)^\mu{}_\nu = \begin{pmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{pmatrix}, \quad (J^3)^\mu{}_\nu = \begin{pmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&i&0&0\\ 0&0&0&0 \end{pmatrix}, \tag{11.101a}$$

$$(K^1)^\mu{}_\nu = \begin{pmatrix} 0&i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad (K^2)^\mu{}_\nu = \begin{pmatrix} 0&0&i&0\\ 0&0&0&0\\ i&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad (K^3)^\mu{}_\nu = \begin{pmatrix} 0&0&0&i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{pmatrix}. \tag{11.101b}$$

1. Show that

$$(M^{23})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad (M^{31})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad (M^{12})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{11.102a}$$

$$(M^{01})^{\mu\nu} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (M^{02})^{\mu\nu} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (M^{03})^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{11.102b}$$

By definition, $(M^{\alpha\beta})^{\mu\nu}$ is antisymmetric under exchange of $\alpha \leftrightarrow \beta$. We have shown that $(M^{\alpha\beta})^{\mu\nu}$ is antisymmetric under exchange of $\mu \leftrightarrow \nu$.

2. Show that

$$(M^{01})^{\mu\nu} = i(g^{0\mu}g^{1\nu} - g^{1\mu}g^{0\nu}), \tag{11.103a}$$
$$(M^{02})^{\mu\nu} = i(g^{0\mu}g^{2\nu} - g^{2\mu}g^{0\nu}), \tag{11.103b}$$
$$(M^{03})^{\mu\nu} = i(g^{0\mu}g^{3\nu} - g^{3\mu}g^{0\nu}), \tag{11.103c}$$
$$(M^{12})^{\mu\nu} = i(g^{1\mu}g^{2\nu} - g^{2\mu}g^{1\nu}), \tag{11.103d}$$
$$(M^{23})^{\mu\nu} = i(g^{2\mu}g^{3\nu} - g^{3\mu}g^{2\nu}), \tag{11.103e}$$
$$(M^{31})^{\mu\nu} = i(g^{3\mu}g^{1\nu} - g^{1\mu}g^{3\nu}). \tag{11.103f}$$

In summary, the explicit values of the matrix elements $(M^{\alpha\beta})^{\mu}{}_{\nu}$ is given by

$$(M^{\alpha\beta})^{\mu\nu} = i(g^{\alpha\mu}g^{\beta\nu} - g^{\beta\mu}g^{\alpha\nu}), \tag{11.104a}$$
$$(M^{\alpha\beta})^{\mu}{}_{\nu} = i(g^{\alpha\mu}g^{\beta}{}_{\nu} - g^{\beta\mu}g^{\alpha}{}_{\nu}). \tag{11.104b}$$

3. By making use of the relation

$$(M^{\alpha\beta})^{\mu\nu} = i(g^{\alpha\mu}g^{\beta\nu} - g^{\beta\mu}g^{\alpha\nu}), \tag{11.105}$$

show that

$$\begin{aligned} [M^{\mu\nu}, M^{\alpha\beta}]^{\rho\sigma} &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} - M^{\nu\beta}g^{\mu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} - M^{\nu\alpha}g^{\mu\beta}\Big)\Big]^{\rho\sigma} \\ &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} + g^{\mu\beta}M^{\nu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}M^{\nu\beta}\Big)\Big]^{\rho\sigma}, \end{aligned} \tag{11.106a}$$

$$\begin{aligned} [M^{\mu\nu}, M^{\alpha\beta}]^{\rho}{}_{\sigma} &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} - M^{\nu\beta}g^{\mu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} - M^{\nu\alpha}g^{\mu\beta}\Big)\Big]^{\rho}{}_{\sigma} \\ &= i\Big[\Big(M^{\mu\beta}g^{\nu\alpha} + g^{\mu\beta}M^{\nu\alpha}\Big) - \Big(M^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}M^{\nu\beta}\Big)\Big]^{\rho}{}_{\sigma}. \end{aligned} \tag{11.106b}$$

4. The commutation relations for the generators $M^{\alpha\beta}$ of the Lorentz transformation constructs a Lie algebra:

$$[M^{\mu\nu}, M^{\alpha\beta}]^{\rho\sigma} = f^{\mu\nu\alpha\beta}{}_{\kappa\lambda}(M^{\kappa\lambda})^{\rho\sigma}. \tag{11.107}$$

Show that the structure constant is given by

$$f^{\mu\nu\alpha\beta\kappa\lambda} = \left(g^{\nu\alpha}g^{\mu\kappa}g^{\beta\lambda} - g^{\mu\alpha}g^{\nu\kappa}g^{\beta\lambda}\right) - \left(g^{\nu\beta}g^{\mu\kappa}g^{\alpha\lambda} - g^{\mu\beta}g^{\nu\kappa}g^{\alpha\lambda}\right). \tag{11.108}$$

11.6 Orbital Angular Momentum $\boldsymbol{L^{\mu\nu}}$

Problem 11.17 We generalize quantum mechanical orbital angular momentum operator $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$ in the $3+1$ Minkowski space:

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{11.109}$$

Note that $L^{\alpha\beta}$ is not a matrix for a given α and β, while $(M^{\alpha\beta})^\mu{}_\nu$ is a matrix. We define

$$L^{12} = -L^{21} = L^3, \tag{11.110a}$$

$$L^{23} = -L^{32} = L^{[[1]]}, \tag{11.110b}$$

$$L^{31} = -L^{13} = L^{[[2]]}, \tag{11.110c}$$

$$L^{[[01]]} = -L^{[[10]]} = K^1, \tag{11.110d}$$

$$L^{[[02]]} = -L^{[[20]]} = K^2, \tag{11.110e}$$

$$L^{[[03]]} = -L^{[[30]]} = K^3, \tag{11.110f}$$

1. Show that

$$[L^i, L^j] = i\epsilon^{ijk}L^k, \tag{11.111a}$$

$$[L^i, K^j] = i\epsilon^{ijk}K^k, \tag{11.111b}$$

$$[K^i, K^j] = -i\epsilon^{ijk}K^k. \tag{11.111c}$$

2. We define

$$(A^i)^\mu{}_\nu \equiv \frac{1}{2}(L^i + iK^i)^\mu{}_\nu, \tag{11.112a}$$

$$(B^i)^\mu{}_\nu \equiv \frac{1}{2}(L^i - iK^i)^\mu{}_\nu. \tag{11.112b}$$

Show that

$$[A^i, B^j]^\mu{}_\nu = 0, \tag{11.113a}$$

$$[A^i, A^j]^\mu{}_\nu = i\epsilon^{ijk}(A^k)^\mu{}_\nu, \tag{11.113b}$$

$$[B^i, B^j]^\mu{}_\nu = i\epsilon^{ijk}(B^k)^\mu{}_\nu. \tag{11.113c}$$

3. Show that

$$[x^\mu, x^\nu] = 0, \tag{11.114a}$$

$$[p^\mu, p^\nu] = 0, \tag{11.114b}$$

$$[x^\mu, p^\nu] = -ig^{\mu\nu}. \tag{11.114c}$$

4. By making use of the identity

$$[A, BC] = [A, B]C + B[A, C], \tag{11.115}$$

show that

$$[L^{\mu\nu}, p^\alpha] = i([[p^\mu]]g^{\nu\alpha} - [[p^\nu]]g^{\mu\alpha}), \tag{11.116a}$$

$$[L^{\mu\nu}, x^\alpha] = i(x^\mu g^{\nu\alpha} - x^\nu g^{\mu\alpha}). \tag{11.116b}$$

5. Show that $L^{\mu\nu}$ satisfies the commutation relation that is identical to that of $M^{\mu\nu}$:

$$\begin{aligned}[L^{\mu\nu}, L^{\alpha\beta}] &= i\Big[\Big(L^{\mu\beta}g^{\nu\alpha} - L^{\nu\beta}g^{\mu\alpha}\Big) - \Big(L^{\mu\alpha}g^{\nu\beta} - L^{\nu\alpha}g^{\mu\beta}\Big)\Big] \\ &= i\Big[\Big(L^{\mu\beta}g^{\nu\alpha} + g^{\mu\beta}L^{\nu\alpha}\Big) - \Big(L^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}L^{\nu\beta}\Big)\Big]. \end{aligned} \tag{11.117a}$$

11.7 Pauli-Lubanski operator $\boldsymbol{W^\mu}$

Problem 11.18 The Pauli-Lubanski operator W^μ is defined by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu M_{\alpha\beta}, \tag{11.118}$$

where $\epsilon_{\mu\nu\alpha\beta}$ is a completely antisymmetric tensor and conventionally $\epsilon_{0123} = -\epsilon^{0123} = 1$ and

$$M^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \tag{11.119a}$$

$$L^{\mu\nu} \equiv x^\mu p^\nu - p^\mu x^\nu = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{11.119b}$$

Here, $M^{\mu\nu}$ is the generator for the total angular momentum, $L^{\mu\nu}$ is for the orbital angular momentum, and $S^{\mu\nu}$ is for the spin angular momentum.

1. Show that the contribution of $L^{\mu\nu}$ vanishes completely:

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu S_{\alpha\beta}. \tag{11.120}$$

2. Show that $W_\mu p^\mu = 0$.

3. Show that

$$[p^\mu, W^\nu] = 0. \tag{11.121}$$

Therefore, W^μ must be invariant under translation.

