

# **QCD PRACTICE**

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# Contents

<b>1</b>	<b>QED Review</b>	<b>1</b>
1.1	Maxwell's equations . . . . .	1
1.2	Gauge . . . . .	3
1.3	Lagrangian . . . . .	5
1.4	Classical charged-particle Lagrangian . . . . .	7
1.5	Gauge Invariance and Covariant Derivative . . . . .	10
1.6	Scalar particle . . . . .	11
1.7	Time-dependent perturbation theory . . . . .	13
1.8	Propagator . . . . .	14
1.9	Photon propagator . . . . .	16
1.10	Feynman rules . . . . .	19
1.11	Dirac equation . . . . .	21
1.12	Spinor . . . . .	25
1.13	Fermion propagator . . . . .	28
<b>2</b>	<b>QCD Lagrangian</b>	<b>31</b>
2.1	QED Summary . . . . .	31
2.2	QCD Lagrangian . . . . .	32
2.3	Gauge Fixing in QED . . . . .	34
2.4	Gauge-Fixing and Ghost Terms in QCD . . . . .	35
2.5	$SU(N_c)$ algebra summary . . . . .	36
2.6	QCD Feynman rules . . . . .	36
2.6.1	Gluon Vertices . . . . .	37
2.6.2	Ghost Vertices . . . . .	38
<b>3</b>	<b><math>SU(N)</math></b>	<b>39</b>
3.1	Generators and structure constants . . . . .	39
3.2	Derivation of completeness relation . . . . .	41
3.3	Useful trace formulas . . . . .	42
3.4	Adjoint representation . . . . .	44
3.5	$SU(3)$ Clebsch-Gordan coefficients . . . . .	46
3.6	Examples . . . . .	48

<b>4 Tree-Level Calculation</b>	<b>49</b>
4.1 Cross section formula Summary . . . . .	49
4.2 $e^+e^- \rightarrow \mu^+\mu^-$ . . . . .	50
4.3 $e^+e^- \rightarrow q + \bar{q}$ . . . . .	54
4.4 $q\bar{q} \rightarrow g^* \rightarrow q' + \bar{q}'$ . . . . .	54
<b>5 Higher Order Correction</b>	<b>55</b>
5.1 Fermion Self Energy . . . . .	55
5.2 Gluon Self Energy . . . . .	58
5.3 Scalar Integrals . . . . .	59
5.4 Vertex Correction . . . . .	60
5.5 Gloun Self Energy . . . . .	61
5.5.1 Scalar Integrals-Massless fermion loop . . . . .	61
5.5.2 Scalar Integrals-Massive fermion loop . . . . .	63
5.5.3 Sign Convention of $\Pi$ . . . . .	64
5.5.4 Fermion Loop . . . . .	64
5.5.5 Ghost Loop . . . . .	65
5.5.6 Gluon Loop . . . . .	65
5.5.7 Gluon + Ghost . . . . .	65
5.5.8 Gluon Self Energy . . . . .	66
5.6 $Z_Q$ Contribution . . . . .	67
5.7 $Z_g$ Contribution . . . . .	67
5.8 1-gluon Exchange Vertex Contribution $\lambda_1$ . . . . .	68
5.9 tri-gluon Vertex Contribution $\lambda_2$ . . . . .	69
5.10 Sum of vertex corrections . . . . .	69
5.11 Charge renoramlization . . . . .	69
5.12 UV/IR-finite final result . . . . .	70
<b>6 Dimensional Regularization</b>	<b>73</b>
6.1 Minkowski space volume integral . . . . .	73
6.2 Wick Rotation . . . . .	74
6.3 Useful Integrals in the Eucledian Space . . . . .	77
6.4 Feynman Parametrization . . . . .	77
6.5 $\Gamma$ function characteristics . . . . .	78
<b>7 One Loop Recurrence Relations</b>	<b>79</b>
7.1 Basic Scalar Integral Evaluation . . . . .	79
7.2 Feynman Trick . . . . .	80
7.3 primitive recurrence relation . . . . .	80
7.4 Integration by parts . . . . .	81
7.5 $I_n$ for non-integer $n$ . . . . .	83

# Chapter 1

## QED Review

### 1.1 Maxwell's equations

1. Let us define covariant( $x_\mu$ ) and contravariant ( $x^\mu$ ) vectors as

$$x^\mu \equiv (t, +\mathbf{x}), \quad x_\mu \equiv (t, -\mathbf{x}). \quad (1.1)$$

2. Show that

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, +\nabla \right), \quad \partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\nabla \right) \quad (1.2a)$$

3. Show that  $\partial_\mu(\partial^\mu)$  transforms like  $x_\mu(x^\mu)$ .

4. Show

$$\partial^2 = \frac{\partial^2}{\partial t^2} - \nabla^2. \quad (1.3)$$

5. Show

$$\partial \cdot J = \frac{\partial}{\partial t} J^0 + \nabla \cdot \mathbf{J}. \quad (1.4)$$

6. Using the potential  $A^\mu = (\phi, \mathbf{A})$ , express the electromagnetic fields

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad (1.5a)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (1.5b)$$

7. Show that

$$E^i = -\frac{\partial}{\partial x^0} A^i - \frac{\partial}{\partial x^i} A^0 = -\partial^0 A^i + \partial^i A^0, \quad (1.6a)$$

$$B^i = \epsilon^{ijk} \frac{\partial}{\partial x^j} A^k = -\epsilon^{ijk} \partial^j A^k. \quad (1.6b)$$

8. Defining the **field-strength tensor**

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (1.7)$$

show that

$$\begin{aligned} E^i &= \partial^i A^0 - \partial^0 A^i = -F^{0i}, \\ B^i &= -\frac{1}{2}\epsilon^{ijk}F^{jk}. \end{aligned}$$

9. Express the field strength tensor in a matrix form as

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad (1.8a)$$

$$F_{\mu\nu} = g_{\mu\alpha}F^{\alpha\beta}g_{\beta\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix}. \quad (1.8b)$$

10. Show that

$$F_{\mu\nu} = F^{\mu\nu}(\mathbf{E} \rightarrow -\mathbf{E}). \quad (1.9)$$

11. Let us define the **dual field-strength tensor**  $\mathcal{F}^{\mu\nu}$  as

$$\mathcal{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \quad (1.10)$$

where  $\epsilon^{\mu\nu\alpha\beta}$  is a totally antisymmetric tensor and  $\epsilon_{0123} = -\epsilon^{0123} = +1$ .

12. Show that

$$\mathcal{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} = F^{\mu\nu}(\mathbf{E} \rightarrow \mathbf{B}, \mathbf{B} \rightarrow -\mathbf{E}). \quad (1.11)$$

13. Show that

$$\nabla \cdot \mathbf{V} = \partial_i V^i, \quad \left( \frac{\partial}{\partial t} \mathbf{V} \right)^i = \partial^0 V^i \quad (1.12a)$$

$$(\nabla \times \mathbf{V})^i = \epsilon^{ijk}\partial_j V^k = -\epsilon^{ijk}\partial^j V^k \quad (1.12b)$$

$$\epsilon^{ijk}\partial^i\partial^j V^k = 0 \quad (1.12c)$$

$$\epsilon^{ijk}\partial^i F^{jk} = 0 \quad (1.12d)$$

$$(\nabla \times \mathbf{E})^i = \frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk}, \quad \frac{\partial B^i}{\partial t} = -\frac{1}{2}\epsilon^{ijk}\partial^0 F^{jk} \quad (1.12e)$$

$$(\nabla \times \mathbf{B})^i = \partial_j F^{ji}, \quad -\frac{\partial E^i}{\partial t} = \partial_0 F^{0i} \quad (1.12f)$$

14. Show that

$$\nabla \cdot \mathbf{E} = \rho \rightarrow \partial_0 F^{00} + \partial_i F^{i0} = J^0 \quad (1.13a)$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} \rightarrow \partial_0 F^{0i} + \partial_j F^{ji} = J^i \quad (1.13b)$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{1}{2} \epsilon^{ijk} \partial^i F^{jk} \equiv 0 : \partial_i \mathcal{F}^{i0} = 0 \quad (1.13c)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \rightarrow \frac{1}{2} \epsilon^{ijk} \partial^0 F^{jk} - \frac{1}{2} \epsilon^{ijk} \partial^0 F^{jk} = 0 \quad (1.13d)$$

$$: \partial_0 \mathcal{F}^{0i} + \partial_j \mathcal{F}^{ji} = 0 \quad (1.13e)$$

15. You have shown that Maxwell's equations reduce into the form

$$\partial_\mu F^{\mu\nu} = J^\nu, \quad \partial_\mu \mathcal{F}^{\mu\nu} = 0 \quad (1.14)$$

## 1.2 Gauge

16. Show that  $\partial^\mu \partial^\nu \chi - \partial^\nu \partial^\mu \chi = 0$ , where  $\chi$  is a Lorentz scalar function.

17. Show that  $F^{\mu\nu}$  is invariant under the gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi, \quad (1.15)$$

and the electromagnetic fields  $\mathbf{E}$  and  $\mathbf{B}$  are also gauge invariant.

18. Show that the Maxwell's equations are gauge invariant.

19. Let us use the Lorentz gauge  $\partial \cdot A = 0$ . Show that the Maxwell's equations reduce into

$$\partial^2 A^\mu = J^\mu. \quad (1.16)$$

20. We can make another gauge transformation

$$A^\mu \rightarrow A^\mu + \partial^\mu \Lambda. \quad (1.17)$$

Show that  $\Lambda$  satisfies the Lorentz condition  $\partial^2 \Lambda = 0$ .

21. Show that

$$A^\mu = \epsilon^\mu(\mathbf{p}) e^{-ip \cdot x}. \quad (1.18)$$

is a solution to the free photon ( $J^\mu = 0$ ) equation with

$$\mathbf{p}^2 = 0, \quad \epsilon \cdot \mathbf{p} = 0. \quad (1.19)$$

22. Choosing  $\Lambda = iae^{-ip \cdot x}$ , where  $a$  is a scalar, show that the gauge invariance condition ensures that the transformation

$$\epsilon^\mu \rightarrow \epsilon'^\mu = \epsilon^\mu + ap^\mu \quad (1.20)$$

does not change physical results.

23. If we choose  $a$  such that  $\epsilon'^0 = 0$ , show that

$$\boldsymbol{\epsilon} \cdot \mathbf{p} = 0 \quad (1.21)$$

and it is equivalent to the Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ .

24. Show that the transverse condition  $\epsilon \cdot p = 0$  and Coulomb gauge condition  $\boldsymbol{\epsilon} \cdot \mathbf{p} = 0$  restricts the photon to have only two degrees of freedom

$$\epsilon^\mu = (0, 1, 0, 0), \quad (0, 0, 1, 0) \quad (1.22)$$

if  $p^\mu = (E, 0, 0, E)$ .

25. We have shown that there are only two independent parameters describing the polarization vector  $\epsilon^\mu$  of the photon. There are only two polarization(spin) states for a massless spin-1 particle.

26. Show that spin-1 wavefunction is expressed in terms of spherical harmonics for  $|\ell = 1, \ell_z\rangle$  states as

$$Y_1^\pm(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} = \sqrt{\frac{3}{4\pi}} \hat{\mathbf{r}} \cdot \boldsymbol{\epsilon}(\pm), \quad (1.23a)$$

$$Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \hat{\mathbf{r}} \cdot \boldsymbol{\epsilon}(0). \quad (1.23b)$$

where  $\hat{\mathbf{r}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi)$ .

27. We can express the spherical harmonics in Cartesian coordinate system, which is convenient for vector transformation. The polarization vectors  $\boldsymbol{\epsilon}(\lambda)$  are defined by

$$\boldsymbol{\epsilon}(\pm) = \mp \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) = \mp \frac{1}{\sqrt{2}} (1, \pm i, 0) \quad (1.24a)$$

$$\boldsymbol{\epsilon}(0) = \hat{\mathbf{z}} = (0, 0, 1) \quad (1.24b)$$

28. Show that the polarization vectors are expressed using four-vector nontation as

$$\epsilon^\mu(\pm) = \mp \frac{1}{\sqrt{2}} (0, 1, \pm i, 0), \quad (1.25a)$$

$$\epsilon^\mu(0) = (0, 0, 0, 1). \quad (1.25b)$$

29. Show that massless photon's wavefunction is a linear combination of  $\epsilon^\mu(\pm)$ . The longitudinal state  $\epsilon^\mu(0)$  is not allowed.

30. Show the orthogonality conditions

$$\boldsymbol{\epsilon}^*(\lambda) \cdot \boldsymbol{\epsilon}(\lambda') = \delta^{\lambda\lambda'}, \quad (1.26a)$$

$$\boldsymbol{\epsilon}^*(\lambda) \cdot \boldsymbol{\epsilon}(\lambda') = -\delta^{\lambda\lambda'}. \quad (1.26b)$$

31. Show that the spin sum of the polarization tensor is

$$\sum_{\lambda=\pm} \epsilon^{i*}(\lambda) \epsilon^j(\lambda) = \delta_\perp^{ij} = \delta^{ij} - \hat{\mathbf{z}}^i \hat{\mathbf{z}}^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \quad (1.27a)$$

if  $k^\mu = (k, 0, 0, k)$ .

32. Show that the spin sum of the polarization tensor is

$$\sum_{\lambda=\pm} \epsilon^{i*}(\lambda) \epsilon^j(\lambda) = \delta_{\perp}^{ij} = \delta^{ij} - \frac{\mathbf{k}^i \mathbf{k}^j}{\mathbf{k}^2}. \quad (1.28a)$$

if  $k^\mu = (|\mathbf{k}|, \mathbf{k})$ . The polarization sum is for the Coulomb gauge, where  $\nabla \cdot \mathbf{A} = 0$ .

33. Defining  $n = (1, 0, 0, -1)$  when  $p = (E, 0, 0, E)$ , show that we may write it in a Lorentz covariant form

$$\sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda) \epsilon^\nu(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \quad (1.29a)$$

34. The formula

$$\Pi^{\mu\nu} = \sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda) \epsilon^\nu(\lambda) = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \quad (1.30)$$

is valid for any light-like vector  $n^2 = 0$  satisfying  $\epsilon \cdot p \neq 0$ ,  $n \cdot p \neq 0$ , and  $n \cdot \epsilon = 0$ . Show explicitly that

$$\Pi^\mu_\mu = -2, \quad (1.31a)$$

$$n_\mu \Pi^{\mu\nu} = 0, \quad n_\nu \Pi^{\mu\nu} = 0, \quad (1.31b)$$

$$p_\mu \Pi^{\mu\nu} = 0, \quad p_\nu \Pi^{\mu\nu} = 0. \quad (1.31c)$$

The polarization sum is for the light-cone gauge, where  $n \cdot A = 0$  with  $n^2 = 0$ .

35. We can extend our formula to the case  $n^2 \neq 0$  keeping  $\epsilon \cdot p \neq 0$ ,  $n \cdot p \neq 0$ , and  $n \cdot \epsilon = 0$ . Derive

$$\Pi^{\mu\nu} = \sum_{\lambda=\pm} \epsilon^{*\mu}(\lambda) \epsilon^\nu(\lambda) = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} - \frac{n^2 p^\mu p^\nu}{(p \cdot n)^2}. \quad (1.32)$$

from the conditions

$$\Pi^\mu_\mu = -2, \quad (1.33a)$$

$$n_\mu \Pi^{\mu\nu} = 0, \quad n_\nu \Pi^{\mu\nu} = 0, \quad (1.33b)$$

$$p_\mu \Pi^{\mu\nu} = 0, \quad p_\nu \Pi^{\mu\nu} = 0, \quad (1.33c)$$

The polarization sum is for the axial gauge, where  $n \cdot A = 0$  with  $n^2 \neq 0$ .

### 1.3 Lagrangian

36. **Euler-Lagrange equation:** Action is defined by

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (1.34)$$

Show that under the variation  $\phi \rightarrow \phi + \delta\phi$ , where  $\phi$  and  $\partial_\mu\phi$  are fixed at the end points

$$\delta S = \int d^4x \delta\mathcal{L}(\phi, \partial_\mu\phi) = \int d^4x \left[ \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\partial_\mu\phi \right] \quad (1.35a)$$

$$= \int d^4x \left[ \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right] \delta\phi, \quad (1.35b)$$

where we neglected the surface term

$$\int d^4x \partial_\mu \left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right] = \int da_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi. \quad (1.36a)$$

37. Show that  $\delta S = 0$  if  $\phi$  satisfies the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0. \quad (1.37)$$

38. **real scalar field** If

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2), \quad (1.38)$$

show that the Euler-Lagrange equation is the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0. \quad (1.39)$$

This leads to  $p^2 = m^2$ .