4. Show that

$$\epsilon^{0ijk} = -\epsilon^{ijk}, \tag{11.122a}$$

$$\epsilon^{i0jk} = +\epsilon^{ijk}, \tag{11.122b}$$

$$\epsilon^{ij0k} = -\epsilon^{ijk}, \tag{11.122c}$$

$$\epsilon^{ijk0} = +\epsilon^{ijk}. \tag{11.122d}$$

Note that our convention is $\epsilon_{0123} = -\epsilon^{0123} = \epsilon^{123} = \epsilon_{123} = 1$.

5. Show that

$$\epsilon^{jki}\epsilon^{jk\ell} = 2\delta^{i\ell}. \tag{11.123}$$

6. Show that

$$\begin{aligned} W^0 &= \frac{1}{2}(-\epsilon^{ijk})(-p^i)(+S^{jk}) \\ &= \frac{1}{2}\epsilon^{ijk}p^i\epsilon^{jk\ell}S^\ell \\ &= \boldsymbol{p}\cdot\boldsymbol{S}, \end{aligned} \tag{11.124}$$

where the angular-momentum operator $\boldsymbol{J}$ is defined in Eq. (11.101a). This operator is proportional to the helicity operator:

$$\lambda = \hat{\boldsymbol{p}}\cdot\boldsymbol{S} = \frac{\boldsymbol{p}\cdot\boldsymbol{S}}{|\boldsymbol{p}|}. \tag{11.125}$$

7. Show that

$$\begin{aligned} W^i &= \frac{1}{2}\epsilon^{i\mu\alpha\beta}p_\mu S_{\alpha\beta} \\ &= \frac{1}{2}\left(\epsilon^{i0jk}p_0 S_{jk} + \epsilon^{ij0k}p_j S_{0k} + \epsilon^{ijk0}p_j S_{k0}\right) \\ &= \frac{1}{2}\left[\epsilon^{ijk}p^0 S^{jk} + (-\epsilon^{ijk})(-p^j)(-S^{0k}) + \epsilon^{ijk}(-p^j)(-S^{k0})\right] \\ &= (E\boldsymbol{S}[[-]]\boldsymbol{p}\times\boldsymbol{K})^i, \end{aligned} \tag{11.126}$$

where the operators $\boldsymbol{J} = \boldsymbol{S}$ and $\boldsymbol{K}$ are defined in Eqs. (11.101a) and (11.101b), respectively. Therefore, we have shown that

$$W^\mu = (\boldsymbol{p} \cdot \boldsymbol{S}, E\boldsymbol{S}[[-]]\boldsymbol{p} \times \boldsymbol{K}). \tag{11.127}$$

We observe that W^0 is a scalar and $\boldsymbol{W}$ is a three-vector under rotation.

8. Because $\boldsymbol{p}$ is a three-vector,

$$[S^i, p^j] = i\epsilon^{ijk} p^j. \tag{11.128}$$

Confirm this relation by an explicit computation.

9. Because $W^0 = \boldsymbol{p} \cdot \boldsymbol{S}$ is a scalar under rotation,

$$[\boldsymbol{S}, W^0] = 0. \tag{11.129}$$

Confirm this relation by an explicit computation:

$$[S^i, W^0] = [S^i, p^j S^j] = i\epsilon^{ijk}(p^k S^j + p^j S^k) = 0. \tag{11.130}$$

10. According to our previous calculation, $\boldsymbol{W} = E\boldsymbol{S}[[-]]\boldsymbol{p} \times \boldsymbol{K}$. Provide an argument that the following commutation must be valid:

$$[S^i, W^j] = i\epsilon^{ijk} W^k. \tag{11.131}$$

Confirm this commutation relation that states that $\boldsymbol{W}$ is a three-vector by an explicit calculation.

11. Provide an argument that the following commutation must be valid:

$$[K^i, W^0] = [[-]]iW^i, \tag{11.132a}$$

$$[K^i, W^j] = [[-]]iW^0\delta^{ij} = [[]]ig^{ij}W^0. \tag{11.132b}$$

Confirm this commutation relation by an explicit calculation.

12. Show that

$$[S^{\mu\nu}, W^\alpha] = i(W^\mu g^{\nu\alpha} - W^\nu g^{\mu\alpha}). \tag{11.133}$$

13. Show that

$$[S^{\mu\nu}, W_\alpha W^\alpha] = 0. \tag{11.134}$$

Problem 11.19 We would like to find the commutation relations for $[W^\mu, W^\nu]$. Note that we have derived

$$[P^\mu, W^\nu] = 0, \tag{11.135}$$

$$[L^i, W^0] = 0, \tag{11.136}$$

$$[L^i, W^j] = i\epsilon^{ijk}W^k, \tag{11.137}$$

$$[K^i, W^0] = iW^i, \tag{11.138}$$

$$[K^i, W^j] = iW^0\delta^{ij}. \tag{11.139}$$

1. Show that

$$[W^\mu, W^\mu] = 0, \tag{11.140}$$

where there is no sum over μ.

2. Show that

$$[AB, C] = A[B, C] + [A, C]B, \tag{11.141}$$

$$[A, BC] = [A, B]C + B[A, C] \tag{11.142}$$

3. Show that

$$\begin{aligned}
[W^0, W^i] &= [p^j L^j, W^i] \\
&= p^j[L^j, W^i] + [p^j, W^i]L^j \\
&= i\epsilon^{jik}p^j W^k = -i(\boldsymbol{p}\times\boldsymbol{W})^i = i(\boldsymbol{W}\times\boldsymbol{p})^i \\
&= i\epsilon^{0ijk}p_j W_k = i\epsilon^{0i\alpha\beta}p_\alpha W_\beta, \qquad (11.143)\\
[W^i, W^0] &= i\epsilon^{i0\alpha\beta}p_\alpha W_\beta. \qquad (11.144)
\end{aligned}$$

4. Show that

$$\begin{aligned}
[W^i, W^j] &= [EL^i + \epsilon^{i\ell m}p^\ell K^m, W^i] \\
&= E[L^i, W^j] + \epsilon^{i\ell m}p^\ell[K^m, W^j] \\
&= i\epsilon^{ijk}p^0 W^k + i\epsilon^{i\ell j}p^\ell W^0 \\
&= i(-\epsilon^{0ijk})p_0(-W_k) + i\epsilon^{0ij\ell}(-p_\ell)W_0 \\
&= i\epsilon^{ij0k}p_0 W_k + i\epsilon^{ij\ell 0}p_\ell W_0 \\
&= i\epsilon^{0i\alpha\beta}p_\alpha W_\beta. \qquad (11.145\text{a})
\end{aligned}$$

In summary, we have shown that

$$[W^\mu, W^\nu] = i\epsilon^{\mu\nu\alpha\beta} p_\alpha W_\beta. \tag{11.146}$$

12. Poincaré transformation

Problem 12.1 Lorentz transformation has 6 generators: 3 for rotations and 3 for boosts:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{12.1}$$

The transformation can further be generalized to include 4 generators that generates translational operation:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu. \tag{12.2}$$

This is called the Poincaré transformation and is represented by $\mathcal{P}(\Lambda, a)$. Show that

$$\mathcal{P}(\Lambda_2, b)\mathcal{P}(\Lambda_1, a) = \mathcal{P}(\Lambda, c), \tag{12.3}$$

where

$$\Lambda = \Lambda_2\Lambda_1, \quad c^\mu = \Lambda^\mu_{2\,\nu} a^\nu + b^\mu. \tag{12.4}$$

Therefore, the set of Poincaré transformations forms a group.

Problem 12.2 The translational operation:

$$x'^\mu = x^\mu + a^\mu \tag{12.5}$$

can be obtained by multiplying an operator

$$U(a) = \exp\left[-ia_\alpha p^\alpha\right] \equiv \exp\left[a_\alpha \partial^\alpha\right]. \tag{12.6}$$

where p^α is the relativistic quantum mechanical version of the momentum operator

$$p^\alpha = i\partial^\alpha \equiv i\frac{\partial}{\partial x_\alpha}. \tag{12.7}$$

1. Show also that the generators p^μ satisfy the following commutation relations:

$$[p^\mu, p^\nu] = 0. \tag{12.8}$$

2. Show that for an arbitrary scalar function $\phi(x)$ transforms under the translation like

$$\phi(x) \to U(a)\phi(x) = \phi(x-a). \tag{12.9}$$

Problem 12.3 The Pauli-Lubanski operator W^μ is defined by

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} p_\nu M_{\alpha\beta}. \tag{12.10}$$

where $M_{\alpha\beta}$ is the generators for the Lorentz transformation and

1. Show that W^μ is orthogonal to p^μ:

$$W_\mu p^\mu = 0. \tag{12.11}$$

2. Prove the following commutation relations:

$$[p^\mu, W^\nu] = 0, \tag{12.12a}$$

$$[M^{\mu\nu}, W^\alpha] = i(W^\mu g^{\nu\alpha} - W^\nu g^{\mu\alpha}), \tag{12.12b}$$

$$[W^\mu, W^\nu] = i\epsilon^{\mu\nu\alpha\beta} p_\alpha W_\beta. \tag{12.12c}$$

3. Show that

$$[p^\mu, W^\alpha W_\alpha] = [p^\mu, W^\alpha]W_\alpha + W^\alpha[p^\mu, W_\alpha] = 0, \tag{12.13a}$$

$$\begin{aligned}
[M^{\mu\nu}, W^\alpha W_\alpha] &= [M^{\mu\nu}, W^\alpha]W_\alpha + W_\alpha[M^{\mu\nu}, W^\alpha] \\
&= i(W^\mu g^{\nu\alpha} - W^\nu g^{\mu\alpha})W_\alpha + iW_\alpha(W^\mu g^{\nu\alpha} - W^\nu g^{\mu\alpha}), \\
&= i(W^\mu W^\nu - W^\nu W^\mu) + i(W^\nu W^\mu - W^\mu W^\nu) = 0.
\end{aligned} \tag{12.13b}$$

Therefore, $W^2 = W^\alpha W_\alpha$ is invariant under Poincaré transformation.

4. Show that

$$\epsilon^{\lambda\kappa\sigma\tau}\epsilon_{\lambda\kappa\sigma\tau} = -4! = -24. \tag{12.14}$$

5. Show that

$$\epsilon^{\lambda\kappa\sigma\mu}\epsilon_{\lambda\kappa\sigma\tau} = -3! g^\mu{}_\tau. \tag{12.15}$$

6. Show that

$$\epsilon^{\lambda\kappa\mu\nu}\epsilon_{\lambda\kappa\sigma\tau} = -2! \begin{vmatrix} g^\mu{}_\sigma & g^\mu{}_\tau \\ g^\nu{}_\sigma & g^\nu{}_\tau \end{vmatrix} = -2(g^\mu{}_\sigma g^\nu{}_\tau - g^\mu{}_\tau g^\nu{}_\sigma). \tag{12.16}$$

7. Show that

$$\epsilon^{\lambda\mu\nu\alpha}\epsilon_{\lambda\kappa\sigma\tau} = -\begin{vmatrix} g^{\mu}{}_{\kappa} & g^{\mu}{}_{\sigma} & g^{\mu}{}_{\tau} \\ g^{\nu}{}_{\kappa} & g^{\nu}{}_{\sigma} & g^{\nu}{}_{\tau} \\ g^{\alpha}{}_{\kappa} & g^{\alpha}{}_{\sigma} & g^{\alpha}{}_{\tau} \end{vmatrix}. \tag{12.17}$$

8. Show that

$$\begin{aligned} M_{\mu\nu}M^{\mu\nu} &= M_{0i}M^{0i} + M_{i0}M^{i0} + M_{ij}M^{ij} \\ &= -2M^{i0}M^{i0} + M^{ij}M^{ij} \\ &= 2(\boldsymbol{J}^2 - \boldsymbol{K}^2). \end{aligned} \tag{12.18}$$

9. Show that

$$\begin{aligned} p^i p^j J^j J^i &= p^j \left(p^i J^j \right) J^i \\ &= p^j \left(J^j p^i + i\epsilon^{ijk} p^k \right) J^i \\ &= (\boldsymbol{p} \cdot \boldsymbol{J})^2 + i\boldsymbol{p} \cdot \boldsymbol{p} \times \boldsymbol{J} \\ &= (\boldsymbol{p} \cdot \boldsymbol{J})^2. \end{aligned} \tag{12.19}$$

10. Show that in the rest frame, where $p = (mc, \mathbf{0})$, we have

$$\begin{aligned} p^\alpha p_\beta M_{\mu\alpha}M^{\mu\beta}\big|_{\text{rest}} &= p^i p_j M_{0i}M^{0j} + p^0 p_0 M_{i0}M^{i0} + p^j p_0 M_{ij}M^{i0} + p^0 p_j M_{i0}M^{ij} + p^j p_k M_{ij}M^{ik} \\ &= -p^0 p^0 M^{i0}M^{i0} \\ &= -(mc)^2 \boldsymbol{K}^2. \end{aligned} \tag{12.20}$$

11. Show that

$$
\begin{aligned}
W^2 &\equiv W^\lambda W_\lambda \\
&= -\frac{1}{4} p_\sigma M_{\mu\nu} p_\tau M_{\alpha\beta} \begin{vmatrix} g^{\sigma\tau} & g^{\sigma\alpha} & g^{\sigma\beta} \\ g^{\mu\tau} & g^{\mu\alpha} & g^{\mu\beta} \\ g^{\nu\tau} & g^{\nu\alpha} & g^{\nu\beta} \end{vmatrix} \quad \leftarrow [M_{\mu\nu}, p_\tau] = i(p_\mu g_{\nu\tau} - p_\nu g_{\mu\tau}) \\
&= -\frac{1}{4} p_\sigma \left[p_\tau M_{\mu\nu} + i(p_\mu g_{\nu\tau} - p_\nu g_{\mu\tau})\right] M_{\alpha\beta} \begin{vmatrix} g^{\sigma\tau} & g^{\sigma\alpha} & g^{\sigma\beta} \\ g^{\mu\tau} & g^{\mu\alpha} & g^{\mu\beta} \\ g^{\nu\tau} & g^{\nu\alpha} & g^{\nu\beta} \end{vmatrix} \\
&= -\frac{1}{4} p_\sigma p_\tau \begin{vmatrix} g^{\sigma\tau} & g^{\sigma\alpha} & g^{\sigma\beta} \\ g^{\mu\tau} & g^{\mu\alpha} & g^{\mu\beta} \\ g^{\nu\tau} & g^{\nu\alpha} & g^{\nu\beta} \end{vmatrix} M_{\mu\nu} M_{\alpha\beta} \\
&= -\frac{1}{4} p_\sigma p_\tau \begin{vmatrix} p^2 & p^\alpha & p^\beta \\ p^\mu & g^{\mu\alpha} & g^{\mu\beta} \\ p^\nu & g^{\nu\alpha} & g^{\nu\beta} \end{vmatrix} M_{\mu\nu} M_{\alpha\beta} \\
&= -\frac{1}{4} \Big[p^2(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) - g^{\mu\alpha} p^\nu p^\beta - g^{\nu\beta} p^\mu p^\alpha + g^{\mu\beta} p^\nu p^\alpha + g^{\nu\alpha} p^\mu p^\beta\Big] M_{\mu\nu} M_{\alpha\beta} \\
&= -\frac{1}{4} \big(2p^2 M_{\mu\nu} M^{\mu\nu} - 2p^\alpha p_\beta M_{\mu\alpha} M^{\mu\beta} + 2p^\alpha p_\beta M_{\mu\alpha} M^{\beta\mu}\big) \\
&= -\frac{1}{2} p^2 M_{\mu\nu} M^{\mu\nu} + p^\alpha p_\beta M_{\mu\alpha} M^{\mu\beta}. \qquad (12.21)
\end{aligned}
$$

```
 p.p*( - al.mu*be.nu + al.nu*be.mu)

    + al.mu*be.p*nu.p

    - al.nu*be.p*mu.p

    - al.p*be.mu*nu.p

    + al.p*be.nu*mu.p;

vector p,s,u,v,ta,al,be;

mm:=mat((s.ta,s.al,s.be),

        (u.ta,u.al,u.be),

        (v.ta,v.al,v.be));

dd:=det(mm);

index s,ta;operator m;

xx1:=p.s*p.ta*m(u,v)*m(al,be)*dd;

index u,v;

xx2:=p.s*i*(p.u*v.ta-p.v*u.ta)*dd;% zero
```

12. Show that in the rest frame, we have

$$W^2 = -(mc)^2 \boldsymbol{J}^2. \tag{12.22a}$$

Note that W^2 is invariant.

13. Show that the following operators are Casimir operators:

$$[\mathcal{P}(\Lambda, a), P^2] = 0, \tag{12.23}$$

$$[\mathcal{P}(\Lambda, a), W^2] = 0, \tag{12.24}$$

where $P^2 = P^\mu P_\mu$ and $W^2 = W^\mu W_\mu$.

```
vector al,be,mu,nu;
index al,be,mu,nu;
n4:=eps(al,be,mu,nu)*eps(al,be,mu,nu);
x4:=-24;
n3:=eps(al,be,mu,u)*eps(al,be,mu,v);
x3:=-6*u.v;
n2:=eps(al,be,u,v)*eps(al,be,x,y);
m2:=mat((u.x,u.y),
        (v.x,v.y));
x2:=-2det(m2);
y2:=-2*(u.x*v.y-u.y*v.x);

n1:=eps(al,u,v,w)*eps(al,x,y,z);
m1:=mat((u.x,u.y,u.z),
        (v.x,v.y,v.z),
        (w.x,w.y,w.z));
x1:=-det(m1);
n2-x2;
n2-y2;
n1-x1;

remind mu,nu,al,be;
mm1:=mat(( p.p, p.al, p.be),
```

```
        (mu.p,mu.al,mu.be),
        (nu.p,nu.al,nu.be));
zz1:=-det(mm1);
zz1f:= p.p*( - al.mu*be.nu + al.nu*be.mu)
    + al.mu*be.p*nu.p
    - al.nu*be.p*mu.p
    - al.p*be.mu*nu.p
    + al.p*be.nu*mu.p;
```

V. Four-Vector

13. Lorentz Covariance

13.1 Metric tensor

Exercise 13.1 In the 3-dimensional Euclidean space, the distance $d(\boldsymbol{x}, \boldsymbol{y})$ between two points $\boldsymbol{x} = (x^1, x^2, x^3)$ and $\boldsymbol{y} = (y^1, y^2, y^3)$ is defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{(\boldsymbol{x} - \boldsymbol{y})^2} = \sqrt{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}, \tag{13.1}$$

where the scalar product of the ***three-vectors*** $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\boldsymbol{x} \cdot \boldsymbol{y} = x^i y^i. \tag{13.2}$$

This can be expressed in terms of the ***metric tensor*** δ^{ij} in the 3-dimensional Euclidean space:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x^i \delta^{ij} y^j. \tag{13.3}$$

Show that the matrix representation of the metric tensor of the 3-dimensional Euclidean space is

$$\delta^{ij} = (\mathbb{1})^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13.4}$$

Problem 13.2 Show that the metric tensor δ^{ij} of the 3-dimensional Euclean space is invariant under rotation:

$$\delta^{ij} = R^{ia}(\boldsymbol{\theta}) R^{jb}(\boldsymbol{\theta}) \delta^{ab}, \tag{13.5}$$

where

$$R(\boldsymbol{\theta}) = R(\theta \hat{\boldsymbol{n}}) = \exp\left[-i\theta \hat{\boldsymbol{n}} \cdot \boldsymbol{J} \right] \tag{13.6}$$

is the rotation matrix about an axis $\hat{\boldsymbol{n}}$ by an angle θ. For example,

$$R(\theta \hat{\boldsymbol{z}}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \tag{13.7}$$

in three dimensions. Therefore, the metric tensor is the same in any frame of references that are related by rotation.