39. **Symmetry and conserved current** If the Lagrangian is invariant under a transformation  $\phi \rightarrow \phi + \delta\phi$ , show that

$$\delta\mathcal{L}(\phi, \partial_\mu\phi) = \frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\partial_\mu\phi, \quad (1.40a)$$

$$= \left[ \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right] \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\mu(\delta\phi) \quad (1.40b)$$

$$= \partial_\mu \left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right] = 0. \quad (1.40c)$$

If there is symmetry, there is a corresponding conserved quantity.

40. The conserved ( $\partial \cdot J = 0$ ) current

$$J^\mu \propto \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi. \quad (1.41)$$

41. If the current is vanishing on a boundary surface, charge inside the surface is conserved

$$\frac{\partial Q}{\partial t} = - \int d^3x \nabla \cdot \mathbf{J} = - \int da \cdot \mathbf{J} = 0, \quad (1.42a)$$

where  $Q = \int d^3x J^0$ .

42. Show that

$$-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \mathbf{E}^2 - \mathbf{B}^2 \quad (1.43a)$$

43. Show that

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) - \rho\phi + \mathbf{J} \cdot \mathbf{A}$$

44. Show that the Euler-Lagrange equation for the Lagrangian is the Maxwell's equations

$$\partial_\mu F^{\mu\nu} = J^\nu; (\partial^2 - \partial^\mu \partial_\nu) A^\nu = J^\nu \quad (1.44)$$

## 1.4 Classical charged-particle Lagrangian

45. Consider a particle with the charge  $q$  and mass  $m$  moving in an external electromagnetic field. The equation of motion is

$$\frac{dE}{dt} = q\mathbf{v} \cdot \mathbf{E}, \quad \frac{dp}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (1.45a)$$

46. Let us define the four-velocity in Lorentz covariant notation

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(1, \mathbf{v}). \quad (1.46)$$

show that  $u^2 = 1$ .

47. Using the four-velocity in Lorentz covariant notation, show that

$$\frac{dp^\mu}{d\tau} = q\gamma(\mathbf{v} \cdot \mathbf{E}, \mathbf{E} + \mathbf{v} \times \mathbf{B}) = qF^{\mu\nu}u_\nu, \quad (1.47)$$

where  $p^\mu = mu^\mu$  and  $m$  is the rest mass of the particle.

48. Show that

$$F^{\mu\nu}u_\nu = \gamma \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -v^1 \\ -v^2 \\ -v^3 \end{pmatrix} \quad (1.48a)$$

$$= \gamma \begin{pmatrix} v^1 E^1 + v^2 E^2 + v^3 E^3 \\ E^1 + (v^2 B^3 - v^3 B^2) \\ E^2 + (v^3 B^1 - v^1 B^3) \\ E^3 + (v^1 B^2 - v^2 B^1) \end{pmatrix}^\mu. \quad (1.48b)$$

49. **Free particle Lagrangian:** Consider a free particle with a mass  $m$ . The action must be Lorentz invariant and the only available Lorentz invariant scalar is  $m$ .

$$S = \int_{t_i}^{t_f} L dt = \int_{t_i}^{t_f} \gamma L d\tau = f(m). \quad (1.49)$$

and  $L\gamma$  must be a scalar. And the dimension must be order of mass. Let us try

$$L = -\frac{m}{\gamma} = -m\sqrt{1 - \mathbf{v}^2}. \quad (1.50)$$

Show that the Euler-Lagrange equation is

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial v^i} \right) - \frac{\partial L}{\partial x^i} = 0, \quad (1.51a)$$

$$\frac{\partial L}{\partial v^i} = \frac{mv^i}{\sqrt{1 - \mathbf{v}^2}} = \gamma mv^i, \quad (1.51b)$$

$$\frac{\partial}{\partial t} (m\gamma v^i) = 0. \quad (1.51c)$$

Momentum is conserved!  $\leftarrow$  free particle.

50. Show that corresponding Hamiltonian is

$$p^i = \frac{\partial L}{\partial v^i} = -\frac{\partial}{\partial v^i} \sqrt{1 - \mathbf{v}^2} = \frac{mv^i}{\sqrt{1 - \mathbf{v}^2}}, \quad (1.52a)$$

$$H = p^i v^i - L = \frac{m\mathbf{v}^2}{\sqrt{1 - \mathbf{v}^2}} + m\sqrt{1 - \mathbf{v}^2} \quad (1.52b)$$

$$= \frac{m}{\sqrt{1 - \mathbf{v}^2}} - \frac{m(1 - \mathbf{v}^2)}{\sqrt{1 - \mathbf{v}^2}} + m\sqrt{1 - \mathbf{v}^2} \quad (1.52c)$$

$$= \frac{m}{\sqrt{1 - \mathbf{v}^2}} = \gamma m = \sqrt{\mathbf{p}^2 + m^2}. \quad (1.52d)$$

51. **Charged particle Lagrangian:** In nonrelativistic quantum mechanics,  $L_{\text{int}} = -V_{\text{int}}$ . If we consider the electrostatic potential in nonrelativistic quantum mechanics,

$$L_{\text{int}} = -V_{\text{int}} = -q\phi. \quad (1.53)$$

52. Let us construct a Lorentz scalar. As we did for a charged particle,  $\gamma L$  must be Lorentz invariant and in the nonrelativistic limit the Lagrangian must reduce the form shown above.

$$L_{\text{int}} = -\frac{qu \cdot A}{\gamma}, \quad (1.54)$$

where  $u^\mu = \gamma(1, \mathbf{v})$  and  $A^\mu = (\phi, \mathbf{A})$ .

53. Therefore,

$$L = -\frac{m + qv \cdot A}{\gamma} = -m\sqrt{1 - \mathbf{v}^2} - q\phi + q\mathbf{v} \cdot \mathbf{A} \quad (1.55)$$

Conjugate momentum is

$$P^i = \frac{\partial L}{\partial v^i} = \gamma m v^i + q A^i = (\mathbf{p} + q \mathbf{A})^i \rightarrow \mathbf{v} = \frac{\mathbf{P} - q \mathbf{A}}{\gamma m}, \quad (1.56a)$$

$$1 = \gamma^2(1 - \mathbf{v}^2) = \gamma^2 - \frac{(\mathbf{P} - q \mathbf{A})^2}{m^2}, \quad (1.56b)$$

$$\rightarrow \gamma m = \sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2} \rightarrow \mathbf{v} = \frac{\mathbf{P} - q \mathbf{A}}{\sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2}}. \quad (1.56c)$$

54. Show that

$$(\mathbf{P} - q \mathbf{A}) \cdot \mathbf{v} = \frac{(\mathbf{P} - q \mathbf{A})^2}{\sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2}}, \quad (1.57a)$$

$$\frac{m}{\gamma} = \frac{m^2}{\gamma m} = \frac{m^2}{\sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2}}. \quad (1.57b)$$

Then the Hamiltonian is

$$H = \mathbf{P} \cdot \mathbf{v} - L = (\mathbf{P} - q \mathbf{A}) \cdot \mathbf{v} + \frac{m}{\gamma} + q\phi \quad (1.58a)$$

$$= \sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2} + q\phi \quad (1.58b)$$

55. The Hamiltonian for a charged particle

$$H = \sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2} + q\phi. \quad (1.59)$$

is same as that for a free particle

$$H = \sqrt{\mathbf{p}^2 + m^2} \quad (1.60)$$

when we substitute

$$H \rightarrow H - q\phi \quad (1.61a)$$

$$\mathbf{p} \rightarrow \mathbf{P} - q \mathbf{A}. \quad (1.61b)$$

56. Show that mass-shell condition still holds

$$\mathbf{p}^2 = m^2, \quad p^\mu = (E - q\phi, \mathbf{P} - q \mathbf{A}), \quad (1.62)$$

where  $E$  is the total energy of the particle

57. **Nonrelativistic case:** Show that in the nonrelativistic(NR) limit

$$H = \sqrt{(\mathbf{P} - q \mathbf{A})^2 + m^2} + q\phi \rightarrow \frac{(\mathbf{P} - q \mathbf{A})^2}{2m} + q\phi. \quad (1.63)$$

We could have derived the form from the free particle equation

$$H = \frac{\mathbf{p}^2}{2m} \leftarrow [H \rightarrow H - q\phi, \mathbf{p} \rightarrow \mathbf{P} - q \mathbf{A}]. \quad (1.64)$$

58. **Schrödinger equation for a charged particle:** For a free particle, we replace

$$\mathbf{p} \rightarrow -i\nabla, H \rightarrow i\frac{\partial}{\partial t}, \quad (1.65)$$

which generate  $\mathbf{p}$  and  $E$  once they act on the free-particle wavefunction  $e^{-ip \cdot x}$ , and apply the operator to the wavefunction  $\psi$ .

$$i\frac{\partial}{\partial t}\psi = \frac{(-i\nabla)^2}{2m}\psi \quad (1.66)$$

59. Show that the Schrödinger equation for a charged particle is

$$(H - q\phi)\psi = \frac{(\mathbf{P} - q\mathbf{A})^2}{2m}\psi, \quad (1.67a)$$

$$i\left(\frac{\partial}{\partial t} + iqA^0\right)\psi = -\frac{(-\nabla + iq\mathbf{A})^2}{2m}\psi. \quad (1.67b)$$

## 1.5 Gauge Invariance and Covariant Derivative

60. Let us define the covariant derivative

$$\mathcal{D}^\mu \equiv \partial^\mu + iqA^\mu = (D^0, -\mathbf{D}) = \left(\frac{\partial}{\partial t} + iqA^0, -\nabla + iq\mathbf{A}\right). \quad (1.68)$$

Show that the Shrödinger equation becomes

$$iD^0\psi = -\frac{\mathbf{D}^2}{2m}\psi. \quad (1.69)$$

61. Let us replace  $\psi \rightarrow \psi' = U\psi$ , where the transformation keeps the probability

$$\psi^\dagger\psi = \psi'^\dagger\psi' \rightarrow U^\dagger U = 1. \quad (1.70)$$

Therefore  $U$  is unitary.

62. We know physical observables are invariant under the gauge transformation

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu\chi. \quad (1.71)$$

We will find there IS a gauge transformation that keeps the Shrödinger equation invariant under the unitary transformation  $\psi \rightarrow \psi' = U\psi$ .

$$D'^\mu = \partial^\mu + iqA'^\mu = \partial^\mu + iq(A^\mu + \partial^\mu\chi) = D^\mu + iq\partial^\mu\chi \quad (1.72a)$$

$$D'^0 = \partial^0 + iq(A^0 + \partial^0\chi) \quad (1.72b)$$

$$-\mathbf{D}' = -\nabla + iq(\mathbf{A} - \nabla\chi). \quad (1.72c)$$

Show, if  $\partial^\mu\chi = \frac{i}{q}(\partial^\mu U)U^\dagger$ , that

$$D'^0 U\psi = UD^0\psi, \quad \mathbf{D}'U\psi = U\mathbf{D}\psi \quad (1.73a)$$

$$(\mathbf{D}')^2 U\psi = \mathbf{D}'(U\mathbf{D}\psi) = U\mathbf{D}^2\psi \quad (1.73b)$$

Therefore, Schrödinger equation is invariant under the gauge transformation

$$A^\mu \rightarrow A'^\mu = UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger, \quad \psi \rightarrow \psi' = U\psi, \quad U^\dagger U = 1 \quad (1.74)$$

## 1.6 Scalar particle

63. **Scalar particle and EM interaction** Remember

$$H = \sqrt{\mathbf{p}^2 + m^2} \rightarrow H^2 = \mathbf{p}^2 + m^2. \quad (1.75)$$

Klein-Gordon equation is the wave equation for a scalar particle that can be obtained by the replacement  $H \rightarrow i\partial^0$  and  $\mathbf{p} = -i\nabla$ , that is  $p^\mu \rightarrow i\partial^\mu$

$$(\partial^2 + m^2)\phi = 0. \quad (1.76)$$

We have checked that the Lagrangian for the equation of motion is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi\partial_\mu\phi - m^2\phi^2). \quad (1.77)$$

If we introduce a complex scalar field, that can be constructed as a linear combination of two real scalar fields as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad (1.78)$$

$$\phi^* \neq \phi.$$

64. Using the fact that  $\phi_1$  and  $\phi_2$  are satisfying the Klein-Gordon equation, show that  $\phi$  and  $\phi^*$  also satisfies the Klein-Gordon equation.

$$(\partial^2 + m^2)\phi = 0, \quad (\partial^2 + m^2)\phi^* = 0. \quad (1.79)$$

65. Show that the Lagrangian is

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi. \quad (1.80)$$

Now we can introduce the covariant derivative to the complex scalar field.

$$\mathcal{L} = (\mathcal{D}^\mu\phi)^\dagger(\mathcal{D}_\mu\phi) - m^2\phi^\dagger\phi, \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu. \quad (1.81)$$

66. Show that the Lagrangian is invariant under the gauge transformation

$$\phi \rightarrow U\phi, \quad U^\dagger U = 1, \quad (1.82a)$$

$$A^\mu \rightarrow UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger. \quad (1.82b)$$

67. The Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu. \quad (1.83a)$$

Show that the current  $J^\mu$  is

$$J^\mu = +iq \left[ \phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger) \phi \right], \quad (1.84)$$

by expanding the covariant derivative.

68. Show that the  $J^\mu$  is conserved

$$\partial_\mu J^\mu = 0 \quad (1.85)$$

by using the Klein-Gordon equation.

69. Show that

$$Q = \int d^3x J^0 = 0 \quad (1.86)$$

for a real scalar field  $\phi^* = \phi$ .

70. Show that the current becomes

$$J^\mu = q \times |N|^2 2p^\mu. \quad (1.87)$$

if we use the free-particle wavefunction  $\phi = Ne^{-ip \cdot x}$  and  $\phi^* = N^* e^{ip \cdot x}$ , where  $N$  is the normalization factor.

71. Show that, if  $\int d^3x = V$ ,

$$Q = \int d^3x J^0 = q \times |N|^2 2EV. \quad (1.88)$$

72. Choose the covariant normalization  $N = 1/\sqrt{V}$  and show that the charge inside the volume  $V = \int d^3x$  is

$$Q = \int d^3x J^0 = q \times 2E. \quad (1.89)$$

73.  $q$  is the charge of the particle  $\phi$ .

74. Show that  $2E$  is the number of particles in  $V$ .

75. Show that the charge  $Q$  in  $V$  is not Lorentz invariant.

76. The number of particles in  $V$  is  $2m$  if the particle is at rest. Explain why the density is increasing with a factor  $E/m$  compared to that for the rest frame because of the length contraction.

77. **Negative energy solution to the Klein-Gordon equation:** Let us go back to the Klein-Gordon equation

$$(\partial^2 + m^2)\phi = 0. \quad (1.90)$$

Show that there are two solutions

$$\phi_+ = Ne^{-ip \cdot x}, \quad \phi_- = Ne^{+ip \cdot x} \quad (1.91)$$

where  $p = (E, \mathbf{p})$  with  $E = \sqrt{\mathbf{p}^2 + m^2} > 0$ .

78. Show that  $J^\mu(\phi = \phi_+) = q \times 2p^\mu|N|^2$  and  $J^\mu(\phi = \phi_-) = -q \times 2p^\mu|N|^2$ .
79. Show that  $J^\mu(\phi = \phi_-) = q \times 2(-p^\mu)$  means that a negative energy particle with charge  $+q$  is flowing from the future to the past.
80. Show that  $J^\mu(\phi = \phi_-) = (-q) \times 2p^\mu$  means that a positive energy particle with charge  $-q$  is flowing from the past to the future.
81. Combining the two equivalent statements, we conclude as follows. Once we know how to deal with a scalar particle with charge  $+q$ , the wavefunction for a positive-energy scalar particle with charge  $-q$  can be described in terms of the negative-energy scalar particle with charge  $+q$  flowing backward!