Problem 13.3 Show that the scalar product is invariant under rotation:

$$\boldsymbol{x}' \cdot \boldsymbol{y}' = \boldsymbol{x} \cdot \boldsymbol{y}, \tag{13.8}$$

where the primed vector is obtained by rotation:

$$x'^i = R^{ij}(\boldsymbol{\theta}) x^j. \tag{13.9}$$

Exercise 13.4 The $(3+1)$-dimensional Minkowski space consists of a single time component

$$x^0 = ct, \tag{13.10}$$

where c is the speed of light and t is the time, and 3 spatial components defined in the 3-dimensional Euclidean space. An element x of that space is called a ***four-vector***:

$$x = (x^0, x^1, x^2, x^3) = (x^0, \boldsymbol{x}). \tag{13.11}$$

We use a Greek letter to represent an index for the four-vector. For example, the μth component is x^μ, where $\mu = 0$, 1, 2, 3 while $i = 1$, 2, 3 for a three-vector. In the $(3+1)$-dimensional Minkowski space, the distance $d(x, y)$ between two points $x = (x^0, \boldsymbol{x})$ and $y = (y^0, \boldsymbol{y})$ is defined by

$$d(x, y) = \sqrt{(x-y)^2} = \sqrt{(x-y)\cdot(x-y)} = \sqrt{(x^0 - y^0)^2 - (\boldsymbol{x} - \boldsymbol{y})^2}, \tag{13.12}$$

where the scalar product of two four-vectors x and y is

$$x \cdot y = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y} = x^0 y^0 - x^i y^i. \tag{13.13}$$

We introduce two ways to express components of a four-vector. One way is the ***contravariant*** form x^μ and the other way is the ***covariant*** form x_μ:

$$x^\mu = (x^0, +\boldsymbol{x}), \tag{13.14a}$$

$$x_\mu = (x^0, -\boldsymbol{x}). \tag{13.14b}$$

Note that

$$x^0 = x_0 = ct, \tag{13.15a}$$

$$x_1 = -x^1 = -x, \tag{13.15b}$$

$$x_2 = -x^2 = -y, \tag{13.15c}$$

$$x_3 = -x^3 = -z, \tag{13.15d}$$

in $3+1$ dimensions.

1. Show that the scalar product of two four-vectors x and y can be expressed as

$$x \cdot y = x_\mu y^\mu = x^\mu y_\mu. \tag{13.16}$$

2. Show that

$$x \cdot y \neq x^\mu y^\mu,\ x_\mu y_\mu. \tag{13.17}$$

Therefore, in any pair of repeated four-vector indices, one must be covariant and the other must be contravariant.

3. Show that the scalar product $x \cdot y$ can be expressed as

$$x \cdot y = x^\mu y^\nu g_{\mu\nu} = x_\mu y_\nu g^{\mu\nu}, \tag{13.18}$$

where

$$g^{\mu\nu} = g_{\mu\nu} = \begin{cases} +1, & \mu = \nu = 0, \\ -1, & \mu = \nu = \quad 1,\, 2,\, 3, \\ \ \ 0, & \mu \neq \nu. \end{cases} \tag{13.19}$$

4. The tensor $g^{\mu\nu}$ is called the metric tensor of the Minkowski space. Show that its matrix representation is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{\mu\nu}, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}. \tag{13.20}$$

5. Show that the matrix representations of ${g^\mu}_\nu$ and ${g_\mu}^\nu$ are given by

$${g^\mu}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mu}_{\ \nu}, \tag{13.21a}$$

$${g_\mu}^\nu = g_{\mu\alpha} g^{\alpha\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\ \nu}_{\mu}. \tag{13.21b}$$

Note that we must not use the form g_μ^ν which is ambiguous.

14. Lorentz transformation

14.1 Definition

Problem 14.1 ***Lorentz transformation*** represents the rules of the coordinate transformations of a physical quantity f in an inertial reference frame S to the corresponding quantity f' in another inertial reference frame S'. If there is a physical quantity s defined in S that is invariant under Lorentz transformation, $s = s'$, then we call s a ***Lorentz scalar***. Under Lorentz transformation, a four-displacement x^ν defined in S transforms into x'^μ as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{14.1}$$

where the summation over the repeated index ν is assumed for $\mu = 0$, 1, 2, and 3. We restrict ourselves for the $3+1$ Minkowski space. Any physical quantity f^μ that transforms like Eq. (14.1) is a four-vector.

Because we have required that the scalar product of two four-vectors is invariant in any inertial reference frame, the scalar product must be a Lorentz scalar:

$$x' \cdot y' = x \cdot y. \tag{14.2}$$

An implicit way defining Lorentz transformation is to require the transformation matrix Λ in Eq. (14.1) to respect the invariance of the scalar product.

Show that the metric tensor $g^{\mu\nu}$ is invariant under Lorentz transformation:

$$g^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta}. \tag{14.3}$$

Problem 14.2 Let us consider the Lorentz transformation of four-displacement x^ν:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{14.4}$$

We assume that the transformation matrix Λ is independent of the position. By taking appropriate partial derivatives of Eq. (14.4), verify the following identities. Note that the identity (14.3) is particularly useful for that verification.

1. The derivative operator $\frac{\partial}{\partial x_\nu}$ transforms like a contravariant four-vector x^ν:

$$\frac{\partial}{\partial x'_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu}. \tag{14.5}$$

2. The derivative operator $\frac{\partial}{\partial x^\nu}$ transforms like a covariant four-vector x_ν:

$$\frac{\partial}{\partial x'^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu}. \tag{14.6}$$

Therefore, we are justified to write

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{14.7}$$

3. Show that

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \quad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, +\boldsymbol{\nabla}\right). \tag{14.8}$$

14.2 Four-displacement and proper time

Problem 14.3 We recall that the four-displacement $x = (ct, \boldsymbol{x})$ transforms covariantly under Lorentz transforamtion

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{14.9}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix and its square is Lorentz invariant:

$$x'^2 \equiv g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu} x^\mu x^\nu = x^2. \tag{14.10}$$

We define the ***proper time***

$$\tau \equiv \frac{\sqrt{x^2}}{c}, \tag{14.11}$$

which is a Lorentz scalar.

1. Show that in the rest frame S of a particle, the four-displacement of that particle is expressed as

$$x = (c\tau, \boldsymbol{0}). \tag{14.12}$$

2. Suppose that there is a frame S' in which that particle is moving with the constant velocity $\boldsymbol{v}$. We denote x' by the displacement of that particle in the frame S'. Show that the four-diplacement x' of the particle at time t must be

$$x' = (ct, \boldsymbol{x}), \tag{14.13}$$

where

$$\boldsymbol{x} = \boldsymbol{v}t. \tag{14.14}$$

3. Show that the invariance constraint $x'^2 = x^2$ requires

$$t = \gamma\tau, \tag{14.15}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}. \tag{14.16}$$

14.3 Four-velocity

Problem 14.4 We recall that the four-displacement $x = (ct, \boldsymbol{x})$ transforms covariantly under Lorentz transforamtion

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{14.17}$$

We can take the derivative with respect to a Lorentz scalar to keep the transformation rule the same as that of the four-displacement. If we take the derivative with respect to the proper time $\tau = \sqrt{x^2}/c$, then we find that

$$u'^\mu = \Lambda^\mu{}_\nu u^\nu, \tag{14.18}$$

where u is the ***four-velocity***,

$$u = \frac{dx}{d\tau}. \tag{14.19}$$

1. Show that in the rest frame S of a particle, the four-velocity of that particle is expressed as

$$u = (c, \mathbf{0}). \tag{14.20}$$

2. Show that in any inertial reference frame the square of the four-velocity is invariant:

$$u^2 = c^2. \tag{14.21}$$

3. Suppose that there is a frame S' in which that particle is moving with the constant velocity $\boldsymbol{v}$. We denote x' by the displacement of that particle in the frame S'. Show that the four-velocity u' of the particle is

$$u' = (\gamma c, \gamma \boldsymbol{v}). \tag{14.22}$$

4. By squaring u' explicitly, show that

$$u'^2 = c^2 = u^2. \tag{14.23}$$

14.4 Four-momentum

Problem 14.5 We recall that the four-velocity $u = (\gamma c, \gamma \boldsymbol{v})$ transforms covariantly under Lorentz transforamtion

$$u'^{\mu} = \Lambda^{\mu}{}_{\nu} u^{\nu} \tag{14.24}$$

and

$$u'^2 = c^2 = u^2. \tag{14.25}$$

We multiply the rest mass m, the mass of a particle measured when it is at rest, to the four-velocity to define the ***four-momentum***:

$$p = mu. \tag{14.26}$$

1. Show that in the rest frame S of a particle, the four-momentum of that particle is expressed as

$$p = (mc, \mathbf{0}). \tag{14.27}$$

2. Show that in any inertial reference frame the square of the four-momentum is invariant:

$$p^2 = m^2 c^2. \tag{14.28}$$

3. Suppose that there is a frame S' in which that particle is moving with the constant velocity $\boldsymbol{v}$. We denote x' by the displacement of that particle in the frame S'. Show that the four-momentum p' of the particle is

$$p' = (m\gamma c, m\gamma \boldsymbol{v}). \tag{14.29}$$

4. By squaring p' explicitly, show that

$$p'^2 = m^2 c^2 = p^2. \tag{14.30}$$

Problem 14.6 Let us interpret the expression for the four-momentum:

$$p = (m\gamma c, m\gamma \boldsymbol{v}). \tag{14.31}$$

1. Show that the ***three-momentum*** $\boldsymbol{p}$ is of the form

$$\boldsymbol{p} = m\frac{d\boldsymbol{x}}{d\tau} = m\gamma\frac{d\boldsymbol{x}}{dt} = m\gamma\boldsymbol{v}. \tag{14.32}$$