## 1.7 Time-dependent perturbation theory

82. Consider

$$H = H_0 + V, \quad H_0 = \frac{\mathbf{p}^2}{2m}, \quad (1.92a)$$

$$H_0\phi_n = E_n\phi_n, \quad \int d^3\mathbf{x}\phi_m^*(\mathbf{x})\phi_n(\mathbf{x}) = \delta_{mn}. \quad (1.92b)$$

Write the wavefunction  $\psi$  in terms of the eigenfunctions of the unperturbed Hamiltonian as

$$\psi = \sum_n a_n(t)\phi_n e^{-iE_n t}. \quad (1.93)$$

Solve  $a_n$  satisfying

$$i\frac{\partial}{\partial t}\psi = (H_0 + V)\psi \quad (1.94)$$

to get

$$i\sum_n \frac{\partial a_n(t)}{\partial t}\phi_n(\mathbf{x})e^{-iE_n t} = \sum_n a_n(t)V(t, \mathbf{x})\phi_n(\mathbf{x})e^{-iE_n t}. \quad (1.95)$$

83. Show that  $a_n(t \rightarrow -\infty) = \delta_{ni}$  means the initial state is monochromatic

$$\psi(t = -\infty) = \phi_i(\mathbf{x})e^{-iE_i t}. \quad (1.96)$$

84. Convoluting

$$\int d^3\mathbf{x}\phi_f^\dagger(\mathbf{x})e^{iE_f t}, \quad (1.97)$$

show that

$$a_{fi} = \delta_{fi} - i \int dt d^3\mathbf{x}\phi_f^\dagger(\mathbf{x})V(t, \mathbf{x})\phi_i(\mathbf{x})e^{i(E_f - E_i)t} \quad (1.98)$$

85. In short

$$\psi(t \rightarrow \infty) = S\psi(t \rightarrow -\infty), \quad (1.99a)$$

$$S = 1 + i\mathcal{T}, \quad (1.99b)$$

$$i\mathcal{T} = -i \int d^4x \phi_f^*(x)V\phi_i(x) = i \int d^4x \mathcal{L}_{\text{int}} = L_{\text{int}}. \quad (1.99c)$$

86. Show that, if  $\mathcal{T}$  is Hermitian

$$\mathcal{T}^\dagger = \mathcal{T} \leftrightarrow L_{\text{int}}, \quad (1.100)$$

$S$  is unitarity  $S^\dagger S = 1$ . Therefore,

$$\psi^\dagger\psi(t = \infty) = \psi^\dagger\psi(t = -\infty). \quad (1.101)$$

87. **Energy-momentum conservation:** If the potential is independent of time  $V(t, \mathbf{x}) = V(\mathbf{x})$ ,  $i\mathcal{T}_{fi}$  becomes

$$i\mathcal{T}_{fi} = -iV_{fi} \int_{-\infty}^{\infty} dt e^{i(E_f - E_i)t} = -iV_{fi}(2\pi)\delta(E_f - E_i), \quad (1.102a)$$

$$V_{fi} = \int d^3\mathbf{x} \phi_f^\dagger(\mathbf{x}) V(\mathbf{x}) \phi_i(\mathbf{x}), \quad (1.102b)$$

and the energy is conserved.

$$\frac{\mathbf{p}_f^2}{2m} = \frac{\mathbf{p}_i^2}{2m}. \quad (1.103)$$

88. In general,  $\mathbf{p}_f \neq \mathbf{p}_i$ . Where is the lost momentum,  $\mathbf{p}_i - \mathbf{p}_f$ ? It has been transferred to the potential because

$$V(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} V(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (1.104a)$$

$$\phi_f^\dagger(\mathbf{x}) = N_f e^{i\mathbf{p}_f \cdot \mathbf{x}}, \quad (1.104b)$$

$$\phi_i(\mathbf{x}) = N_i e^{-i\mathbf{p}_i \cdot \mathbf{x}}. \quad (1.104c)$$

89. Show that

$$V_{fi} = N_f N_i \int d^3\mathbf{k} V(\mathbf{k}) \delta(\mathbf{k} + \mathbf{p}_f - \mathbf{p}_i) = N_f N_i V(\mathbf{p}_i - \mathbf{p}_f). \quad (1.105)$$

Therefore, momentum is conserved  $\mathbf{k} + \mathbf{p}_f = \mathbf{p}_i$ .

## 1.8 Propagator

90. **Propagator for a scalar particle:** Let us recall the transition matrix

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}}. \quad (1.106)$$

We choose  $\mathcal{L}_{\text{int}} = -\phi^\dagger J$  which is analogous to the electromagnetic interaction lagrangian  $-J_\mu A^\mu$ . Resulting lagrangian for a scalar field is

$$\mathcal{L} = \phi^\dagger (-\partial^2 - m^2) \phi - \phi^\dagger J, \quad (1.107)$$

where we neglect the surface term. Then the wave equation becomes

$$\frac{\partial \mathcal{L}}{\partial \phi^\dagger} = 0 \rightarrow (\partial^2 + m^2) \phi = -J. \quad (1.108)$$

Show that

$$\phi(x) = \int dy^4 \Delta_F(x - y) J(y), \quad (1.109)$$

if

$$(\partial^2 + m^2) \Delta_F(x) = -\delta(x). \quad (1.110)$$

Hint: Act  $\partial^2 + m^2$  to both sides.

91. Show that

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}} = -i \int d^4x J(x) \phi(x) \quad (1.111\text{a})$$

$$= -i \int d^4x d^4y J(x) \Delta_F(x-y) J(y) \quad (1.111\text{b})$$

$$= \int d^4x d^4y [-iJ(x)] [i\Delta_F(x-y)] [-iJ(y)] \quad (1.111\text{c})$$

92.  $i\Delta_F(x-y)$  is the Feynman propagator. It describes the propagation of a scalar field from a space-time point  $y$  to  $x$ .

93. Show that

$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{-ip^0 x^0}}{p^0 - E + i\epsilon} = -i e^{-iEx^0}, \quad \text{if } x^0 > 0, \quad (1.112)$$

Hint: Use Cauchy integral formula. Explain why the above integral is same as the complex integral along a contour closed on the lower-half plain.

94. Show that

$$\int \frac{dp^0}{2\pi} \frac{e^{-ip^0 x^0}}{p^0 + E - i\epsilon} = +i e^{iEx^0}, \quad \text{if } x^0 < 0, \quad (1.113)$$

Explain why the above integral is same as the complex integral along a contour closed on the upper-half plain.

95. Show that

$$\begin{aligned} \int \frac{dp^0}{2\pi} \frac{e^{-ip^0 x^0}}{p^2 - m^2 + i\epsilon} &= \int \frac{dp^0}{2\pi} \frac{e^{-ip^0 x^0}}{(p^0)^2 - (E^2 - i\epsilon)} \\ &= \int \frac{dp^0}{2\pi} \frac{e^{-ip^0 x^0}}{(p^0 + E - i\epsilon)(p^0 - E + i\epsilon)} \\ &= \frac{-i}{2E} \left( \theta(x^0) e^{-iEx^0} + \theta(-x^0) e^{+iEx^0} \right) \end{aligned} \quad (1.114\text{a})$$

where  $E = \sqrt{\mathbf{p}^2 + m^2} > 0$ ,  $\epsilon \rightarrow 0^+$  and

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (1.115)$$

96. Explain why the sign of  $i\epsilon$  is important.

97. Explain what is wrong if  $\epsilon$  is finite.

98. Show that

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot x} \quad (1.116\text{a})$$

$$= \int \frac{d^3\mathbf{p}}{2E(2\pi)^3} \left[ \theta(x^0) e^{-i(Et - \mathbf{p} \cdot \mathbf{x})} + \theta(-x^0) e^{i(Et - \mathbf{p} \cdot \mathbf{x})} \right] \quad (1.116\text{b})$$

$$= \int \frac{d^3\mathbf{p}}{2E(2\pi)^3} \left[ \theta(x^0) e^{-ip \cdot x} + \theta(-x^0) e^{ip \cdot x} \right] \quad (1.116\text{c})$$

where  $E = \sqrt{m^2 + \mathbf{p}^2} > 0$  and  $p^0 = E$  on the last line.

99. Show that the retarded term  $\theta(x^0)$  is for the particle propagating to the future.
100. Show that the advanced term  $\theta(-x^0)$  term is for the particle to the past.
101. Show that
- $$\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon} \quad (1.117)$$

is the solution to the equation

$$(\partial^2 + m^2) \Delta_F(x) = -\delta(x) \quad (1.118)$$

## 1.9 Photon propagator

102. The  $i\mathcal{T}$  matrix for a photon field is

$$i\mathcal{T} = i \int d^4 x \mathcal{L}_{\text{int}} = -i \int d^4 x J^\mu A_\mu, \quad (1.119)$$

where the equation of motion is Maxwell's equation

$$\partial_\nu F^{\nu\mu} = J^\mu; \quad (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu = J^\mu. \quad (1.120)$$

103. We want to find the solution to

$$(\partial^2 g^{\mu\alpha} - \partial^\mu \partial^\alpha) (D_F)_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x). \quad (1.121)$$

$$A^\mu(x) = \int dy^4 D_F^{\mu\nu}(x-y) J_\nu(y) \quad (1.122a)$$

$$\text{if } (\partial^2 g^{\mu\alpha} - \partial^\mu \partial^\alpha) (D_F)_{\alpha\nu} = g^\mu{}_\nu \delta(x). \quad (1.122b)$$

104. Show that

$$\mathcal{T} = i \int d^4 x \mathcal{L}_{\text{int}} = -i \int d^4 x J_\mu(x) A^\mu(x) \quad (1.123)$$

$$= -i \int d^4 x d^4 y J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) \quad (1.124)$$

$$= \int d^4 x d^4 y [-i J_\mu(x)] [i D_F^{\mu\nu}(x-y)] [-i J_\nu(y)] \quad (1.125)$$

$i\Delta_F^{\mu\nu}(x-y)$  is the Feynman propagator. It describes the propagation of a vector field from a space-time point  $y$  to  $x$ .

105. **Propagator in the Feynman gauge:** If we choose the Lorentz gauge,  $\partial \cdot A = 0$ , and we may neglect the term  $\partial^\mu \partial^\nu$  terms in the wave equations. Maxwell's equation becomes

$$\partial^2 A^\mu = J^\mu. \quad (1.126)$$

Show that

$$D_F^{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-g^{\mu\nu}}{p^2 + i\epsilon} e^{-ip \cdot x} = -g^{\mu\nu} \Delta_F(x) \text{ with } m = 0 \quad (1.127)$$

is the solution to the equation

$$\partial^2 g^{\mu\alpha} (D_F)_{\alpha\nu} = g^\mu{}_\nu \delta(x). \quad (1.128)$$

106. Show that the wave equation

$$\partial^2 A^\mu = J^\mu \quad (1.129)$$

is actually from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha} (\partial \cdot A)^2 \quad (1.130)$$

where  $\alpha = 1$ . The term added to the original Lagrangian is the gauge-fixing term.

107. The photon propagator

$$D_F^{\mu\nu}(x) = -g^{\mu\nu} \Delta_F(x) \quad (1.131)$$

is defined in the Feynman gauge, a special case of the Lorentz gauge.

108. **Propagator in the Lorentz gauge** Show that the equation of motion for the photon field in the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha} (\partial \cdot A)^2 \quad (1.132)$$

is

$$\left[ \partial^2 g^{\mu\alpha} + \left( \frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] A_\alpha = J^\mu \quad (1.133)$$

and the propagator  $D$  must satisfy

$$\left[ \partial^2 g^{\mu\alpha} + \left( \frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x). \quad (1.134)$$

Note that we are using the Lorentz gauge  $\partial \cdot A = 0$ .

109. **Propagator in the Lorentz gauge** Show that

$$D^{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-g^{\mu\nu} + (1-\alpha) \frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon} e^{-ip \cdot x} \quad (1.135)$$

$$= \int \frac{d^4 p}{(2\pi)^4} \left[ -g^{\mu\nu} + (1-\alpha) \frac{p^\mu p^\nu}{p^2} \right] \Delta_F(p) e^{-ip \cdot x} \quad (1.136)$$

$$\Delta_F(p) = \frac{1}{p^2 + i\epsilon} \quad (1.137)$$

is the solution to the equation

$$\left[ \partial^2 g^{\mu\alpha} + \left( \frac{1}{\alpha} - 1 \right) \partial^\mu \partial^\alpha \right] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x) \quad (1.138)$$

110. **Propagator in the axial gauge** We can choose the axial gauge  $n \cdot A = 0$ . In this case, gauge fixing term can be written in the form to have our Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu - \frac{1}{2\alpha}(n \cdot A)^2. \quad (1.139)$$

111. Show that

$$iD^{\mu\nu}(x) = i \int \frac{d^4p}{(2\pi)^4} \left[ -g^{\mu\nu} + \frac{n^\mu p^\nu + p^\mu n^\nu}{n \cdot p} - (n^2 + \alpha p^2) \frac{p^\mu p^\nu}{(n \cdot p)^2} \right] \Delta_F(p) e^{-ip \cdot x} \quad (1.140)$$

$$\Delta_F(p) = \frac{1}{p^2 + i\epsilon} \quad (1.141)$$

is the gluon propagator in the axial gauge.

112. **Massive spin-1 propagator** The Lagrangian for a massive spin-1 field is just like that for the photon except that the particle has nonvanishing mass. We insert the mass term  $\frac{m^2}{2}B^\mu B_\mu$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}B^\mu B_\mu - J^\mu B_\mu \quad (1.142)$$

$$F^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \quad (1.143)$$

113. Show that the equation of motion is

$$[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu] B_\nu = J^\mu. \quad (1.144)$$

114. The  $\mathcal{T}$  matrix for a massive spin-1 field is

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}} = -i \int d^4x J^\mu B_\mu, \quad (1.145)$$

where the equation of motion is

$$[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu] B_\nu = J^\mu. \quad (1.146)$$

115. We want to find the solution to

$$[(\partial^2 + m^2)g^{\mu\alpha} - \partial^\mu \partial^\alpha] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x). \quad (1.147)$$

$$B^\mu(x) = \int dy^4 D^{\mu\nu}(x-y) J_\nu(y) \quad (1.148)$$

$$\text{if } \quad [(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu] D_{\alpha\nu}(x) = g^\mu{}_\nu \delta(x) \quad (1.149)$$

Show that

$$i\mathcal{T} = i \int d^4x \mathcal{L}_{\text{int}} = -i \int d^4x J_\mu(x) B^\mu(x) \quad (1.150)$$

$$= -i \int d^4x d^4y J_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \quad (1.151)$$

$$= \int d^4x d^4y [-iJ_\mu(x)] [iD^{\mu\nu}(x-y)] [-iJ_\nu(y)] \quad (1.152)$$

116. Show that

$$D^{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{-g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}}{p^2 - m^2 + i\epsilon} e^{-ip \cdot x} \quad (1.153)$$

$$= \int \frac{d^4 p}{(2\pi)^4} \left[ -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right] \Delta_F(p) e^{-ip \cdot x} \quad (1.154)$$

$$\Delta_F(p) = \frac{1}{p^2 - m^2 + i\epsilon} \quad (1.155)$$

is the solution to the equation

$$[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu] D_{\alpha\nu}(x) = g^\mu_\nu \delta(x) \quad (1.156)$$

Note that  $p^2 \neq m^2$  in general.

## 1.10 Feynman rules

**Momentum-space Feynman rule** Let us go back to the  $\mathcal{T}$  matrix

$$i\mathcal{T} = i \int d^4 x \mathcal{L}_{\text{int}} \quad (1.157)$$

$$= \int d^4 x d^4 y [-iJ(x)] \cdot [iD(x-y)] \cdot [-iJ(y)] \quad (1.158)$$

where indices are suppressed for vector particles.

117. Expand the currents as

$$J(y) = \int \frac{d^4 k}{(2\pi)^4} J(k) e^{-ik \cdot y} \quad (1.159)$$

and show that

$$i\mathcal{T} = i \int d^4 x \mathcal{L}_{\text{int}} \quad (1.160)$$

$$= \int d^4 x d^4 y [-iJ(x)] \cdot [iD(x-y)] \cdot [-iJ(y)] \quad (1.161)$$

$$= \int \frac{d^4 p}{(2\pi)^4} [-iJ(-p)] \cdot [iD(p)] \cdot [-iJ(p)]. \quad (1.162)$$

118. Consider monochromatic currents

$$J(x) = J(p_1) = N^2 \hat{J}(p_1) e^{-ip_1 \cdot x} \quad (1.163)$$

$$J(y) = J(p_2) = N^2 \hat{J}(p_2) e^{-ip_2 \cdot y} \quad (1.164)$$

where  $N^2$  came from the normalization of the initial and final states involving the current.

Show that in this case

$$i\mathcal{T} = i \int d^4 x \mathcal{L}_{\text{int}} = N^4 (2\pi)^4 \delta^4(p_1 + p_2) \mathcal{M} \quad (1.165)$$

$$\mathcal{M} = [-i\hat{J}(p_2)] \cdot [iD(p = p_1 = -p_2)] \cdot [-i\hat{J}(p_1)] \quad (1.166)$$

119. Show that the propagators for a scalar field and massless/massive vector fields are even functions;

$$D(x) = D(-x), \quad D(p) = D(-p) \quad (1.167)$$

Hint: Look into the partial differential equation for the propagator.

120. Show that

$$|\mathcal{T}|^2 = \left| i \int d^4x \mathcal{L}_{\text{int}} \right|^2 = N^8 [(2\pi)^4 \delta^4(p_1 + p_2)]^2 |\mathcal{M}|^2 \quad (1.168)$$

$$= N^8 \times VT \times (2\pi)^4 \delta^4(p_1 + p_2) |\mathcal{M}|^2 \quad (1.169)$$

$$\mathcal{M} = [-i\hat{J}(p_2)] \cdot [iD(p = \pm p_1 \text{ or } \pm p_2)] \cdot [-i\hat{J}(p_1)] \quad (1.170)$$

Hint:  $(2\pi)^4 \delta^4(0) = \int dt d^3\mathbf{x}$ .

121. Show that the probability of the transition per unit volume per unit time is

$$P = \frac{|\mathcal{T}|^2}{TV} = \frac{1}{V^4} \times (2\pi)^4 \delta^4(p_1 + p_2) |\mathcal{M}|^2 \quad (1.171)$$

$$\mathcal{M} = [-i\hat{J}(p_2)] \cdot [iD(p = \pm p_1 \text{ or } \pm p_2)] \cdot [-i\hat{J}(p_1)] \quad (1.172)$$

where we used  $N = 1/\sqrt{V}$ .

122. **Cross section** Show that the number density of a particle with energy  $E$  in  $V$  is

$$\frac{2E}{V}. \quad (1.173)$$

123. Show that the flux of the two colliding particle is

$$F = |\mathbf{v}_a - \mathbf{v}_b| \times \frac{2E_a}{V} \times \frac{2E_b}{V} \quad (1.174)$$

$$= 4(|\mathbf{p}_a|E_b + |\mathbf{p}_b|E_a)/V^2 \quad (1.175)$$

$$= 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}/V^2 \quad (1.176)$$

where  $\mathbf{v}_i$ ,  $\mathbf{p}_i$ , and  $E_i$  are the velocity, momentum and energy of the  $i$ -th particle.

124. Show that the number of states in the final state is

$$\begin{aligned} dN_{\text{final states}} &= V\theta(p_1^0)\delta(p_1^2 - m_1^2) \frac{d^4 p_1}{(2\pi)^4} V\theta(p_2^0)\delta(p_2^2 - m_2^2) \frac{d^4 p_2}{(2\pi)^4} \\ &= \frac{Vd^3\mathbf{p}_1}{2E_1(2\pi)^3} \frac{Vd^3\mathbf{p}_2}{2E_2(2\pi)^3} \end{aligned} \quad (1.177)$$

if there are two final particles.

125. Cross section is defined by

$$\begin{aligned}
d\sigma &= \frac{P}{F} \times dN_{\text{final states}} \\
&= \frac{1}{V^4} (2\pi)^4 \delta(p_a + p_b - p_1 - p_2) |\mathcal{M}|^2 \\
&\times \frac{V^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} \times \frac{V d^3 \mathbf{p}_1}{2E_1(2\pi)^3} \frac{V d^3 \mathbf{p}_2}{2E_2(2\pi)^3} \\
&= \frac{|\mathcal{M}|^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} (2\pi)^4 \delta(p_a + p_b - p_1 - p_2) \frac{d^3 \mathbf{p}_1}{2E_1(2\pi)^3} \frac{d^3 \mathbf{p}_2}{2E_2(2\pi)^3}. \quad (1.178)
\end{aligned}$$

126. We usually define the phase space  $d\Phi$  after including the energy-momentum delta function

$$d\Phi = (2\pi)^4 \delta \left( P - \sum_i p_i \right) dN_{\text{final states}}, \quad (1.179)$$

where  $P$  is the sum of initial momenta  $p_i$  is the momentum of the  $i$ -th final-state particle.

127. We find  $V$  dependence exactly cancels. If we redefine the phase space

$$d\Phi = (2\pi)^4 \delta(p_a + p_b - p_1 - p_2) \prod_{i=1}^n \frac{d^3 \mathbf{p}_i}{2E_i(2\pi)^3} \quad (1.180)$$

we have

$$d\sigma = \frac{|\mathcal{M}|^2}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} d\Phi \quad (1.181)$$

128. Show that the mass term  $\frac{1}{2}m^2 A^\mu A_\mu$  for a gauge field is NOT invariant under gauge transformation

$$A^\mu \rightarrow U A^\mu U^\dagger + \frac{i}{q} (\partial^\mu U) U^\dagger, \quad \psi \rightarrow U \psi, \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu \quad (1.182)$$

This guarantees that the gauge field is massless.

129. Show that gauge field is travelling with the speed of light.

## 1.11 Dirac equation

130. We would like to construct a relativistically covariant theory for a fermion. If we ignore the spin, the equation must reduce to the Klein-Gordon equation. But we want to have an equation which has linear time derivative instead of  $\partial/\partial t^2$ , which appears in the Klein-Gordon equation.

$$i \frac{\partial \psi}{\partial t} = H\psi, \quad H = -i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m \quad (1.183)$$

131. Show that  $\psi$  must include the four states

$$|1\rangle = |\uparrow, E > 0\rangle, |2\rangle = |\downarrow, E > 0\rangle, \quad (1.184)$$

$$|3\rangle = |\uparrow, E < 0\rangle, |4\rangle = |\downarrow, E < 0\rangle \quad (1.185)$$

132. Show that  $H$  is Hermitian.

133. Show that

$$\langle m|H|n\rangle = \sqrt{\mathbf{p}^2 + m^2}, \text{ if } m = n = 1, 2 \quad (1.186)$$

$$\langle m|H|n\rangle = -\sqrt{\mathbf{p}^2 + m^2}, \text{ if } m = n = 3, 4 \quad (1.187)$$

$$\langle m|H|n\rangle = 0, \text{ if } m \neq n \quad (1.188)$$

$$\sum_n \langle n|H|n\rangle = 0 \quad (1.189)$$

134. Show that  $\alpha^1, \alpha^2, \alpha^3$ , and  $\beta$  are Hermitian.

135. If the Dirac equation is equivalent to the Klein-Gordon equation if we neglect the spin dependence and the sign of the energy, show

$$H^2 = -\nabla^2 + m^2 \rightarrow (\partial^2 + m^2)\psi = 0 \quad (1.190)$$

136. Show that the condition requires

$$\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i) = \delta^{ij} \quad (1.191)$$

$$\alpha^i\beta + \beta\alpha^i = 0 \quad (1.192)$$

$$\beta^2 = 1 \quad (1.193)$$

137. Show that the conditions  $(\alpha^i)^2 = 1$  and  $\beta^2 = 1$  require that the eigenvalues for the matrices are  $\pm 1$ .

From now on we will use the notation

$$\{A, B\} \equiv AB + BA. \quad (1.194)$$

138. Let us choose the basis so that the first two components are positive-energy components and the other two are negative-energy components. By taking  $\mathbf{p} = 0$ , show that

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.195)$$

139. From now on we express any  $4 \times 4$  matrix in spinor space in terms of  $2 \times 2$  block matrices.

140. Show that  $\beta^2 = 1$ .

141. Let us try arbitrary  $4 \times 4$  matrices

$$\alpha^i = \begin{pmatrix} a^i & b^i \\ c^i & d^i \end{pmatrix} \quad (1.196)$$

142. Show that the condition  $\{\beta, \alpha^i\} = 0$  requires

$$\alpha^i = \begin{pmatrix} 0 & b^i \\ c^i & 0 \end{pmatrix} \quad (1.197)$$

143. Show that the condition  $\alpha^\dagger = \alpha$  requires  $c = b^\dagger$ .

144. Show that the condition  $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$  requires

$$b^i(b^j)^\dagger + b^j(b^i)^\dagger = 2\delta^{ij} \quad (1.198)$$

$$(b^i)^\dagger b^j + (b^j)^\dagger b^i = 2\delta^{ij} \quad (1.199)$$

145. Choose the term  $i = j$  to find  $b^i$  is unitary

$$b^i(b^i)^\dagger = (b^i)^\dagger b^i = 1 \quad (1.200)$$

The solution is the  $3 = 2^2 - 1$  SU(2) generators, Pauli matrices;  $b^i = \sigma^i$ , where Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.201)$$

Therefore, we have

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad (1.202)$$

146. Show that

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk} \sigma^k \quad (1.203)$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij} \quad (1.204)$$

$$[\sigma^i, \sigma^j] \equiv \sigma^i \sigma^j - \sigma^j \sigma^i = +2i\epsilon^{ijk} \sigma^k \quad (1.205)$$

147. Show that

$$\mathbf{a} \cdot \boldsymbol{\sigma} \mathbf{b} \cdot \boldsymbol{\sigma} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma} \quad (1.206)$$

148. Show that

$$\mathbf{a} \cdot \boldsymbol{\alpha} \mathbf{b} \cdot \boldsymbol{\alpha} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\Sigma}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad (1.207)$$

149. Defining  $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma}) = (\beta, \beta\boldsymbol{\alpha})$ , show that

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (1.208)$$

150. Show that the anticommutation relations for  $\beta$  and  $\alpha^i$ 's reduce to a single formula

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \quad (1.209)$$

where the  $4 \times 4$  identity matrix

151. Let us define

$$\not{a} \equiv \gamma^\mu a_\mu \quad (1.210)$$

Show that

$$\not{a}\not{b} + \not{b}\not{a} = 2a \cdot b, \quad \not{a}^2 = a^2 \quad (1.211)$$

152. Show that

$$(\gamma^\mu)^\dagger = \gamma_\mu, \quad (\gamma_\mu)^\dagger = \gamma^\mu \quad (1.212)$$

153. Show that

$$\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu, \quad \gamma^0(\gamma_\mu)^\dagger\gamma^0 = \gamma_\mu \quad (1.213)$$

Multiply  $\beta$  to the original Dirac equation and find

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad (1.214)$$

Taking the Hermitian adjoint and find

$$-i\partial_\mu \bar{\psi} \gamma^\mu - \bar{\psi} m = 0 \quad (1.215)$$

where  $\bar{\psi} = \psi^\dagger \gamma^0$ .

154. Show that the Euler-Lagrange equation for the  $\psi$  field in the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad (1.216)$$

gives the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad (1.217)$$

155. Show that the Lagrangian can also be written as

$$\mathcal{L} = -i(\partial_\mu \bar{\psi})\gamma^\mu \psi - m\bar{\psi}\psi \quad (1.218)$$

if we neglect the surface term.

156. Show that the Euler-Lagrange equation for  $\psi^\dagger$  field is

$$-i\partial_\mu \bar{\psi} \gamma^\mu - \bar{\psi} m = 0 \quad (1.219)$$

157. Show that the Lagrangian for the Quantum electrodynamics

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1.220)$$

$$= -i(\mathcal{D}_\mu\psi)^\dagger \gamma^0 \gamma^\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1.221)$$

is invariant under the gauge transformation

$$\psi \rightarrow U\psi, A^\mu \rightarrow UA^\mu U^\dagger + \frac{i}{q}(\partial^\mu U)U^\dagger, \mathcal{D}^\mu = \partial^\mu + iqA^\mu \quad (1.222)$$

158. Introducing the covariant derivative  $\mathcal{D}^\mu = \partial^\mu + iqA^\mu$ , show that

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - J^\mu A_\mu \quad (1.223)$$

$$J^\mu = q\bar{\psi}\gamma^\mu\psi \quad (1.224)$$

159. Making use of the Dirac equations

$$i\gamma^\mu \partial_\mu \psi = m\psi, -i(\partial_\mu \bar{\psi})\gamma^\mu = m\bar{\psi} \quad (1.225)$$

show that the current is conserved

$$\partial_\mu J^\mu = 0 \quad (1.226)$$

## 1.12 Spinor

160. Let us consider an electron with momentum  $p^\mu$  and  $z$ -component spin  $s$ , where  $p^0 = E > 0$ . Show that the wavefunction  $\psi(x) = u(p, s)e^{-ip\cdot x}$  should be normalized to be

$$\psi^\dagger(x)\psi(x) = 2E \rightarrow u^\dagger(p, s)u(p, s') = 2E\delta_{ss'}. \quad (1.227)$$

161. Using the length contraction, show that the number of electron in the whole space of volume  $V$  is 1.

162. Show that

$$\bar{u}(p)u(p) = 2m \quad (1.228)$$

using  $u^\dagger(p, s)u(p, s') = 2m\delta_{ss'}$  and Lorentz covariance only.

163. Remind the fact that  $\bar{\psi}\gamma^\mu\psi$  is transforming like a four vector. Using the result  $u^\dagger(p, s)u(p, s) = 2E$  and Lorentz covariance only, show that

$$\bar{u}(p)\gamma^\mu u(p) = 2p^\mu. \quad (1.229)$$

164. Using spinors for an electron at rest  $p = (m, \mathbf{0})$ , show that

$$\sum_s u(p, s)\bar{u}(p, s) = 2m \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2m \times \frac{1}{2}(1 + \gamma^0). \quad (1.230)$$

165. Using Lorentz covariance, show that

$$\sum_s u(p, s) \bar{u}(p, s) = 2m \times \frac{\not{p} + m}{2m} = \not{p} + m. \quad (1.231)$$

166. Let us consider a negative-energy electron with momentum  $-p^\mu$  and  $z$ -component spin  $s$ , where  $p^0 = E > 0$ . Show that the wavefunction  $\psi(x) = u(-p, s)e^{+ip\cdot x}$  should be normalized to be

$$\psi^\dagger(x)\psi(x) = 2E \rightarrow u^\dagger(-p, s)u(-p, s') = 2E\delta_{ss'}. \quad (1.232)$$

167. Explain why it is not proportional to  $-2E < 0$  but proportional to  $2E > 0$ .

168. Using the length contraction, show that the number of electron in the whole space of volume  $V$  is 1.

169. Show that

$$\bar{u}(-p, s)u(-p, s') = -2m\delta_{ss'} \quad (1.233)$$

using  $u^\dagger(-p, s)u(-p, s') = 2m\delta_{ss'}$  and Lorentz covariance only.

170. Show that

$$\sum_s u(-p, s) \bar{u}(-p, s) = 2m \times \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = 2m \times \frac{1}{2}(-1 + \gamma^0). \quad (1.234)$$

171. Using Lorentz covariance, show that

$$\sum_s u(-p, s) \bar{u}(-p, s) = 2m \times \frac{\not{p} - m}{2m} = \not{p} - m. \quad (1.235)$$

172. Explain why this formula cannot be obtained if we substitute  $p \rightarrow -p$  to the positive energy case

$$\sum_s u(p, s) \bar{u}(p, s) = \not{p} + m. \quad (1.236)$$

173. Following the results for the charged scalar field, we would like to make use of the negative-energy solution for the positive energy antiparticle with momentum  $-(-p) = p$ . Then the wavefunction for the antiparticle with momentum  $p$  can be  $\bar{v}(p) \equiv \bar{u}(-p)$ . Show that

$$v^\dagger(p, s)v(p, s') = 2E\delta_{ss'}, \quad (1.237a)$$

$$\bar{v}(p, s)v(p, s') = -2m\delta_{ss'}, \quad (1.237b)$$

$$\sum_s v(p, s)\bar{v}(p, s) = \not{p} - m, \quad (1.237c)$$

where  $p = (E, \mathbf{p})$  and  $E = \sqrt{\mathbf{p}^2 + m^2} > 0$ .

174. Wavefunction for a positive-energy antiparticle with momentum  $p^\mu$  is

$$\bar{v}(p)e^{-ip\cdot x}. \quad (1.238)$$

175. For the final state, we use

$$v(p)e^{+ip \cdot x}. \quad (1.239)$$

176. Using Dirac equation, show that

$$(\not{p} - m)u(p) = 0, \quad (1.240a)$$

$$\bar{u}(p)(\not{p} - m) = 0, \quad (1.240b)$$

$$(\not{p} + m)v(p) = 0, \quad (1.240c)$$

$$\bar{v}(p)(\not{p} + m) = 0. \quad (1.240d)$$

177. Using the fact that  $-p \rightarrow p$  in the replacement  $v(p) = u(-p)$ , explain why the spin-up positive-energy positron state is expressed in terms of spin-down negative-energy electron state.

178. Formal way to prove this is involving charge-conjugation operation with the transformation matrix

$$i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (1.241)$$

to the spin-half Lagrangian.

179. Replacing the derivative in Dirac equation by the covariant derivative  $\mathcal{D}^\mu = \partial^\mu + iqA^\mu$ , show that

$$(i\not{\partial} - q\not{A} - m)\psi = 0. \quad (1.242)$$

Taking complex conjugate, show that

$$(-i\not{\partial}^* - q\not{A}^* - m)\psi^* = 0 \rightarrow [-\gamma^{\mu*}(i\partial_\mu + qA_\mu) - m]\psi^* = 0, \quad (1.243)$$

where  $A_\mu^* = A_\mu$ .