Therefore, the three-momentum is the product of $m\gamma$ and velocity $\boldsymbol{v}$. Here, the mass of a moving particle is $m\gamma$ that is greater than the rest mass m:

$$m\gamma = \frac{m}{\sqrt{1-\beta^2}}. \tag{14.33}$$

2. We can compute the force $\boldsymbol{F}$ on a particle of rest mass m. Show that

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}\left(\frac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}}\right). \tag{14.34}$$

3. We can compute the kinetic energy T by evaluating the work done on the massive particle from the instant at rest to the instant when the velocity reaches $\boldsymbol{v}$:

$$T = \int \boldsymbol{F}\cdot d\boldsymbol{x} = \int \boldsymbol{v}\cdot d\boldsymbol{p}. \tag{14.35}$$

Show that

$$T = \frac{1}{2m}\int \sqrt{1-(v/c)^2}\,d\boldsymbol{p}^2. \tag{14.36}$$

4. Show that

$$\begin{aligned} T &= \frac{mc^2}{2}\int \frac{d\left(\frac{\boldsymbol{p}^2}{m^2c^2}\right)}{\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}} \\ &= mc^2\int d\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}} \\ &= \sqrt{(mc^2)^2+(\boldsymbol{p}c)^2} - mc^2. \end{aligned} \tag{14.37}$$

5. Show that the time-component p^0 of the four-momentum is

$$p^0 = m\gamma c = \sqrt{(mc)^2+\boldsymbol{p}^2} = \frac{mc^2+T}{c}. \tag{14.38}$$

Therefore, it is natural to interpret this result as

$$p^0 = \frac{E}{c}, \tag{14.39}$$

where E is the energy of a particle:

$$E = m\gamma c^2 = mc^2 + T. \tag{14.40}$$

In addition, a particle of rest mass m has the intrinsic energy mc^2 when it is at rest. We call mc^2 the ***rest energy*** of that particle. We conclude that the four-momentum of a free particle of rest mass m is

$$p = \left(\frac{E}{c}, \boldsymbol{p}\right), \quad E = m\gamma c^2. \tag{14.41}$$

15. Four-vectors in Electrodynamics

15.1 Maxwell's equations

Problem 15.1 Let us derive Maxwell's equations of differential form from the integral form. We first consider the expressions in the MKSA unit system.

1. Gauss law for the electrostatic field $\boldsymbol{E}$ in free space is given by

$$\oint_{\partial V} \boldsymbol{E} \cdot d\boldsymbol{\sigma} = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho dV, \tag{15.1}$$

where ϵ_0 is the electric permittivity of free space, $d\boldsymbol{\sigma}$ is the differential surface element on the closed surface ∂V which is the boundary of a connected volume V. Q is the net charge contained in the volume V and ρ is the charge density at a point inside the region V.

   (a) By making use of the divergence theorem, show that the differential form of this equation is

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}. \tag{15.2}$$

   (b) Show that the electric field at a point $\boldsymbol{r}$ due to a point charge q at the origin is given by

$$\boldsymbol{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{r}}}{r^2}, \tag{15.3}$$

   where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$ and $r = |\boldsymbol{r}|$.

2. Gauss law for the magnetic field $\boldsymbol{B}$ in free space is given by

$$\oint_{\partial V} \boldsymbol{B} \cdot d\boldsymbol{\sigma} = 0, \tag{15.4}$$

where $d\boldsymbol{\sigma}$ is the differential surface element on the closed surface ∂V which is the boundary of a connected volume V.

(a) Explain why this equation implies that there is no magnetic monopole.

(b) By making use of the divergence theorem, show that the differential form of this equation is

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{15.5}$$

(c) Show that there must exist a vector field $\boldsymbol{A}$ that satisfies

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{15.6}$$

We denote $\boldsymbol{A}$ by the ***vector potential***.

(d) Show that the vector potential $\boldsymbol{A}$ that gives the magnetic field $\boldsymbol{B}$ is not unique. You can check this by computing $\boldsymbol{\nabla} \times \boldsymbol{A}'$, where

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla}\chi. \tag{15.7}$$

Here, χ is an arbitrary scalar field.

3. Faraday's law of induced electric field in free space is given by

$$\oint_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot d\boldsymbol{\sigma}, \tag{15.8}$$

where $d\boldsymbol{\ell}$ is the differential displacement on a closed curve ∂S which is the boundary of a connected surface S and $d\boldsymbol{\sigma}$ is the differential surface element on S.

(a) By making use of the Stokes' theorem, show that the differential form of this equation is

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{15.9}$$

(b) We can choose a vector potential $\boldsymbol{A}$ that satisfies

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{15.10}$$

Show that there must exist ***electrostatic potential*** ϕ which is a scalar field such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{15.11}$$

(c) Show that the transformation of the vector potential

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{\nabla}\chi, \tag{15.12}$$

requires the simultaneous transformation of the scalar potential ϕ as

$$\phi' = \phi + \frac{\partial \chi}{\partial t}. \tag{15.13}$$

Therefore, we have a freedom to choose the scalar and vector potentials ϕ and $\boldsymbol{A}$ that yield given electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, that are physical and uniquely defined. This is called the ***gauge degree of freedom*** and under the electromagnetic ***gauge transformation***,

$$(\phi, \boldsymbol{A}) \to (\phi', \boldsymbol{A}') = \Big(\phi + \frac{\partial \chi}{\partial t}, \boldsymbol{A} - \boldsymbol{\nabla}\chi\Big), \tag{15.14}$$

the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant.

4. Maxwell-Ampere's law of induced magnetic field in free space is given by

$$\oint_{\partial S} \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{\sigma} + \mu_0 \frac{d}{dt} \int_S \epsilon_0 \boldsymbol{E} \cdot d\boldsymbol{\sigma}, \tag{15.15}$$

where $d\boldsymbol{\ell}$ is the differential displacement on a closed curve ∂S which is the boundary of a connected surface S and $d\boldsymbol{\sigma}$ is the differential surface element on S. Here, μ_0 is the magnetic permeability, $\boldsymbol{J} = \rho \boldsymbol{v}$ is the electric current density and $\epsilon_0 \boldsymbol{E}$ is the electric displacement vector of free space whose time derivative is called the displacement current density.

(a) Explain the mechanism of inducing magnetic field when there is no physical flow of electric charge in space by making use of the displacement current.

(b) By making use of the Stokes' theorem, show that the differential form of this equation is

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \Big(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\Big). \tag{15.16}$$

(c) Show that the magnetic field induced by the electric current I flowing around a closed circuit C is

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}}{r^2}, \tag{15.17}$$

which is called ***Biot-Savart's law***. Here, $d\boldsymbol{\ell}$ is the differential line element of C, $\boldsymbol{r}$ is the displacement vector from the line element to the field point, and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$.

(d) Show that the Lorentz force on a charged particle in an electromagnetic field is

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{15.18}$$

15.2 Propagation of electromagnetic field

Problem 15.2 Let us consider the propagation of the electromagnetic field in free space. Thus we set $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$. Then the Maxwell's equations reduce into the form:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \tag{15.19a}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{15.19b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{15.19c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{15.19d}$$

1. Show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \times \boldsymbol{B} = -\mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{15.20a}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \tag{15.20b}$$

   which lead to

$$\left(\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) \boldsymbol{E} = \boldsymbol{0}, \tag{15.21a}$$

$$\left(\mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right) \boldsymbol{B} = \boldsymbol{0}. \tag{15.21b}$$

2. It has been experimentally confirmed that the electromagnetic fields propagate in free space with the speed $c \equiv 299\,792\,458$ m/s, which is exact. Show that

$$\mu_0 \epsilon_0 = \frac{1}{c^2}. \tag{15.22}$$

3. Show that the following set of plane waves are solutions to the wave equations:

$$\boldsymbol{E} = \hat{\boldsymbol{\epsilon}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{15.23a}$$

$$\boldsymbol{B} = \frac{\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\epsilon}}}{c}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{15.23b}$$

   where

$$\frac{\omega^2}{c^2} = \boldsymbol{k}^2, \tag{15.24}$$

   and

$$\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{k}} = 0. \tag{15.25}$$

   Therefore, the electromagnetic fields are perpendicular to the propagation so that there are two degrees of freedom in choosing $\hat{\boldsymbol{\epsilon}}$: The ***polarization vector*** $\hat{\boldsymbol{\epsilon}}$ is on the two-dimensional plane that is perpendicular to $\hat{\boldsymbol{k}}$.

15.3 Heaviside-Lorentz unit system

Problem 15.3 In particle physics, it is conventional to use the Heaviside-Lorentz unit which is most natural. In this unit system, Lorentz force, Coulomb's law, and Biot-Savart law are written in the form

$$\boldsymbol{F} = q\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\Big). \tag{15.26a}$$

$$\boldsymbol{E} = \frac{q}{4\pi}\frac{\hat{\boldsymbol{r}}}{r^2}, \tag{15.26b}$$

$$\boldsymbol{B} = \frac{I}{4\pi c}\oint \frac{d\boldsymbol{\ell} \times \hat{\boldsymbol{r}}}{r^2}, \tag{15.26c}$$

1. Show that the $\boldsymbol{E}$ and $\boldsymbol{B}$ are of the same physical dimension in the Heaviside-Lorentz unit system.

2. Show in the Heaviside-Lorentz unit system that

$$\text{charge} = \text{current} \times c \times \text{time}. \tag{15.27}$$

3. Show in the Heaviside-Lorentz unit system that the Maxwell's equations in free space are expressed as

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \tag{15.28a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{15.28b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{15.28c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\Big(\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t}\Big). \tag{15.28d}$$

4. Show in the Heaviside-Lorentz unit system that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{15.29a}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{15.29b}$$

5. Show in the Heaviside-Lorentz unit system that the electromagnetic fields are invariant under gauge transformation:

$$\phi \to \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}, \tag{15.30a}$$

$$\boldsymbol{A} \to \boldsymbol{A} - \boldsymbol{\nabla}\chi, \tag{15.30b}$$

   where χ is an arbitrary scalar field. From now on, we employ the Heaviside-Lorentz unit system instead of MKSA unit system.

Problem 15.4 Let us reconsider the propagation of the electromagnetic field in free space in the Heaviside-Lorentz unit system. Then the Maxwell's equations reduce into the form:

$$\nabla \cdot \boldsymbol{E} = 0, \tag{15.31a}$$
$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{15.31b}$$
$$\nabla \cdot \boldsymbol{B} = 0, \tag{15.31c}$$
$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{15.31d}$$

1. Show that

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \boldsymbol{B} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \tag{15.32a}$$
$$\nabla \times (\nabla \times \boldsymbol{B}) = \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \boldsymbol{E} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \tag{15.32b}$$

   which lead to

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{E} = \boldsymbol{0}, \tag{15.33a}$$
$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{B} = \boldsymbol{0}. \tag{15.33b}$$

2. Show that the following set of plane waves are solutions to the wave equations:

$$\boldsymbol{E} = \hat{\boldsymbol{\epsilon}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{15.34a}$$
$$\boldsymbol{B} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\epsilon}}\, e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{15.34b}$$

   where

$$\frac{\omega^2}{c^2} = \boldsymbol{k}^2, \tag{15.35}$$

   and

$$\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{k}} = 0. \tag{15.36}$$

15.4 Field-strength tensor

Problem 15.5 We define the ***electromagnetic field strength tensor*** $F^{\mu\nu}$ as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{15.37}$$

where A^μ is the ***electromagnetic four-vector potential***:

$$A = (\phi, \boldsymbol{A}). \tag{15.38}$$

1. Show that $A = (\phi/c, \boldsymbol{A})$ in the MKSA unit system.

2. Show that $F^{\mu\nu}$ is antisymmetric so that

$$F^{00} = F^{11} = F^{22} = F^{33}. \tag{15.39}$$

3. Show that the antisymmetricity of $F^{\mu\nu}$ requires that there are only 6 independent elements.

$$\begin{aligned} &F^{01} = -F^{10}, \quad F^{02} = -F^{20}, \quad F^{03} = -F^{30}, \\ &F^{12} = -F^{21}, \quad F^{23} = -F^{32}, \quad F^{31} = -F^{13}. \end{aligned} \tag{15.40}$$

4. By making use of the fact that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{15.41a}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{15.41b}$$

show that

$$F^{0i} = -E^i, \quad i = 1, 2, 3. \tag{15.42a}$$

$$F^{12} = -B^3, F^{23} = -B^1, F^{31} = -B^2. \tag{15.42b}$$

Therefore,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ +E^1 & 0 & -B^3 & +B^2 \\ +E^2 & +B^3 & 0 & -B^1 \\ +E^3 & -B^2 & +B^1 & 0 \end{pmatrix}^{\mu\nu}. \tag{15.43}$$

We have not proved that A is a four-vector. This can be confirmed in the next problem.

15.5 Covariant form of Maxwell's equations

Problem 15.6 We would like to show that the equation for the Lorentz force reduces into the form,

$$\frac{dp^\mu}{d\tau} = \frac{q}{c} F^{\mu\nu} u_\nu. \tag{15.44}$$

Here, τ is the proper time, u is the four-velocity, and $F^{\mu\nu}$ is the electromagnetic field strength tensor.

1. Derive the relation (15.44).

2. By making use of the fact that both p and u are four-vectors, verify that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ must be a Lorentz covariant tensor.

3. Show that A must be a four-vector.

Problem 15.7 Show that $F^{\mu\nu}$ transforms like

$$F'^{\alpha\beta} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F^{\mu\nu} \tag{15.45}$$

under Lorentz transformation.

Problem 15.8 In classical electrodynamics, dynamincs of the electromagnetic field is described in terms of the ***Lagrangian density***

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{c}J \cdot A, \tag{15.46}$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{15.47}$$

Here, A is the electromagnetic four-vector potential:

$$A = (\phi, \boldsymbol{A}), \tag{15.48}$$

and J is the ***electromagnetic four-current***:

$$J = (c\rho, \boldsymbol{J}). \tag{15.49}$$

In the Heaviside-Lorentz unit system, Maxwell's equations are written in the form:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho, \tag{15.50a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{15.50b}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{15.50c}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{c}\Big(\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t}\Big). \tag{15.50d}$$

1. Show that the ***Euler-Lagrange equation*** for the field A^ν that minimizes the action

$$S = \int d^4x\, \mathcal{L} \tag{15.51}$$

is

$$\partial^\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \right] - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0, \tag{15.52}$$

which leads to

$$\partial_\mu F^{\mu\nu} = \frac{J^\nu}{c}. \tag{15.53}$$

2. Show that $\partial_\mu F^{\mu\nu} = J^\nu/c$ is equivalent to two of the Maxwell equations:

$$\nabla \cdot \boldsymbol{E} = \rho, \tag{15.54a}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\Big(\boldsymbol{J} + \frac{\partial \boldsymbol{E}}{\partial t}\Big). \tag{15.54b}$$

3. Let us define the ***dual field strength tensor***

$$\mathcal{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{15.55}$$

Show that

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{15.56}$$

is an identity that follows directly from the definition (15.55). Show also that this equation is equivalent to the two remaining Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0, \tag{15.57a}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}. \tag{15.57b}$$

Problem 15.9 We recall that the electromagnetic fields are invariant under the gauge transformation

$$\phi \to \phi + \frac{1}{c}\frac{\partial \chi}{\partial t}, \tag{15.58a}$$

$$\boldsymbol{A} \to \boldsymbol{A} - \nabla \chi. \tag{15.58b}$$

We also have found that non-vanishing elements of the field strength tensor $F^{\mu\nu}$ are electromagnetic fields.

1. Show that the gauge transformation (15.58) is equivalent to the following covariant form:

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi. \tag{15.59}$$

2. Show that the field strength tensor $F^{\mu\nu}$ is invariant under gauge transformation:

$$F'^{\mu\nu} = \partial^\mu(A^\nu + \partial^\nu\chi) - \partial^\nu(A^\mu + \partial^\mu\chi) = F^{\mu\nu}. \tag{15.60}$$

16. Four-momentum and mass

16.1 Momentum and Mass

Exercise 16.1 Suppose $p = (p^0, p^1, p^2, p^3) = (E/c, \boldsymbol{p})$ is a four-vector. Show that p^2 is a Lorentz scalar and its value is given by

$$p^2 \equiv p_\mu p^\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2, \tag{16.1}$$

where m is the rest mass. For convenience, we set the speed of light to be unity: $c = 1$.

16.2 Invariant Mass

Problem 16.2 Let us consider the collision of two particles with momenta p_1 and p_2 with $p_i^2 = m_i^2$. The ***invariant mass*** of the two particles is defined by

$$m_{12} \equiv \sqrt{(p_1 + p_2)^2}. \tag{16.2}$$

We define $p \equiv p_1 + p_2$.