180. Using the fact

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (1.244)$$

and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.245)$$

show that

$$-\gamma^{\mu*} = \begin{cases} -\gamma^\mu & \text{if } \mu \neq 2, \\ +\gamma^\mu & \text{if } \mu = 2. \end{cases} \quad (1.246)$$

181. Show for

$$U = i\gamma^2, \quad (1.247)$$

that

$$U^2 = 1 \rightarrow U^{-1} = U^\dagger = U, \quad (1.248a)$$

$$U^{-1}\gamma^\mu U = -\gamma^\mu \rightarrow U\gamma^\mu = -\gamma^\mu U \text{ if } \mu \neq 2, \quad (1.248b)$$

$$U^{-1}\gamma^\mu U = \gamma^\mu \rightarrow U\gamma^\mu = \gamma^\mu U \text{ if } \mu = 2, \quad (1.248c)$$

Therefore,

$$i\gamma^2(-\gamma^{\mu*}) = \gamma^\mu(i\gamma^2). \quad (1.249)$$

182. Show that

$$i\gamma^2[-\gamma^{\mu*}(i\partial_\mu + qA_\mu) - m]\psi^* = 0 \rightarrow [\gamma^\mu(i\partial_\mu + qA_\mu) - m](i\gamma^2\psi^*) = 0. \quad (1.250)$$

Therefore,  $i\gamma^2\psi^*$  is the wavefunction for the antiparticle.

### 1.13 Fermion propagator

183. Show that the Dirac

$$(i\gamma^\mu\partial_\mu - m)\psi = J \quad (1.251)$$

Be careful with the sign in front of the source term on the right-hand side;  $H = i\partial^0$ .

184. Show that the propagator satisfies the equation

$$(i\gamma^\mu\partial_\mu - m)S_F(x) = I\delta(x) \quad (1.252)$$

where  $I$  is the  $4 \times 4$  identity matrix in spinor space.

185. Replacing  $S_F(x) = (i\gamma^\mu\partial_\mu + m)f(x)$ , show that  $f(x)$  satisfies the equation

$$(\partial^2 + m^2)f(x) = -\delta(x) \quad (1.253)$$

186. Show that the solution to the above equation is

$$S_F(x) = (i\gamma^\mu\partial_\mu + m)\Delta_F(x) \quad (1.254)$$

187. Show that

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{(\not{p} + m)e^{-ip\cdot x}}{p^2 - m^2 + i\epsilon} \quad (1.255)$$

188. Show that the Dirac

$$\bar{\psi} \left( -i\gamma^\mu \not{\partial}_\mu - m \right) = J \quad (1.256)$$

where  $A \not{\partial}_\mu \equiv \partial_\mu A$ . Be careful with the sign in front of the source term on the right-hand side;  $H = i\partial^0$ .

189. Show that the propagator satisfies the equation

$$S'_F(x) \left( -i\gamma^\mu \not{\partial}_\mu - m \right) = I\delta(x) \quad (1.257)$$

where  $S'_F$  is the propagator for the  $\bar{\psi}$  field. We will show that  $S'_F(x) \neq S_F(x)$

190. Replacing  $S'_F(x) = g(x) \left( -i\gamma^\mu \overleftrightarrow{\partial}_\mu + m \right)$ , show that  $g(x)$  satisfies the equation

$$(\partial^2 + m^2) g(x) = -\delta(x) \quad (1.258)$$

191. Show that the solution to the above equation is

$$S'_F(x) = \Delta_F(x) \left( -i\gamma^\mu \overleftrightarrow{\partial}_\mu + m \right) \quad (1.259)$$

192. Show that

$$S'_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{(-\not{p} + m) e^{-ip \cdot x}}{p^2 - m^2 + i\epsilon} \quad (1.260)$$

$$= \int \frac{d^4 p}{(2\pi)^4} \frac{(\not{p} + m) e^{ip \cdot x}}{p^2 - m^2 + i\epsilon} = S_F(-x) \neq S_F(x) \quad (1.261)$$

$$(e^{-ip \cdot x} \neq e^{ip \cdot x})$$

193. Show that  $S_F$  is not an even function

$$S_F(-x) \neq S_F(x), S_F(-p) \neq S_F(p) \quad (1.262)$$

194. Show that

$$\psi(x) = \int d^4 y S_F(x-y) J(y) = \int d^4 y S_F(x-y) [q\gamma^\mu A_\mu(y)] \psi(y), \quad (1.263)$$

$$\bar{\psi}(x) = \int d^4 y J(y) S_F(y-x) = \int d^4 y \bar{\psi}(y) [q\gamma^\mu A_\mu(y)] S_F(y-x) \quad (1.264)$$

195. Show that the positive-energy component propagates as

$$S_F(x) = \theta(x^0) S_{\text{ret.}}(x) + \theta(-x^0) S_{\text{adv.}}(x) \quad (1.265)$$

$$S_{\text{ret.}}(x) = -i \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\not{p} + m}{2E} e^{-ip \cdot x} \quad (1.266)$$

$$S_{\text{adv.}}(x) = -i \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{-\not{p} + m}{2E} e^{ip \cdot x} \quad (1.267)$$

where  $p^0 = E = \sqrt{m^2 + \mathbf{p}^2} > 0$ . From now on we neglect the normalization factor  $N = 1/\sqrt{V}$  which does not affect the invariant measurables like cross section.

196. **Transition amplitude  $\mathcal{M}$**  Once we know the transition amplitude  $\mathcal{M}$  in momentum space, we can calculate the cross section of a process.

197. For a scalar-exchange process

$$\mathcal{M} = \left[ -i\hat{J}(-p) \right] \frac{i}{p^2 - m^2 + i\epsilon} \left[ -i\hat{J}(p) \right] \quad (1.268)$$

198. For a photon-exchange process

$$\mathcal{M} = \left[ -i\hat{J}_\mu(-p) \right] \frac{i \left[ -g^{\mu\nu} + (1-\alpha) \frac{p^\mu p^\nu}{p^2} \right]}{p^2 + i\epsilon} \left[ -i\hat{J}_\nu(p) \right] \quad (1.269)$$

199. For a massive-vector-exchange process

$$\mathcal{M} = \left[ -i\hat{J}_\mu \right] \frac{i \left[ -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \right]}{p^2 - m^2 + i\epsilon} \left[ -i\hat{J}_\nu \right] \quad (1.270)$$

200. For a fermion-exchange process

$$\mathcal{M} = \left[ -i\hat{J} \right] \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \left[ -i\hat{J} \right] \quad (1.271)$$

# Chapter 2

## QCD Lagrangian

### 2.1 QED Summary

1. In the previous chapter we wrote the QED interaction Lagrangian, derived Feynman rules, and learned how to write the amplitude. QED Lagrangian is given by

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.1)$$

where the covariant derivative is defined by

$$\mathcal{D}^\mu = \partial^\mu + iqA^\mu \quad (2.2)$$

with  $q = Qe$ ,  $e = \sqrt{4\pi\alpha} > 0$  and  $Q = -1$  for the electron.

2. Wavefunctions for incoming positive-energy electron and positron with momentum  $p$  are

$$e^- : u(p)e^{-ip\cdot x}, \quad e^+ : \bar{v}(p)e^{-ip\cdot x}. \quad (2.3)$$

3. Wavefunctions for outgoing positive-energy electron and positron with momentum  $p$  are

$$e^- : \bar{u}(p)e^{+ip\cdot x}, \quad e^+ : v(p)e^{+ip\cdot x}. \quad (2.4)$$

4. Wavefunction for incoming and outgoing photons with momentum  $k$  are

$$\text{in} : \epsilon^\mu(k)e^{-ik\cdot x}, \quad \text{out} : \epsilon^{\mu*}(k)e^{+ik\cdot x}. \quad (2.5)$$

5. Propagator for the electron with momentum  $p$  is

$$iS_F(p) = \frac{i}{\not{p} - m + i\epsilon}. \quad (2.6)$$

6. Propagator for the photon has various forms depending on the gauge, which does not change the observables. In the Feynman gauge, the propagator for a photon with momentum  $k$  is

$$iD_F^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2 + i\epsilon}. \quad (2.7)$$

7. The vertex factor can be read from the interaction Lagrangian  $i\mathcal{L}_{\text{int}}$  as

$$\bar{e}A^\mu e = +iq\gamma^\mu, \quad q = -e. \quad (2.8)$$

8. Show that once the coupling  $q$  is defined by the covariant derivative, the coupling is same including sign for both electron and positron.

9. We can re-express the field strength tensor in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{1}{iq} [\mathcal{D}^\mu, \mathcal{D}^\nu], \quad \mathcal{D}^\mu = \partial^\mu + iqA^\mu. \quad (2.9)$$

10. Check the gauge invariance of the Lagrangian.

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{4(iq)^2} \text{Tr} ([\mathcal{D}^\mu, \mathcal{D}^\nu] [\mathcal{D}_\mu, \mathcal{D}_\nu]), \quad (2.10)$$

under the gauge transformation

$$\psi \rightarrow U\psi, \quad \mathcal{D}^\mu \rightarrow U\mathcal{D}^\mu U^\dagger. \quad (2.11)$$

Note that the trace is for the  $1 \times 1$  matrix  $U$ .

## 2.2 QCD Lagrangian

11. It is known that there are three ( $N_c = 3$ ) color states for a quark, which is a spin-half particle. The color is independent of spin and momentum.
12. We can extend QED to treat this new degree of freedom by introducing gauge fields mediating the color force between any two quarks. This can be done by declaring the spinor field  $\psi$  has the wavefunction of the form

$$\psi_{\text{quark}} = \psi_{\text{quark}}(\text{spin}) \times \psi_{\text{quark}}(\text{color}). \quad (2.12)$$

13. We can introduce the gauge transform for the quark just like that for the electron as

$$\psi \rightarrow U\psi, \quad \mathcal{D}^\mu \rightarrow U\mathcal{D}^\mu U^\dagger. \quad (2.13)$$

Note that the matrix  $U$  is now a  $3 \times 3$  matrix and acts only on the color wavefunction.

$$U\psi_{\text{quark}} = \psi_{\text{quark}}(\text{spin}) \times U\psi_{\text{quark}}(\text{color}). \quad (2.14)$$

14. We know that  $U$  must be unitary. Show that the matrix  $U$  can be parametrized by

$$U = e^{-i \sum_{a=1}^{N_c^2-1} T^a \alpha^a}, \quad (2.15)$$

where  $\alpha^a$  is real and  $T^a$ 's are the  $SU(N_c)$  generators.

15. Show that  $T^a$  is traceless and Hermitian.

16. Show that the number of generators for the  $SU(N_c)$  is  $N_c^2 - 1$ .

17. The covariant derivative can be generalized to the  $SU(N_c)$  as

$$\mathcal{D}^\mu = \partial^\mu + ig_s A^\mu, \quad (2.16)$$

where  $A^\mu$  is the matrix-valued gluon field

$$A^\mu = A_a^\mu T^a. \quad (2.17)$$

Note that  $g_s$  is the strong coupling. Therefore, gluons can have  $N_c^2 - 1 = 8$  different color states.

18. We can imagine the QCD Lagrangian should be of the form

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a, \quad (2.18)$$

because there are 8 gluons. Note that  $G_a^{\mu\nu}$  is the field strength tensor for the gluon with color  $a$ , which is a QCD analogy of the photon field strength tensor.

19. Let us check the gauge invariance of the Lagrangian. We can use the fact

$$\text{Tr} (T^a T^b) = \frac{1}{2} \delta^{ab}, \quad (2.19)$$

to derive

$$-\frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a = -\frac{1}{2} \text{Tr} (G^{\mu\nu} G_{\mu\nu}). \quad (2.20)$$

Again,  $G^{\mu\nu}$  is a matrix

$$G^{\mu\nu} = G_a^{\mu\nu} T^a. \quad (2.21)$$

20. Show that the covariant derivative transforms as

$$\mathcal{D}^\mu \rightarrow U \mathcal{D}^\mu U^\dagger, \quad \mathcal{D}^\mu \mathcal{D}^\nu \rightarrow U \mathcal{D}^\mu \mathcal{D}^\nu U^\dagger, \quad (2.22)$$

under the gauge transformation

$$\psi \rightarrow U \psi, \quad A^\mu \rightarrow U A^\mu U^\dagger - \frac{1}{ig_s} (\partial^\mu U) U^\dagger. \quad (2.23)$$

21. Show that the QCD Lagrangian

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{2} \text{Tr} (G^{\mu\nu} G_{\mu\nu}) \quad (2.24)$$

is invariant under the gauge transformation. We have constructed the QCD Lagrangian, which is gauge invariant and Lorentz invariant.

22. In the previous section, we have written the QED Lagrangian as the following.

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{4} G_a^{\mu\nu} G_{\mu\nu}^a. \quad (2.25)$$

Show that

$$G^{\mu\nu} = \frac{1}{ig_s} [\mathcal{D}^\mu, \mathcal{D}^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu + ig_s [A^\mu, A^\nu]. \quad (2.26a)$$

23. Show that

$$G_a^{\mu\nu} = 2\text{Tr}(G^{\mu\nu}T^a) = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + 2ig_s A_b^\mu A_c^\nu \text{Tr}\left(T^a[T^b, T^c]\right). \quad (2.27)$$

24. Using the identity  $[T^b, T^c] = if^{abc}T^a$ , show that

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_s f^{abc} A_b^\mu A_c^\nu. \quad (2.28)$$

25. Show that there exists three-gluon coupling in QCD.

26. Show that there exists four-gluon coupling in QCD.

## 2.3 Gauge Fixing in QED

27. We learned that we need gauge-fixing term such as

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \quad (2.29)$$

in the Lorentz gauge in order to derive the photon propagator.

28. Show in QED that the gauge transform changes the photon field as

$$A^\mu + \partial^\mu \chi, \quad (2.30)$$

if we choose

$$U = e^{-iq\chi}. \quad (2.31)$$

29. Show that the gauge condition  $\partial_\mu A^\mu = 0$  transforms as

$$\partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0. \quad (2.32)$$

There appears unpleasant term  $\partial_\mu \partial^\mu \chi$ .

30. Show that

$$\partial_\mu \partial^\mu \chi = 0, \quad (2.33)$$

if we want to keep the gauge condition  $\partial_\mu A^\mu = 0$ . Under the transform, gauge field is shifted to another gauge field which still satisfies the condition  $\partial_\mu A^\mu = 0$ .

31. Once we introduce a gauge-fixing term, our gauge transform cannot be completely general. The transform must keep the gauge condition in order for the Lagrangian to be gauge invariant. The gauge invariance is valid only in the group of gauges satisfying the same gauge condition.

32. Show that

$$\partial_\mu \partial^\mu \chi = 0, \quad (2.34)$$

is the equation of motion for the Lagrangian

$$\mathcal{L} = \partial^\mu \chi^\dagger \partial_\mu \chi. \quad (2.35)$$

We can include this term into the Lagrangian in order to correct the gauge invariance of  $\partial_\mu A^\mu$ .

33. Show that inclusion of this scalar field does not make any physical contribution in reality because it does not have any interaction term with any other fields.

## 2.4 Gauge-Fixing and Ghost Terms in QCD

34. We can introduce the same kind of gauge-fixing term like that of the Lorentz gauge in QED. We can try the QCD analogy of this

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\alpha} (\partial_\mu A_a^\mu)^2. \quad (2.36)$$

Note that there are  $N_c^2 - 1$  conditions  $\partial_\mu A_a^\mu = 0$  for  $a = 1, 2, \dots, N_c^2 - 1$ .

35. We learned that in QCD the gauge transform changes the gluon field as

$$A^\mu \rightarrow U A^\mu U^\dagger - \frac{1}{ig_s} (\partial^\mu U) U^\dagger, \quad (2.37)$$

where  $A^\mu = A_a^\mu T^a$ . Let us find how each gluon field  $A_a^\mu$  transforms in QCD.

36. It is convenient to choose the parametrization as

$$U = e^{-ig_s \alpha^a T^a}. \quad (2.38)$$

Note that  $\alpha^a$  should be real to preserve  $U$  unitary. Expanding the matrix  $U$  in powers of  $\alpha$  upto corrections of order  $\alpha^2$ , show that

$$A^\mu \rightarrow (1 - ig_s \alpha^c T^c) A_x^\mu T^x (1 + ig_s \alpha^c T^c) + \partial^\mu \alpha_a T^a \quad (2.39a)$$

$$= (A_a^\mu + \partial^\mu \alpha_a) T^a + ig_s A_x^\mu [T^x, T^c] \alpha_c \quad (2.39b)$$

$$= (A_a^\mu + \partial^\mu \alpha_a) T^a + ig_s A_x^\mu (if^{axc}) \alpha_c T^a. \quad (2.39c)$$

Therefore,  $A_a^\mu$  transforms as

$$A_a^\mu \rightarrow A_a^\mu + [\partial^\mu \delta_{ac} + ig_s A_x^\mu (if^{axc})] \alpha_c. \quad (2.40)$$

37. Let us introduce a matrix in the adjoint representation

$$(t^x)_{ac} \equiv if^{axc}. \quad (2.41)$$

Show that

$$A_a^\mu \rightarrow A_a^\mu + [\partial^\mu \delta_{ac} + ig_s A_x^\mu (t^x)_{ac}] \alpha_c = A_a^\mu + \tilde{\mathcal{D}}_{ac}^\mu \alpha_c, \quad (2.42)$$

where the covariant derivative  $\tilde{\mathcal{D}}_{ac}$  in the adjoint representation has the same form

$$\tilde{\mathcal{D}}_{ac}^\mu \equiv \partial^\mu \delta_{ac} - g_s A_x^\mu f^{axc} = \partial^\mu \delta_{ac} + ig_s \tilde{A}_{ac}^\mu, \quad \tilde{A}_{ac}^\mu \equiv A_x^\mu (t^x)_{ac} = A_x^\mu (if^{axc}). \quad (2.43)$$

as that in the fundamental representation.