1. Show in the rest frame of p that

$$p = (\sqrt{p^2}, \mathbf{0}). \tag{16.3}$$

2. Show in any inertial reference frame that

$$p_1 \cdot p_2 = \frac{1}{2}(p^2 - p_1^2 - p_2^2) = \frac{1}{2}(m_{12}^2 - m_1^2 - m_2^2). \tag{16.4}$$

3. Show that the energy E_i^* of particle i in the rest frame of p is given by

$$E_i^* = \frac{p \cdot p_i}{\sqrt{p^2}}. \tag{16.5}$$

4. (a) Show that

$$p \cdot p_1 = p_2 \cdot p_1 + m_1^2 = \frac{1}{2}(m_{12}^2 + m_1^2 - m_2^2), \tag{16.6a}$$

$$p \cdot p_2 = p_1 \cdot p_2 + m_2^2 = \frac{1}{2}(m_{12}^2 - m_1^2 + m_2^2). \tag{16.6b}$$

Therefore,

$$E_1^* = \frac{m_{12}^2 + m_1^2 - m_2^2}{2m_{12}}, \tag{16.7a}$$

$$E_2^* = \frac{m_{12}^2 - m_1^2 + m_2^2}{2m_{12}}. \tag{16.7b}$$

(b) For $m_1 = m_2$, show that

$$E_1^* = E_2^* = \frac{1}{2}m_{12}. \tag{16.8}$$

(c) For $m_1 = m$ and $m_2 = 0$, show that

$$E_1^* = \frac{m_{12}^2 + m_1^2}{2m_{12}}, \tag{16.9a}$$

$$E_2^* = \frac{m_{12}^2 - m_1^2}{2m_{12}}. \tag{16.9b}$$

5. Show that the magnitude of the momentum $|\boldsymbol{p}_i|$ of particle i in the rest frame of p is given by

$$|\boldsymbol{p}_i| = \sqrt{E_i^2 - m_i^2} = \sqrt{\frac{(p \cdot p_i)^2 - p^2 p_i^2}{p^2}}. \tag{16.10}$$

6. It is trivial to show that $\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{0}$ in the p rest frame. Therefore, $\boldsymbol{p}_1 = -\boldsymbol{p}_2 = \boldsymbol{p}^*$ in this frame.

(a) By making use of this fact, show that

$$\sqrt{p^2} = \sqrt{m_1^2 + \boldsymbol{p}^{*2}} + \sqrt{m_2^2 + \boldsymbol{p}^{*2}}. \tag{16.11}$$

(b) Show that

$$|\boldsymbol{p}^*| = \frac{\sqrt{p^2}}{2}\sqrt{\lambda\left(1, \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)}, \tag{16.12}$$

where

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \tag{16.13}$$

7. Let us investigate the mass dependence of the formula.

(a) Show that

$$\lambda(1, a, b) = 1 + a^2 + b^2 - 2ab - 2b - 2a, \tag{16.14a}$$

$$\lambda(1, a, a) = 1 - 4a, \tag{16.14b}$$

$$\lambda(1, a, 0) = (1 - a)^2, \tag{16.14c}$$

$$\lambda(1, 0, 0) = 1. \tag{16.14d}$$

(b) For $m_1 = m_2 = m$, show that

$$\frac{2|\boldsymbol{p}^*|}{\sqrt{p^2}} = \lambda^{1/2}\left(1, \frac{m^2}{p^2}, \frac{m^2}{p^2}\right) = \sqrt{1 - \frac{4m^2}{p^2}}, \tag{16.15}$$

(c) For $m_1 = m$ and $m_2 = 0$, show that

$$\frac{2|\boldsymbol{p}^*|}{\sqrt{p^2}} = \lambda^{1/2}\left(1, \frac{m^2}{p^2}, 0\right) = 1 - \frac{m^2}{p^2}, \tag{16.16}$$

(d) For $m_1 = m_2 = 0$, show that

$$\frac{2|\boldsymbol{p}^*|}{\sqrt{p^2}} = \lambda^{1/2}(1, 0, 0) = 1. \tag{16.17}$$

Problem 16.3 Show that

$$\lambda(a^2, b^2, c^2) = \left[a^2 - (b+c)^2\right]\left[a^2 - (b-c)^2\right] = (a+b+c)(a-b-c)(a+b-c)(a-b+c). \tag{16.18}$$

16.3 2 → 2 reaction and Mandelstam variables

Problem 16.4 Let us consider the $2 \to 2$ scattering $1(p_1) + 2(p_2) \to 3(p_3) + 4(p_4)$ with

$$p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_3^2 = m_3^2, \quad p_4^2 = m_4^2. \tag{16.19}$$

We define Mandelstam variables that are invariant under Lorentz transformation:

$$s = (p_1 + p_2)^2 = (p_3 + p_4), \tag{16.20a}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4), \tag{16.20b}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3). \tag{16.20c}$$

1. Show in any inertial reference frame that

$$s = m_1^2 + 2E_1E_2 - 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + m_2^2 = m_3^2 + 2E_3E_4 - 2\boldsymbol{p}_3 \cdot \boldsymbol{p}_4 + m_4^2, \tag{16.21a}$$

$$t = m_1^2 - 2E_1E_3 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_3 + m_3^2 = m_2^2 - 2E_2E_4 + 2\boldsymbol{p}_2 \cdot \boldsymbol{p}_4 + m_4^2, \tag{16.21b}$$

$$u = m_1^2 - 2E_1E_4 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_4 + m_4^2 = m_2^2 - 2E_2E_3 + 2\boldsymbol{p}_2 \cdot \boldsymbol{p}_3 + m_3^2, \tag{16.21c}$$

where $p_i = (E_i, \boldsymbol{p}_i)$.

2. Show that

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{16.22}$$

17. Phase space

17.1 Two-body phase space

Problem 17.1 Let us consider the decay of a particle A into two massive particles 1 and 2:

$$A(p) \to 1(p_1) + 2(p_2), \tag{17.1}$$

where p, p_1, and p_2 are the momenta for A, 1, and 2, respectively. We assume that the particles are ***on their mass shells***:

$$p^2 = M^2, \quad p_1^2 = m_1^2, \quad p_2^2 = m_2^2. \tag{17.2}$$

1. The conservation of energy and momentum is equivalent to the ***four-momentum conservation***:

$$p = p_1 + p_2. \tag{17.3}$$

Show that the following factor

$$\int d^4x \, e^{-i(p-p_1-p_2)\cdot x} = (2\pi)^4 \delta^{(4)}(p - p_1 - p_2) \tag{17.4}$$

guarantees the four-momentum conservation and is invariant under Lorentz transformation. Here,

$$\delta^{(4)}(p - p_1 - p_2) = \delta(p^0 - p_1^0 - p_2^0)\delta(p^1 - p_1^1 - p_2^1)\delta(p^2 - p_1^2 - p_2^2)\delta(p^3 - p_1^3 - p_2^3). \tag{17.5}$$

2. Show that the following factor

$$\int dp_i^0 \theta(p_i^0) \delta(p_i^2 - m_i^2) \tag{17.6}$$

guarantees that i is on its mass shell and the expression is invariant under Lorentz transformation. Explain the role of the factor $\theta(p_i^0)$.

3. The ***phase space*** of the two-body final state $d\Phi_2(p \to p_1 + p_2)$ is defined by the product of phase-space elements $d^3p_1/(2\pi)^3$ and $d^3p_2/(2\pi)^3$ for the two final-state particles multiplied by the four-momentum conservation factor and the on-shell condition factor:

$$d\Phi_2(p \to p_1 + p_2) = (2\pi)^4 \delta^{(4)}(p - p_1 - p_2) \frac{d^4p_1}{(2\pi)^3} \theta(p_1^0) \delta(p_1^2 - m_1^2) \frac{d^4p_2}{(2\pi)^3} \theta(p_2^0) \delta(p_2^2 - m_2^2). \tag{17.7}$$

Show that this expression is valid in any inertial reference frame because the phase space is invariant under Lorentz transformation.

Problem 17.2 Let us continue to consider the two-body phase space (17.7).

1. By integrating over p_1^0 and p_2^0, show that

$$\begin{aligned} d\Phi_2(p \to p_1 + p_2) &= (2\pi)^4\delta^{(4)}(p - p_1 - p_2)\frac{d^3p_1}{(2\pi)^3 2\sqrt{m_2^2 + \boldsymbol{p}_1^2}}\frac{d^3p_2}{(2\pi)^3 2\sqrt{m_2^2 + \boldsymbol{p}_2^2}} \\ &= \frac{1}{(2\pi)^2}\delta\left[p^0 - \sqrt{m_1^2 + \boldsymbol{p}_1^2} - \sqrt{m_2^2 + (\boldsymbol{p} - \boldsymbol{p}_1)^2}\right] \\ &\quad \times \frac{d^3p_1}{4\sqrt{m_1^2 + \boldsymbol{p}_1^2}\sqrt{m_2^2 + (\boldsymbol{p} - \boldsymbol{p}_1)^2}}, \end{aligned} \tag{17.8}$$

   where

$$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2. \tag{17.9}$$

2. Let us choose the p rest frame, where $p^0 = \sqrt{p^2}$, $\boldsymbol{p} = \boldsymbol{0}$, and $\boldsymbol{p}_2 = -\boldsymbol{p}_1$. Show that

$$\begin{aligned} d\Phi_2(p \to p_1 + p_2)\Big|_{p\ \text{rest}} &= \frac{1}{(2\pi)^2}\delta\left[p^0 - \sqrt{m_1^2 + \boldsymbol{p}_1^2} - \sqrt{m_2^2 + \boldsymbol{p}_1^2}\right]\frac{d^3p_1}{4\sqrt{m_1^2 + \boldsymbol{p}_1^2}\sqrt{m_2^2 + \boldsymbol{p}_1^2}} \\ &= \frac{1}{(2\pi)^2}\left[\frac{|\boldsymbol{p}_1|}{\sqrt{m_1^2 + \boldsymbol{p}_1^2}} + \frac{|\boldsymbol{p}_1|}{\sqrt{m_2^2 + \boldsymbol{p}_1^2}}\right]^{-1}\frac{|\boldsymbol{p}_1|^2 d\Omega}{4\sqrt{m_1^2 + \boldsymbol{p}_1^2}\sqrt{m_2^2 + \boldsymbol{p}_1^2}} \\ &= \frac{1}{(2\pi)^2}\frac{\sqrt{m_1^2 + \boldsymbol{p}_1^2}\sqrt{m_2^2 + \boldsymbol{p}_1^2}}{|\boldsymbol{p}_1|\left(\sqrt{m_1^2 + \boldsymbol{p}_1^2} + \sqrt{m_2^2 + \boldsymbol{p}_1^2}\right)}\frac{|\boldsymbol{p}_1|^2 d\Omega}{4\sqrt{m_1^2 + \boldsymbol{p}_1^2}\sqrt{m_2^2 + \boldsymbol{p}_1^2}} \\ &= \frac{1}{8\pi}\frac{2|\boldsymbol{p}_1|}{\sqrt{p^2}}\frac{d\Omega}{4\pi} \\ &= \frac{1}{8\pi}\sqrt{\lambda\left(1, \frac{m_1^2}{p^2}, \frac{m_2^2}{p^2}\right)}\frac{d\phi}{2\pi}\frac{d\cos\theta}{2}, \end{aligned} \tag{17.10}$$

   where θ and ϕ are the polar and azimuthal angles of particle 1 in the p rest frame.