38. The problem is resolved if we include the ghost term to the Lagrangian. One can find detailed discussion in most field theory text book such as Peskin.

$$\mathcal{L}_{\text{ghost}} = (\partial_\mu \bar{\eta}_a) \tilde{\mathcal{D}}_{ac}^\mu \eta_c = (\partial_\mu \bar{\eta}_a) (\partial^\mu \delta_{ac} - g_s f^{axc} A_x^\mu) \eta_c. \quad (2.44)$$

39. Ghost  $\eta$  is a complex scalar field. However, it behaves like a fermion in statistical sense. We will find later.

## 2.5 $SU(N_c)$ algebra summary

$SU(N_c)$  Generator  $T^a$ ,  $a = 1, \dots, N_c^2 - 1$

$$[T^a, T^b] = if^{abc}T^c \quad (2.45)$$

$$(t^b)_{ac} = if^{abc} \quad (2.46)$$

$$F_{ij} = F^a T_{ij}^a \leftarrow F^a = 2 \text{Tr}(FT^a), \quad \text{Tr}(T^a, T^b) = \frac{1}{2}\delta^{ab} \quad (2.47)$$

$$\begin{aligned} f^{axy}f^{bxy} &= N_c \delta^{ab} \\ &= -(t^a)_{xy}(t^b)_{xy} = (t^a)_{xy}(t^b)_{yx} = \text{Tr}(t^a t^b) \end{aligned} \quad (2.48)$$

## 2.6 QCD Feynman rules

40. we are ready to derive QCD Feynman rules from the QCD Lagrangian.

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \mathcal{D}^\mu - m) \psi - \frac{1}{4} (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_s f^{abc} A_b^\mu A_c^\nu) (\partial_\mu A_a^\mu - \partial_\nu A_a^\nu - g_s f^{apq} A_p^\mu A_q^\nu). \quad (2.49)$$

where the covariant derivatives are defined by

$$\mathcal{D}_{ij}^\mu = \delta_{ij} \partial^\mu + ig_s A_a^\mu T_{ij}^a \quad (2.50)$$

$$\mathcal{D}_{ac}^\mu = \delta_{ab} \partial^\mu + ig_s A_b^\mu (t^b)_{ac} = \delta_{ab} \partial^\mu - g_s G_b^\mu f^{abc} \leftarrow (t^b)_{ac} = if^{abc} \quad (2.51)$$

where indices  $i, j, k, \dots$  and  $a, b, c, \dots$  denote color indices for quark and gluon, respectively.

41. Let us recall the followings. Field strength tensor from fundamental representation

$$\begin{aligned} G^{\mu\nu} &= -\frac{i}{g_s} [\mathcal{D}^\mu, \mathcal{D}^\nu] = -\frac{i}{g_s} [\partial^\mu + ig_s A_a^\mu T_a, \partial^\nu + ig_s A_b^\nu T_b] \\ &= \partial^\mu A_a^\nu T_a - \partial^\nu A_a^\mu T_a + ig_s A_x^\mu A_y^\nu [T_x, T_y] \\ &= \partial^\mu A_a^\nu T^a - \partial^\nu A_a^\mu T^a - g_s A_x^\mu A_y^\nu f_{xya} T^a \leftarrow [T^a, T^b] = if^{abc} T^c \end{aligned} \quad (2.52)$$

$$G_a^{\mu\nu} = 2\text{Tr}(G^{\mu\nu} T^a) = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - g_s f_{abc} A_b^\mu A_c^\nu \quad (2.53)$$

$$(t^b)_{ac} = if^{abc} \quad (2.54)$$

42. Derivative

$$\partial_\mu \rightarrow -ip_\mu (\text{incoming line}), \quad \partial_\mu \rightarrow +ip_\mu (\text{outgoing line}) \quad (2.55)$$

43. Vertex

$$+i \mathcal{L} \quad (2.56)$$

44. Wavefunctions for incoming positive-energy quark and antiquark with momentum  $p$  are

$$q : u(p) e^{-ip \cdot x}, \quad \bar{q} : \bar{v}(p) e^{-ip \cdot x}, \quad (2.57)$$

where we have suppressed the color wavefunction. The color wavefunction is of the form

$$c_i, \quad c_i^\dagger c_j = \delta_{ij}, \quad i, i = 1, 2, 3. \quad (2.58)$$

45. Wavefunctions for outgoing positive-energy quark and antiquark with momentum  $p$  are

$$q : \bar{u}(p)e^{+ip\cdot x}, \quad \bar{q} : v(p)e^{+ip\cdot x}. \quad (2.59)$$

46. Wavefunction for incoming and outgoing gluons with momentum  $k$  are

$$\text{in} : \epsilon^\mu(k)e^{-ik\cdot x}, \quad \text{out} : \epsilon^{\mu*}(k)e^{+ik\cdot x}. \quad (2.60)$$

The color wavefunction for the gluon with color index  $a$  is

$$c_a, \quad c_a^\dagger c_b = \delta_{ab}, \quad i, i = 1, 2, \dots, 8. \quad (2.61)$$

47. Propagator for the quark with momentum  $p$  with initial and final color indices  $i$  and  $j$  is

$$iS_F(p) \times \delta_{ji} = \frac{i\delta_{ji}}{p - m + i\epsilon}. \quad (2.62)$$

The factor shows the color is preserved  $\delta_{ji}$ .

48. Propagator for the photon has various forms depending on the gauge, which does not change the observables. In the Feynman gauge, the propagator for a photon with momentum  $k$  is

$$iD_F^{\mu\nu}(k) \times \delta_{ba} = \frac{-ig^{\mu\nu}\delta_{ba}}{k^2 + i\epsilon}. \quad (2.63)$$

49. The  $\bar{q}_j A_a^\mu q_i$  vertex factor can be read from the interaction Lagrangian  $i\mathcal{L}_{\text{int}}$  as

$$\bar{q}_j A_a^\mu q_i : -ig_s T_{ji}^a \gamma^\mu. \quad (2.64)$$

50. Coupling  $g_s$  is universal for any quark.

### 2.6.1 Gluon Vertices

51. From the QCD Lagrangian

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)^2 - \frac{1}{2\alpha}(\partial \cdot A^a)^2 \\ & + \frac{1}{2}g_s f^{abc}(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu)A_\nu^b A_\nu^c - \frac{1}{4}g_s^2 f^{xab} f^{xcd} A_a^\mu A_b^\nu A_\mu^c A_\nu^d, \end{aligned} \quad (2.65)$$

derive the following Feynman rules.

52. gluon propagator

$$g : -i \frac{g^{\mu\nu} - \frac{\lambda-1}{\lambda} p^\mu p^\nu / p^2}{p^2 + i\epsilon} \quad (2.66)$$

53. Three-gluon vertex  $g_{c_1}^{\mu_1}(p_1 : \text{in}) - g_{c_2}^{\mu_2}(p_2 : \text{in}) - g_{c_3}^{\mu_3}(p_3 : \text{in})$  is

$$ggg : -g_s f^{c_1 c_2 c_3} ((p_1 - p_2)^{\mu_3} g^{\mu_1 \mu_2} + (p_2 - p_3)^{\mu_1} g^{\mu_2 \mu_3} + (p_3 - p_1)^{\mu_2} g^{\mu_3 \mu_1}). \quad (2.67)$$

54. Solution:

$$i\mathcal{L}_{ggg} = i \left[ -\frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} \right]_{ggg} = ig f^{abc} A_b^\mu A_c^\nu \partial_\mu A_\nu^a \quad (2.68)$$

$$= g \sum_{\text{perm.}\{1,2,3\}} f^{abc} \int d^4 p_1 d^4 p_2 d^4 p_3 e^{-i(p_1+p_2+p_3)\cdot x} (p_1 \cdot A_2^a) (A_1^b \cdot A_3^c) \quad (2.69)$$

$$\begin{aligned} &= -g \sum_{\text{perm.}\{1,2,3\}} f^{abc} \int d^4 p_1 d^4 p_2 d^4 p_3 e^{-i(p_1+p_2+p_3)\cdot x} \\ &\quad \times \left[ -(p_1 \cdot A_2^a) (A_1^b \cdot A_3^c) + (p_3 \cdot A_2^a) (A_1^b \cdot A_3^c) \dots \right] \\ &= \int d^4 p_1 d^4 p_2 d^4 p_3 e^{-i(p_1+p_2+p_3)\cdot x} A_{\mu_1}^a(p_1) A_{\mu_2}^b(p_2) A_{\mu_3}^c(p_3) \\ &\quad \times (-gf^{abc}) \cdot [g^{\mu_1\mu_2}(p_1-p_2)^{\mu_3} + g^{\mu_2\mu_3}(p_2-p_3)^{\mu_1} + g^{\mu_3\mu_1}(p_3-p_1)^{\mu_2}]. \quad (2.70) \end{aligned}$$

$(abc) = (c_1 c_2 c_3)$  If we choose the momentum direction into the vertex(annihilation at  $x$ ), momentum dependence is  $e^{-ik\cdot x}$  style and derivative can be replaced by  $-ik$  in momentum space.

55. Following the above way, derive the four-gluon vertex  $g_{c_1}^{\mu_1}(p_1 : \text{in}) - g_{c_2}^{\mu_2}(p_2 : \text{in}) - g_{c_3}^{\mu_3}(p_3 : \text{in}) - g_{c_4}^{\mu_4}(p_4 : \text{in})$  is

$$\begin{aligned} gggg : -ig_s^2 & [ +f^{c_1 c_2 x} f^{c_3 c_4 x} (g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3}) \\ & - f^{c_1 c_3 x} f^{c_2 c_4 x} (g^{\mu_1\mu_4} g^{\mu_2\mu_3} - g^{\mu_1\mu_2} g^{\mu_3\mu_4}) \\ & + f^{c_1 c_4 x} f^{c_2 c_3 x} (g^{\mu_1\mu_2} g^{\mu_3\mu_4} - g^{\mu_1\mu_3} g^{\mu_2\mu_4}) ]. \quad (2.71) \end{aligned}$$

56. Show the Bose symmetry of three- and four-gluon vertices explicitly.

## 2.6.2 Ghost Vertices

57. From the ghost term

$$\mathcal{L} = -\delta^{ac} \bar{\eta}_a \partial^2 \eta_c + g_s f_{abc} \bar{\eta}_a \partial_\mu A_b^\mu \eta_c, \quad (2.72)$$

derive the following rules

58.  $ghost(p)$  propagator

$$gh : \frac{i}{p^2 + i\epsilon} \quad (2.73)$$

59.  $ghost(p_f \text{out}, c_f) - g(\mu, c_g) - ghost(p_i \text{in}, \text{color} = c_i)$

$$gh_f - g - gh_i : +p_f^\mu g_s f^{c_f c_g c_i} \quad (2.74)$$

# Chapter 3

## SU(N)

In this chapter, we review the properties of the special unitary group, SU(N), which is useful in calculating color factors in various QCD processes.

### 3.1 Generators and structure constants

1. Consider a transform  $N \times N$  matrix  $U$  which transforms a matrix  $\mathcal{O}$  and a vector  $\psi$  in complex field as

$$\psi \rightarrow \psi' = U\psi \quad \text{and} \quad \mathcal{O} \rightarrow \mathcal{O}' = U\mathcal{O}U^{-1}. \quad (3.1)$$

Show that  $\psi^\dagger\psi$  is invariant under this transformation

$$\psi^\dagger\mathcal{O}\psi \rightarrow \psi^\dagger U^\dagger U \mathcal{O} U^{-1} U \psi = \psi^\dagger \mathcal{O} \psi, \quad (3.2)$$

if the transformation operator is unitary:

$$U^{-1} = U^\dagger. \quad (3.3)$$

When the transform is infinitesimal,  $U$  may be expressed as

$$U = e^{-i\epsilon_a T_a} = 1 - i\epsilon_a T_a, \quad (3.4)$$

where  $\epsilon_a$ 's are the infinitesimal real parameters and  $T_a$ 's are the generators of the transformation.

2. Show that  $iT^a$  must be antihermitian.
3. Show that  $T^a$  must be hermitian.
4. Show that  $T^a$  must be traceless. Since  $U$  is unitary,  $T_a$ 's are Hermitian. If we restrict  $\det U = +1$  as the case of the identity transformation, the  $T_a$  matrices are restricted to be traceless as

$$\begin{aligned} \det U &= \epsilon_{i_1 i_2, \dots, i_N} U_{1i_1} U_{2i_2} \dots U_{Ni_N} \\ &= \epsilon_{i_1 i_2, \dots, i_N} (\delta_{1i_1} - i\epsilon_a T_a^{1i_1})(\delta_{2i_2} - i\epsilon_a T_a^{2i_2}) \dots (\delta_{Ni_N} - i\epsilon_a T_a^{1i_N}) \\ &= \epsilon_{12, \dots, N} - i\epsilon_a (\epsilon_{i_1 i_2, \dots, i_N} (T_a^{1i_1} \delta_{1i_1} \dots \delta_{Ni_N} + \dots + \delta_{1i_1} \dots \delta_{N-1i_{N-1}} T_a^{1i_1})) \\ &= 1 - i\epsilon_a (\epsilon_{i_1 i_2, \dots, N} T_a^{1i_1} + \dots + \epsilon_{12, \dots, i_N} T_a^{Ni_N}) \\ &= 1 - i\epsilon_a \text{Tr} T_a \quad \rightarrow \text{Tr} T_a = 0 \quad \leftarrow \quad \det U = +1. \end{aligned} \quad (3.5)$$

5. Show that there are  $N_c^2 - 1$  free real parameters for  $\epsilon^a$ . Consider how many  $T_a$ 's are independent. Since a transformation matrix  $U$  is an  $N \times N$  complex matrix, there are  $2N^2$  parameters at first. Hermitian constraint discards  $N^2$  parameters and traceless condition does one more parameter so that we now have  $N^2 - 1$  independent  $T_a$ 's.
6. If  $N_2 = 2$ , the problem is the same as that for a spin-1/2 particle. Show that the generators for the SU(2) are Pauli sigma matrices. Are there  $N_c^2 - 1 = 3$  generators?
7. Consider a commutator  $-i[T_a, T_b]$ . It is a traceless and hermitian operator. Therefore it can be expressed as a linear combination of the generators as

$$[T_a, T_b] = i f_{abc} T_c, \quad (3.6)$$

where  $f^{abc}$  is the  $ab$  anti-symmetric evidently from the definition using commutator.

8. Show that  $f^{abc}$  is totally anti-symmetric.

$$\begin{aligned} f^{abc} &= f^{bca} = f^{cab} \\ &= -f^{bac} = -f^{cba} = -f^{acb} = \frac{1}{3!} (f^{abc} + f^{bca} + f^{cab} - f^{bac} - f^{cba} - f^{acb}). \end{aligned} \quad (3.7a)$$

You can prove it if you use the relations

$$\text{Tr}[AB] = \text{Tr}[BA] \rightarrow \text{Tr}[ABC] = \text{Tr}[BCA] = \text{Tr}[CAB], \quad (3.8)$$

which are valid for any matrices.