3. For $m_1 = m_2 = m$, show that

$$d\Phi_2(p \to p_1 + p_2)\Big|_{p\ \text{rest}} = \frac{1}{8\pi}\sqrt{1 - \frac{4m^2}{p^2}}\frac{d\phi}{2\pi}\frac{d\cos\theta}{2}. \tag{17.11}$$

4. For $m_1 = m$ and $m_2 = 0$, show that

$$d\Phi_2(p \to p_1 + p_2)\Big|_{p\ \text{rest}} = \frac{1}{8\pi}\left(1 - \frac{m^2}{p^2}\right)\frac{d\phi}{2\pi}\frac{d\cos\theta}{2}. \tag{17.12}$$

5. For $m_1 = m_2 = 0$, show that

$$d\Phi_2(p \to p_1 + p_2)\Big|_{p\ \text{rest}} = \frac{1}{8\pi}\frac{d\phi}{2\pi}\frac{d\cos\theta}{2}. \tag{17.13}$$

Problem 17.3 We can define one-body phase space:

$$d\Phi_1(p \to p_1) = (2\pi)^4 \delta^{(4)}(p - p_1) \frac{d^4 p_1}{(2\pi)^3} \theta(p_1^0) \delta(p_1^2 - m_1^2). \tag{17.14}$$

Show that

$$d\Phi_1(p \to p_1) = 2\pi \delta(p^2 - m_1^2). \tag{17.15}$$

Here, we do not need $\theta(p^0)$ because $p^0 \geq 0$ is manifest since p is the initial-state momentum that is physical.

17.2 Three-body phase space

We consider a three-body decay $A(p) \to 1(p_1) + 2(p_2) + 3(p_3)$.

Problem 17.4 The phase space of the three-body final state is defined by

$$\begin{aligned} d\Phi_3(p \to p_1 + p_2 + p_3) \;=\; & (2\pi)^4 \delta^{(4)}(p - p_1 - p_2 - p_3) \frac{d^4 p_1}{(2\pi)^3} \theta(p_1^0) \delta(p_1^2 - m_1^2) \\ & \times \frac{d^4 p_2}{(2\pi)^3} \theta(p_2^0) \delta(p_2^2 - m_2^2) \frac{d^4 p_3}{(2\pi)^3} \theta(p_3^0) \delta(p_3^2 - m_3^2). \end{aligned} \tag{17.16}$$

1. Show that

$$1 = \int d^4 p_{12} \delta^{(4)}(p_{12} - p_1 - p_2) \int dm_{12}^2 \theta(p_{12}^0) \delta(p_{12}^2 - m_{12}^2). \tag{17.17}$$

2. Show that the three-body phase space can be expressed as

$$\begin{aligned} & d\Phi_3(p \to p_1 + p_2 + p_3) \\ & = \frac{1}{2\pi} \int dm_{12}^2 (2\pi)^4 \delta^{(4)}(p_{12} - p_1 - p_2) \\ & \quad \times \frac{d^4 p_1}{(2\pi)^3} \theta(p_1^0) \delta(p_1^2 - m_1^2) \frac{d^4 p_2}{(2\pi)^3} \theta(p_2^0) \delta(p_2^2 - m_2^2) \\ & \quad \times (2\pi)^4 \delta^{(4)}(p - p_{12} - p_3) \frac{d^4 p}{(2\pi)^3} \theta(p^0) \delta(p_{12}^2 - m_{12}^2) \frac{d^4 k}{(2\pi)^3} \theta(p_3^0) \delta(p_3^2 - m_3^2). \end{aligned} \tag{17.18}$$

3. Show that

$$d\Phi_3(p \to p_1 + p_2 + p_3) = \frac{1}{2\pi} \int dm_{12}^2 d\Phi_2(p \to p_{12} + p_3) d\Phi_2(p_{12} \to p_1 + p_2). \tag{17.19}$$

4. Show that the physical range of the invariant mass m_{12} is

$$m_1 + m_2 \leq m_{12} \leq \sqrt{p^2} - m_3. \tag{17.20}$$

We can generalize the result for the three-body phase space to the phase space calculation for an n-body system.

Problem 17.5 We consider a four-body decay $A(p) \to 1(p_1) + 2(p_2) + 3(p_3) + 4(p_4)$. If $1+2$ and $3+4$ are decay products of X and Y, respectively, then it is convenient to break the phase space into the following form:

$$d\Phi_4(p \to p_1+p_2+p_3+p_4) \propto dX^2 dY^2 d_2\Phi(p \to X+Y) d_2\Phi(X \to p_1+p_2) d_2\Phi(Y \to p_3+p_4). \quad (17.21)$$

For convenience, we choose the rest frame of A.

1. Show that

$$d\Phi_4(p \to p_1 + p_2 + p_3 + p_4) = \frac{dX^2 dY^2}{(2\pi)^8} \frac{|\boldsymbol{X}| d\Omega_X^*}{4\sqrt{p^2}} \frac{|\boldsymbol{p}_1^*| d\Omega_1^*}{4\sqrt{X^2}} \frac{|\boldsymbol{p}_3^*| d\Omega_3^*}{4\sqrt{Y^2}}, \quad (17.22)$$

where $\boldsymbol{X}^*$ and $d\Omega_X^* = d\phi_X^* d\cos\theta_X^*$ are the three-momentum and the solid angle of $\boldsymbol{X}$ in the A rest frame, respectively, $\boldsymbol{p}_1^*$ and $d\Omega_1^* = d\phi_1^* d\cos\theta_1^*$ are the three-momentum and the solid angle of 1 in the $X = p_1 + p_2$ rest frame, respectively, $\boldsymbol{p}_3^*$ and $d\Omega_3^* = d\phi_3^* d\cos\theta_3^*$ are the three-momentum and the solid angle of 3 in the $Y = p_3 + p_4$ rest frame, respectively.

2. Show that the physical ranges of the integration variables are given by

$$m_1 + m_2 \le \sqrt{X^2} \le \sqrt{p^2} - (m_3 + m_4), \quad (17.23a)$$

$$m_3 + m_4 \le \sqrt{Y^2} \le \sqrt{p^2} - \sqrt{X^2}, \quad (17.23b)$$

$$0 \le \theta_i^* \le \pi, \quad (17.23c)$$

$$0 \le \phi_i^* \le 2\pi, \quad (17.23d)$$

for $i = X$, 1, and 3.

Problem 17.6 Let us consider the three-body decay $A(p) \to 1(p_1) + 2(p_2) + 3(p_3)$, where

$$p^2 = M^2, \quad p_1^2 = m_1^2, \quad p_2^2 = m_2^2, \quad p_3^2 = m_3^2. \quad (17.24)$$

We define

$$p_{ij} = p_i + p_j, \quad m_{ij} = \sqrt{p_{ij}^2}. \quad (17.25)$$

We recall that the three-body phase space reduces into the form

$$d\Phi_3(p \to p_1 + p_2 + p_3) = \frac{1}{2\pi} \int dm_{12}^2 d\Phi_2(p \to p_{12} + p_3) d\Phi_2(p_{12} \to p_1 + p_2). \quad (17.26)$$

1. Show that

$$m_{12}^2 + m_{23}^2 + m_{31}^2 = M^2 + m_1^2 + m_2^2 + m_3^2. \tag{17.27}$$

2. Show that

$$m_{12}^2 = (p - p_3)^2 = M^2 + m_3^2 - 2ME_3, \tag{17.28}$$

where E_3 is the energy of the particle 3 in the A rest frame.

3. Show that

$$d\Phi_3(p \to p_1 + p_2 + p_3) = \frac{dm_{12}|\boldsymbol{p}_1^*||\boldsymbol{p}_3|d\Omega_1^* d\Omega_3}{8M(2\pi)^5}, \tag{17.29}$$

where $\boldsymbol{p}_1^*$ and Ω_1^* are the three-momentum and its direction of particle 1 in the p_{12} rest frame. $\boldsymbol{p}_3$ and Ω_3 are the three-momentum and its direction of particle 3 in the A rest frame.

4. Show that

$$|\boldsymbol{p}_1^*| = \frac{\sqrt{[m_{12}^2 - (m_1 + m_2)^2][m_{12}^2 - (m_1 - m_2)^2]}}{2m_{12}}, \tag{17.30a}$$

$$|\boldsymbol{p}_3| = \frac{\sqrt{[M^2 - (m_{12} + m_3)^2][M^2 - (m_{12} - m_3)^2]}}{2M}. \tag{17.30b}$$