9. Since the trace of any two generator product is symmetric under the exchange of the color indices,

$$\text{Tr}[T^a T^b] = \text{Tr}[T^b T^a] = \frac{1}{2} (\text{Tr}[T^a T^b] + \text{Tr}[T^b T^a]), \quad (3.9)$$

$\text{Tr}[T^a T^b]$  is proportional to  $\delta^{ab}$  and the normalization can be chosen freely. Conventionally  $\lambda$  is set to 1/2.

$$\text{Tr}[T_a T_b] = \frac{1}{2} \delta_{ab}. \quad (3.10)$$

10. We can infer that the antisymmetric term in  $T_a T_b$  is proportional to  $i f_{abc} T_c$  from the commutation relation. Therefore we can express the product of any two generators as

$$T_a T_b = \frac{1}{2} [(A f_{abc} + B d_{abc}) T_c + C \delta_{ab} I], \quad (3.11)$$

where  $d_{abc}$  is  $ab$ -symmetric and  $A$ ,  $B$  and  $C$  are to be determined as follows. Directly from the commutation relation, we can find  $A = i$  and from the normalization condition,  $\text{Tr}[T_a T_b] = \delta_{ab}/2$ , we can find  $C = 1/N$ . Setting  $B = 1$  also, we get

$$\begin{aligned} T_a T_b &= \frac{1}{2} \left[ (i f_{abc} + d_{abc}) T_c + \frac{1}{N} \delta_{ab} I \right], \\ \{T_a, T_b\} &= d_{abc} T_c + \frac{1}{N} \delta_{ab} I. \end{aligned} \quad (3.12)$$

11. Then we find that  $d_{abc}$  is fully symmetric as

$$\begin{aligned}\text{Tr}[\{T_a, T_b\} T_c] &= d_{abd} \text{Tr}[T_d T_c] = \frac{1}{2} d_{abc} \leftarrow ab - \text{symmetric} \\ &= \text{Tr}[T_a T_b T_c + T_b T_a T_c] = \text{Tr}[T_b T_c T_a + T_b T_a T_c] \\ &= \text{Tr}[T_b \{T_a, T_c\}] \leftarrow ac - \text{symmetric}.\end{aligned}\quad (3.13)$$

12. And these are the useful relations

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}, \quad (3.14)$$

$$\text{Tr}[T^a T^b T^c] = \frac{1}{4} [d^{abc} + i f^{abc}], \quad (3.15)$$

$$d^{abc} = 2 \text{Tr}[\{T^a, T^b\} T^c], \quad (3.16)$$

$$f^{abc} = -2i \text{Tr}[[T^a, T^b] T^c], \quad (3.17)$$

$$\text{Tr}[T^a T^b T^c T^d] = \frac{1}{4} \left[ \frac{1}{N} \delta^{ab} \delta^{cd} + \frac{1}{2} (d^{abe} + i f^{abe})(d^{cde} + i f^{cde}) \right]. \quad (3.18)$$

## 3.2 Derivation of completeness relation

Any  $N \times N$  traceless and hermitian matrix  $\mathcal{A}$  can be expressed in the linear combination of the generators as

$$\mathcal{A} = \sum_{a=1}^{N^2-1} \alpha^a T^a, \quad (3.19)$$

where  $\alpha^a$ 's are real. There are  $N^2$  degrees of freedom in arbitrary  $N \times N$  hermitian matrices and there are  $N^2 - 1$  generators,  $T^a$ 's. Therefore, we need one more  $N \times N$  matrix to form a basis of the hermitian matrix other than the  $N^2 - 1$  generators,  $T^a$ 's. Since identity matrix is an hermitian and it is independent of the generators, we can form a basis by adding the identity matrix.

13. Show that any  $N \times N$  Hermitian matrix is expressed in a linear combination of the identity matrix and  $SU(N)$  generators.

$$\mathcal{H} = \alpha^0 I + \sum_{a=1}^{N^2-1} \alpha^a T^a, \quad (3.20)$$

where  $\alpha^i$  for  $i = 0, 1, \dots, N^2 - 1$  are all real. Then we can obtain useful relation from this completeness condition. By choosing the normalization

$$\text{Tr}[T_a T_b] = \frac{1}{2} \delta_{ab}, \quad (3.21)$$

we obtain the explicit values of the coefficients as

$$\text{Tr} \mathcal{H} = \alpha^0 N \quad \text{and} \quad \text{Tr}[T^a \mathcal{H}] = \frac{1}{2} \alpha^a. \quad (3.22)$$

14. Rewriting the hermitian matrix  $\mathcal{H}$

$$\mathcal{H} = \frac{1}{N} \text{Tr}[\mathcal{H}] \mathbf{I} + 2 \sum_{a=1}^{N^2-1} \text{Tr}[T^a \mathcal{H}] T^a, \quad (3.23)$$

in matrix representation, show that

$$\begin{aligned} \mathcal{H}_{\mu\nu} &= \frac{1}{N} \mathcal{H}_{\alpha\alpha} \delta_{\mu\nu} + 2 \sum_{a=1}^{N^2-1} T_{\mu\nu}^a T_{\alpha\beta}^a \mathcal{H}_{\beta\alpha}, \\ \mathcal{H}_{\beta\alpha} (\delta_{\mu\beta} \delta_{\nu\alpha}) &= \mathcal{H}_{\beta\alpha} \left( \frac{1}{N} \delta_{\alpha\beta} \delta_{\mu\nu} + 2 \sum_{a=1}^{N^2-1} T_{\mu\nu}^a T_{\alpha\beta}^a \right). \end{aligned} \quad (3.24)$$

15. Using the fact that  $\mathcal{H}_{\mu\nu}$  is an arbitrary complex number for any  $\mu > \nu$  and an arbitrary real number for any  $\mu = \nu$ , prove the completeness relation

$$\sum_{a=1}^{N^2-1} T_{\mu\nu}^a T_{\alpha\beta}^a = \frac{1}{2} \left( \delta_{\mu\beta} \delta_{\nu\alpha} - \frac{1}{N} \delta_{\mu\nu} \delta_{\alpha\beta} \right). \quad (3.25)$$

With this relation, we can calculate any color factor involving  $SU(N)$  gauge theory.

### 3.3 Useful trace formulas

We can derive various trace formulas and relations among the structure constants which are very useful in practical calculations concerning perturbative QCD. In this section, we derive these practically useful relations in detail, by using the results shown in previous sections.

From now on, we use summation convention, where any two repeated indices are assumed to be summed over color indices.

16. By using the completeness relation, we can show that the sum of squared generators is proportional to the identity matrix as

$$\begin{aligned} \left( \sum_{a=1}^{N^2-1} T^a T^a \right)_{\mu\nu} &= \sum_{a=1}^{N^2-1} T_{\mu\alpha}^a T_{\alpha\nu}^a = \frac{1}{2} \left( \delta_{\mu\nu} \delta_{\alpha\alpha} - \frac{1}{N} \delta_{\mu\alpha} \delta_{\alpha\nu} \right) \\ &= \frac{1}{2} \left( N \delta_{\mu\nu} - \frac{1}{N} \delta_{\mu\nu} \right), \\ \rightarrow \sum_{a=1}^{N^2-1} T^a T^a &= \frac{N^2 - 1}{2N} \mathbf{I}. \end{aligned} \quad (3.26)$$

For  $SU(N = 3)$ ,

$$C_F = \frac{N^2 - 1}{2N} = \frac{4}{3}. \quad (3.27)$$

The color factor appears in the quark wavefunction renormalization factor.

17. When we calculate gluon-loop corrections to fermion-gluon-fermion vertices, we need to evaluate the matrix such as

$$\sum_a T^a T^b T^a. \quad (3.28)$$

Using the completeness relation to re-order the matrix product and using the formula

$$\sum_a T^a T^a = \frac{N^2 - 1}{2N} I, \quad (3.29)$$

show that

$$T^a T^a T^b = \frac{N^2 - 1}{2N} T^b, \quad (3.30)$$

$$T^a T^b T^a = -\frac{1}{2N} T^b, \quad (3.31)$$

$$T^a T^a T^b T^b = \frac{(N^2 - 1)^2}{4N^2} I, \quad (3.32)$$

$$T^a T^b T^a T^b = -\frac{N^2 - 1}{4N^2} I, \quad (3.33)$$

$$T^a T^b T^b T^a = \frac{(N^2 - 1)^2}{4N^2} I. \quad (3.34)$$

18. We can derive the following trace formulas using the same method used above. Here are the SU(N) trace formulas up to 4 pairs of indices

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}, \quad (3.35)$$

$$\text{Tr}[T^a T^a T^b] = 0, \quad (3.36)$$

$$\text{Tr}[T^a T^b] \text{Tr}[T^a T^c] = \frac{1}{4} \delta^{bc}, \quad (3.37)$$

$$\text{Tr}[T^a T^a T^b T^c] = \frac{N^2 - 1}{4N} \delta^{bc}, \quad (3.38)$$

$$\text{Tr}[T^a T^b T^a T^c] = -\frac{1}{4N} \delta^{bc}, \quad (3.39)$$

$$\text{Tr}[T^a T^b T^c] \text{Tr}[T^a T^b T^d] = -\frac{1}{4N} \delta^{cd}, \quad (3.40)$$

$$\text{Tr}[T^a T^b T^c T^a T^b T^d] = \frac{N^2 + 1}{8N^2} \delta^{cd}, \quad (3.41)$$

$$\text{Tr}[T^a T^b T^c] \text{Tr}[T^b T^a T^d] = \frac{N^2 - 2}{8N} \delta^{cd}, \quad (3.42)$$

$$\text{Tr}[T^a T^b T^c T^b T^a T^d] = \frac{1}{8N^2} \delta^{cd}, \quad (3.43)$$

$$\text{Tr}[T^a T^a T^b T^b T^c T^d] = \frac{(N^2 - 1)^2}{8N^2} \delta^{cd}, \quad (3.44)$$

$$\text{Tr}[T^a T^a T^b T^c T^b T^d] = -\frac{N^2 - 1}{8N^2} \delta^{cd}, \quad (3.45)$$

$$\text{Tr}[T^a T^b T^a T^b T^c T^d] = -\frac{N^2 - 1}{8N^2} \delta^{cd}. \quad (3.46)$$

### 3.4 Adjoint representation

In this section we investigate the properties of the structure constant  $f^{abc}$ 's and symmetric  $d^{abc}$ 's. And we will look into the adjoint representation which is made up of the structure constant  $f^{abc}$  itself.

Compare the above identity with the one driven in previous section,

$$T^a T^b = \frac{1}{2} \left[ \frac{1}{N} \delta^{ab} \mathbf{I} + (d^{abc} + i f^{abc}) T^c \right]. \quad (3.47)$$

19. If we multiply  $\delta^{ab}$  and sum over color indices, then we get the properties of the structure constant and symmetric  $d^{abc}$  as

$$f^{aab} = 0 \quad \text{and} \quad d^{aab} = 0. \quad (3.48)$$

20. If we use Eq.(3.47) two times we get

$$\begin{aligned} T^a T^b T^c &= \frac{1}{4N} [d^{abc} + i f^{abc}] \mathbf{I} \\ &+ \frac{1}{2} \left[ \frac{1}{N} \delta^{ab} \delta^{ce} + \frac{1}{2} (d^{abd} + i f^{abd}) (d^{dce} + i f^{dce}) \right] T^e. \end{aligned} \quad (3.49)$$

Let us consider triple product  $T^a T^b T^c$ . By using Eq. (3.26) we get

$$\begin{aligned} T^a T^a T^b &= \frac{N^2 - 1}{2N} T^b \quad \text{and} \quad T^a T^b T^a = -\frac{1}{2N} T^b \\ \rightarrow T^a [T^a, T^b] &= \frac{N}{2} T^b \quad \text{and} \quad T^a \{T^a, T^b\} = \frac{N^2 - 2}{2N} T^b. \end{aligned} \quad (3.50)$$

Comparing the result by using the Eqs.(3.6), (3.12)

$$\begin{aligned} T^a [T^a, T^b] &= T^a i f^{abc} T^c = \frac{i}{2} f^{abc} [T^a, T^c] = \frac{i}{2} f^{abc} i f^{ace} T^e = \frac{1}{2} f^{acb} f^{ace} T^e, \\ T^a \{T^a, T^b\} &= T^a \left( d^{abc} T^c + \frac{1}{N} \delta^{ab} \mathbf{I} \right) = \frac{1}{2} d^{abc} \{T^a, T^c\} + \frac{1}{N} T^b \\ &= \frac{1}{2} d^{abc} d^{ace} T^e + \frac{1}{N} T^b, \end{aligned} \quad (3.51)$$

show that

$$f^{abc} f^{abd} = N \delta^{cd} \quad (3.52a)$$

$$f^{abc} d^{abd} = 0 \quad (3.52b)$$

$$d^{abc} d^{abd} = \frac{N^2 - 4}{N} \delta^{cd}. \quad (3.52c)$$

21. With the identities and Eqs. (3.16) and (3.17) we obtain other relations

$$\begin{aligned} d^{abc} f^{abd} &= 0, \\ d^{abc} d^{abc} &= \frac{(N^2 - 1)(N^2 - 4)}{N}, \\ f^{abc} f^{abc} &= N(N^2 - 1). \end{aligned} \quad (3.53)$$

22. The structure constants  $f^{abc}$  themselves define adjoint representation

$$F_{ac}^b \equiv i f_{abc}. \quad (3.54)$$

23. Show that  $F^b$ 's are traceless, hermitian, and anti-symmetric  $(N^2 - 1) \times (N^2 - 1)$  matrices.

24. By using the Jacobi identity, show that

$$\begin{aligned} [[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] &= 0, \\ i f^{abd} [T^d, T^c] + i f^{bcd} [T^d, T^a] + i f^{cad} [T^d, T^b] &= 0, \\ i f^{abd} i f^{dce} T^e + i f^{bcd} i f^{dae} T^e + i f^{cad} i f^{dbe} T^e &= 0, \\ (-i f^{abd}) (-i f^{dce}) + (-i f^{bcd}) (-i f^{dae}) + (-i f^{cad}) (-i f^{dbe}) &= 0, \\ (-i f^{abd}) (-i f^{dce}) - (-i f^{adc}) (-i f^{ebd}) &= i f^{aed} (-i f^{dbc}), \\ (F^a F^e)_{bc} - (F^e F^a)_{bc} &= i f^{aed} F_{bc}^d \\ \rightarrow [F^a, F^b] &= i f^{abc} F^c. \end{aligned} \quad (3.55)$$

25. Defining the fully symmetric  $d^{abc}$  likewise ( $D_{bc}^a = d^{abc}$ ), we get

$$\begin{aligned} [\{T^a, T^b\}, T^c] + [\{T^b, T^c\}, T^a] + [\{T^c, T^a\}, T^b] &= 0, \\ d^{abd} [T^d, T^c] + d^{bcd} [T^d, T^a] + d^{cad} [T^d, T^b] &= 0, \\ d^{abd} i f^{dce} T^e + d^{bcd} i f^{dae} T^e + d^{cad} i f^{dbe} T^e &= 0, \\ d^{abd} i f^{edc} - d^{adc} i f^{ebd} &= -d^{bcd} i f^{dae}, \\ -(D^a F^e)_{bc} + (F^e D^a)_{bc} &= i f^{ead} D_{bc}^d, \\ \rightarrow [F^a, D^b] &= i f^{abc} D^c. \end{aligned} \quad (3.56)$$

26. Formulas related to the structure constants can be rewritten in terms of adjoint representation as

$$f^{aab} = 0 \rightarrow \text{Tr } F^b = 0, \quad (3.57)$$

$$d^{aab} = 0 \rightarrow \text{Tr } D^b = 0, \quad (3.58)$$

$$f^{abc} f^{abd} = (-i f^{cab}) (-i f^{dba}) = \text{Tr}[F^c F^d] = N \delta^{cd}, \quad (3.59)$$

$$f^{abc} f^{abd} = (-i f^{acb}) (-i f^{abd}) = \sum_{a=1}^{N^2-1} (F^a F^a)_{cd} = NI_{cd}, \quad (3.60)$$

$$d^{abc} f^{abd} = 0 \rightarrow \text{Tr}[D^c F^d] = 0, \quad (3.61)$$

$$d^{abc} f^{abd} = 0 \rightarrow \sum_{a=1}^{N^2-1} (D^a F^a)_{cd} = 0_{cd}, \quad (3.62)$$

$$d^{abc} d^{abd} = d^{cab} d^{dba} = \text{Tr}[D^c D^d] = \frac{N^2 - 4}{N} \delta^{cd}, \quad (3.63)$$

$$d^{abc} d^{abd} = d^{acb} d^{abd} = \sum_{a=1}^{N^2-1} (D^a D^a)_{cd} = \frac{N^2 - 4}{N} I_{cd}. \quad (3.64)$$

Now we summarize the results concerning the two representations. Regardless of the two representations

$$T(R) \delta_{ab} = \text{Tr}[T_a T_b] \quad \text{and} \quad C_2(R) I = \sum_{a=1}^{N^2-1} T_a^{(R)} T_a^{(R)}, \quad (3.65)$$

where  $I$  is the identity matrix in the representation  $R$  and the label  $R$  is given as

$$T_a^{(F)} = T_a \quad \text{and} \quad T_a^{(A)} = F_a. \quad (3.66)$$

We can set the normalization of the generators by setting the value of  $T(R)$ . Standard normalizations are given as

$$T(R) = \frac{1}{2} \quad \text{and} \quad T(R) = N. \quad (3.67)$$

Then the values of  $C_2(R)$  are fixed as

$$C_2(R) = \frac{N^2 - 1}{2N} \quad \text{and} \quad C_2(R) = N. \quad (3.68)$$

### 3.5 SU(3) Clebsch-Gordan coefficients

27. As the meson formed from  $q\bar{q}$  into singlet and octet in flavor SU(3), the color state formed by  $Q$  and  $\bar{Q}$  with color  $i$  and  $j$  form a color singlet and a color octet states as

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}. \quad (3.69)$$

Obviously the singlet state is the identity matrix and octet states are the eight generators of SU(3) in the fundamental representation up to normalization as

$$\langle 3i; \bar{3}j | 1 \rangle = N_1 \delta_{ij} \quad \text{and} \quad \langle 3i; \bar{3}j | 8a \rangle = N_8 T_{ij}^a. \quad (3.70)$$

The octet coefficient is proportional to the physical vertex ; gluon with color  $a$  goes to a color  $i$  quark and a color  $j$  antiquark. One should be cautious not to confuse the order of indices  $i$  and  $j$  in above equations.

Normalizing the nine states as

$$\begin{aligned} \langle 1 | 1 \rangle &= \sum_{ij} \langle 1 | 3i; \bar{3}j \rangle \langle 3i; \bar{3}j | 1 \rangle \\ &= |N_1|^2 \sum_{ij} \delta_{ji} \delta_{ij} \\ &= |N_1|^2 \text{Tr}[II] = 3|N_1|^2 = 1, \end{aligned} \quad (3.71)$$

$$\begin{aligned} \langle 8a | 8a \rangle &= \sum_{ij} \langle 1 | 3i; \bar{3}j \rangle \langle 3i; \bar{3}j | 1 \rangle \quad (\text{not sum over index } a) \\ &= |N_8|^2 \sum_{ij} T_{ij}^{a*} T_{ij}^a \\ &= |N_8|^2 \sum_{ij} T_{ji}^a T_{ij}^a \\ &= |N_8|^2 \text{Tr}[T^a T^a] = \frac{1}{2} |N_8|^2 = 1. \end{aligned} \quad (3.72)$$

If we perform the same derivation for the case of SU(N), then we get

$$\langle 3i; \bar{3}j | 1 \rangle = \frac{1}{\sqrt{N}} \delta_{ij} \quad \text{and} \quad \langle 3i; \bar{3}j | 8a \rangle = \sqrt{2} T_{ij}^a, \quad (3.73)$$

provided that the generators of the fundamental representation are normalized as

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}. \quad (3.74)$$

28. To find the color factor for singlet and octet in this case, we use this kind of method: Since any product of color matrices are expressed as

$$T_n \equiv T^{a_1} T^{a_2} T^{a_3} \cdots T^{a_n} = A(a_1, a_2, \dots, a_n) + B^a(a_1, a_2, \dots, a_n) T^a \quad (3.75)$$

$$A(a_1, a_2, \dots, a_n) = \frac{1}{N} \text{Tr}(T_n) \quad (3.76)$$

$$B^a(a_1, a_2, \dots, a_n) = 2 \text{Tr}(T^a T_n) \quad (3.77)$$

Then the total color factor is

$$\begin{aligned} C_T \equiv \text{Tr} T_n T_n^\dagger &= A(a_1, a_2, \dots, a_n) A^*(a_1, a_2, \dots, a_n) \text{Tr}(I) \\ &+ B^a(a_1, a_2, \dots, a_n) B^{*b}(a_1, a_2, \dots, a_n) \text{Tr}(T^a T^b) \\ &= A(a_1, a_2, \dots, a_n) A^*(a_1, a_2, \dots, a_n) N \\ &+ B^a(a_1, a_2, \dots, a_n) B^{*b}(a_1, a_2, \dots, a_n) \frac{1}{2} \delta^{ab} \\ &= N A(a_1, a_2, \dots, a_n) A^*(a_1, a_2, \dots, a_n) \\ &+ \frac{1}{2} B^a(a_1, a_2, \dots, a_n) B^{*a}(a_1, a_2, \dots, a_n) \end{aligned} \quad (3.78)$$

And the singlet and octet factors are

$$\begin{aligned} C_1 &= A(a_1, a_2, \dots, a_n) A^*(a_1, a_2, \dots, a_n) \\ &= \frac{1}{N^2} \text{Tr}(T_n) \text{Tr}(T_n) \end{aligned} \quad (3.79)$$

$$\begin{aligned} C_8 \delta^{ab} &= B^a(a_1, a_2, \dots, a_n) B^b(a_1, a_2, \dots, a_n) \\ &= \frac{\delta^{ab}}{N^2 - 1} B^a(a_1, a_2, \dots, a_n) B^a(a_1, a_2, \dots, a_n) \end{aligned} \quad (3.80)$$

$$\begin{aligned} C_8 &= \frac{1}{N^2 - 1} B^a(a_1, a_2, \dots, a_n) B^a(a_1, a_2, \dots, a_n) \\ &= \frac{4}{N^2 - 1} \text{Tr}(T^a T_n) \text{Tr}(T^a T_n) \end{aligned} \quad (3.81)$$

Therefore there is a relation among the total color factor, singlet and octet color factor

$$C_T = NC_1 + \frac{1}{2}(N^2 - 1)C_8 \quad (3.82)$$

## 3.6 Examples

29. In calculations of QCD amplitudes such as  $gg \rightarrow gg$ , we have to calculate color factors such as

$$f^{abc} f^{axy} f^{dbx} f^{dcz}, \quad (3.83)$$

which is very involved.

$$f^{abc} f^{axy} f^{dbx} f^{dcz} = +\frac{1}{2} N^2 \delta^{yz} \quad (3.84)$$

$$d^{abc} f^{axy} f^{dbx} f^{dcz} = 0 \quad (3.85)$$

$$d^{abc} d^{axy} f^{dbx} f^{dcz} = +\frac{1}{2} (N^2 - 4) \delta^{yz} \quad (3.86)$$

$$d^{abc} f^{axy} d^{dbx} f^{dcz} = -\frac{1}{2} (N^2 - 4) \delta^{yz} \quad (3.87)$$

$$d^{abc} d^{axy} d^{dbx} f^{dcz} = 0 \quad (3.88)$$

$$d^{abc} d^{axy} d^{dbx} d^{dcz} = +\frac{1}{2N^2} (N^2 - 4)(N^2 - 12) \delta^{yz} \quad (3.89)$$

where the last formula is not derived directly from the relations given above, instead, with

$$\begin{aligned} & 4^4 \times \text{Tr}(T^a T^b T^c) \text{Tr}(T^a T^x T^y) \text{Tr}(T^d T^b T^x) \text{Tr}(T^d T^c T^z) \\ &= (d^{abc} + i f^{abc})(d^{axy} + i f^{axy})(d^{dbx} + i f^{dbx})(d^{dcz} + i f^{dcz}) \end{aligned} \quad (3.90)$$

and the remaining formulas. And the relation is invariant under cyclic rotations such as

$$\begin{aligned} A^{abc} B^{axy} C^{dbx} D^{dcz} &= B^{abc} C^{axy} D^{dbx} A^{dcz} \\ &= C^{abc} D^{axy} A^{dbx} B^{dcz} \\ &= D^{abc} A^{axy} B^{dbx} C^{dcz} \end{aligned} \quad (3.91)$$

30. The completeness relation is the most powerful tool in calculating color factors. You are advised to transform any color factors into the fundamental representation first. Then using the completeness relation to reorder the color matrices so that you can express the color factor as a linear combination of products of color factors made of

$$\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}. \quad (3.92)$$

Then the remaining calculations are products of

$$\delta^{aa} = N^2 - 1 \quad \text{and/or} \quad \delta^{ii} = N. \quad (3.93)$$

# Chapter 4

## Tree-Level Calculation

### 4.1 Cross section formula Summary

1. Discuss the difference between the two matrices  $\mathcal{T}$  and  $\mathcal{M}$ .

$$S = 1 + i\mathcal{T}, \quad (4.1)$$

2. In the transition matrix  $\mathcal{T}$ , scalar wavefunction factor

$$e^{-ip \cdot x} \quad (4.2)$$

for each particle is included.

3. When we obtain the transition amplitude  $\mathcal{M}$ , position variable in  $\mathcal{T}$  for each vertex is integrated out to give energy-momentum conservation delta function

$$\int d^4x e^{i \sum_i p_i \cdot x} = (2\pi)^4 \delta^{(4)} \left( \sum_i p_i \right). \quad (4.3)$$

4. When we square the matrix  $\mathcal{T}$ , we treat the space-time volume finite  $TV$ .

$$\left[ (2\pi)^4 \delta^{(4)} \left( \sum_i p_i \right) \right]^2 = TV \times (2\pi)^4 \delta^{(4)} \left( \sum_i p_i \right) \quad (4.4)$$

5. In the transition matrix  $\mathcal{T}$ , each external particles have the normalization to get

$$\int \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) d^3x = \frac{2E}{V} \times V = 2E, \quad (4.5)$$

6. Show that the cross section formula for the following process

$$A(p_1, M_1) + B(p_2, M_2) \rightarrow f_1(q_1, m_1) + f_2(q_2, m_2) + \cdots + f_n(q_n, m_n), \quad (4.6)$$

is given by

$$d\sigma = \frac{\sum \mathcal{M}^2}{\text{Flux}} d\Phi_n. \quad (4.7a)$$

$$\text{Flux} = 4\sqrt{(p_1 + p_2)^2 - M_1^2 M_2^2}, \quad (4.7b)$$

$$d\Phi_n = (2\pi)^4 \delta^{(4)} \left( p_1 + p_2 - \sum_{k=1}^n q_k \right) \Pi_{k=1}^n \frac{d\mathbf{q}_k}{2E_i(2\pi)^3}, \quad (4.7c)$$

$$E_k = \sqrt{m_k^2 + \mathbf{q}_k^2}. \quad (4.7d)$$

The notation  $\overline{\sum}$  stands for

$$\frac{1}{(2J_A + 1)(2J_B + 1)N_A(\text{color})N_B(\text{color})} \sum_{\text{spin}_A} \sum_{\text{color}_A} \sum_{\text{spin}_B} \sum_{\text{color}_B} \Pi_{k=1}^n \sum_{\text{spin}_k} \sum_{\text{color}_k} \quad (4.8)$$

summation over all the final states and averaging over all the initial states. The states includes all possible degress of freedom such as spin and color.

7. Show that the following average factors

$$e^+ e^- \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{e^+ \text{-spin}} \left(\frac{1}{2}\right)_{e^- \text{-spin}} = \frac{1}{4}, \quad (4.9a)$$

$$e^- \gamma \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{e^- \text{-spin}} \left(\frac{1}{2}\right)_{\gamma \text{-spin}} = \frac{1}{4}, \quad (4.9b)$$

$$\gamma\gamma \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{\gamma \text{-spin}} \left(\frac{1}{2}\right)_{\gamma \text{-spin}} = \frac{1}{4}, \quad (4.9c)$$

$$q\bar{q} \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{q \text{-spin}} \left(\frac{1}{3}\right)_{q \text{-color}} \left(\frac{1}{2}\right)_{\bar{q} \text{-spin}} \left(\frac{1}{3}\right)_{\bar{q} \text{-color}} = \frac{1}{36}, \quad (4.9d)$$

$$qg \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{q \text{-spin}} \left(\frac{1}{3}\right)_{q \text{-color}} \left(\frac{1}{2}\right)_{g \text{-spin}} \left(\frac{1}{8}\right)_{g \text{-color}} = \frac{1}{96}, \quad (4.9e)$$

$$gg \rightarrow f_1 + \dots + f_n = \left(\frac{1}{2}\right)_{g \text{-spin}} \left(\frac{1}{8}\right)_{g \text{-color}} \left(\frac{1}{2}\right)_{g \text{-spin}} \left(\frac{1}{8}\right)_{g \text{-color}} = \frac{1}{256}. \quad (4.9f)$$

## 4.2 $e^+ e^- \rightarrow \mu^+ \mu^-$

8. Neglecting masses, show that

$$\sigma(e^+ e^- \rightarrow \mu^+ \mu^-) = \frac{4\pi\alpha^2}{3s}. \quad (4.10)$$

9. Draw the Feynman diagram in the leading order in  $\alpha$ .

10. Draw the Feynman diagram in the next-to-leading order in  $\alpha$ .

11. Write the transition amplitude for the process  $e^+(p_1)e^-(p_2) \rightarrow \mu^+(q_1)\mu^-(q_2)$  in the light-cone gauge,  $n \cdot A = 0$  with  $n^2 = 0$ .

$$\begin{aligned} -i\mathcal{M} &= [(+ie)\bar{v}(p_1)\gamma_\mu u(p_2)] \times (+i) \frac{-g^{\mu\nu} + \frac{(p_1+p_2)^\mu n^\nu + n^\mu (p_1+p_2)^\nu}{n \cdot (p_1+p_2)}}{(p_1+p_2)^2 + i\epsilon} \\ &\quad \times [(+ie)\bar{u}(q_2)\gamma_\nu v(q_1)]. \end{aligned} \quad (4.11a)$$

12. Write the same amplitude in the covariant gauge  $\partial \cdot A = 0$ .

$$\begin{aligned} -i\mathcal{M} &= [(+ie)\bar{v}(p_1)\gamma_\mu u(p_2)] \times (+i) \frac{-g^{\mu\nu} + (\frac{1}{\alpha} - 1) \frac{(p_1+p_2)^\mu (p_1+p_2)^\nu}{(p_1+p_2)^2}}{(p_1+p_2)^2 + i\epsilon} \\ &\quad \times [(+ie)\bar{u}(q_2)\gamma_\nu v(q_1)]. \end{aligned} \quad (4.12a)$$

13. Write the same amplitude in the Feynman gauge  $\partial \cdot A = 0$  with gauge-fixing term  $-\frac{1}{2\alpha}(\partial \cdot A)^2$  and  $\alpha = 1$ .

$$\begin{aligned} -i\mathcal{M} &= [(+ie)\bar{v}(p_1)\gamma_\mu u(p_2)] \times (+i) \frac{-g^{\mu\nu}}{(p_1+p_2)^2 + i\epsilon} \\ &\quad \times [(+ie)\bar{u}(q_2)\gamma_\nu v(q_1)]. \end{aligned} \quad (4.13a)$$

14. Explain why the three amplitudes must give the same answer.

15. What happens to the gauge-dependent terms proportional to

$$(p_1+p_2)^\mu (p_1+p_2)^\nu \quad (4.14a)$$

$$(p_1+p_2)^\mu n^\nu + n^\mu (p_1+p_2)^\nu \quad (4.14b)$$

in the propagator in the long run?

16. Show that the squared amplitude in the Feynman gauge becomes

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{[(p_1+p_2)^2]^2} \text{Tr} [\gamma^\mu (\not{p}_2 + m_e) \gamma^\nu (\not{p}_1 - m_e)] \text{Tr} [\gamma^\mu (\not{q}_2 + m_\mu) \gamma^\nu (\not{q}_1 - m_\mu)]. \quad (4.15)$$

17. Using

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad (4.16)$$

show that

$$\text{Tr} [\not{a}\not{b}] = 4a \cdot b, \quad (4.17a)$$

$$\text{Tr} [\not{a}\not{b}\not{c}\not{d}] = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]. \quad (4.17b)$$

18. Neglecting  $m_e$  and  $m_\mu$ , show that

$$\sum_{\text{spin}} |\mathcal{M}|^2 = \frac{16e^4}{[(p_1+p_2)^2]^2} [p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu} p_1 \cdot p_2] [q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - g^{\mu\nu} q_1 \cdot q_2]. \quad (4.18)$$

19. Using four-momentum conservation

$$p_1 + p_2 = q_1 + q_2, \quad (4.19)$$

show that

$$s \equiv (p_1 + p_2)^2 = (q_1 + q_2)^2, \quad (4.20a)$$

$$t \equiv (p_1 - q_1)^2 = (p_2 - q_2)^2, \quad (4.20b)$$

$$u \equiv (p_1 - q_2)^2 = (p_2 - q_1)^2, \quad (4.20c)$$

$$s + t + u = 2m_e^2 + 2m_\mu^2 \rightarrow 0. \quad (4.20d)$$

20. Show that

$$p_1^\mu p_2^\nu [q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - g^{\mu\nu} q_1 \cdot q_2] = \frac{1}{4}(t^2 + u^2 - s^2), \quad (4.21a)$$

$$p_2^\mu p_1^\nu [q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - g^{\mu\nu} q_1 \cdot q_2] = \frac{1}{4}(t^2 + u^2 - s^2), \quad (4.21b)$$

$$-g^{\mu\nu} p_1 \cdot p_2 [q_1^\mu q_2^\nu + q_2^\mu q_1^\nu - g^{\mu\nu} q_1 \cdot q_2] = \frac{s}{2} \left( -\frac{s}{2} - \frac{s}{2} + 4 \times \frac{s}{2} \right) = \frac{s^2}{2}. \quad (4.21c)$$

21. Using above relations, show that

$$\begin{aligned} \sum |\mathcal{M}|^2 &= \frac{16e^4}{s^2} \frac{1}{2}(t^2 + u^2) \\ &= \frac{8(4\pi)^2 \alpha^2}{s^2} (t^2 + u^2) \leftarrow e^2 = 4\pi\alpha, \end{aligned} \quad (4.22a)$$

$$\overline{\sum} |\mathcal{M}|^2 = \frac{1}{4} \sum |\mathcal{M}|^2 = \frac{2(4\pi)^2 \alpha^2}{s^2} (t^2 + u^2). \quad (4.22b)$$

22. Show that the colliding flux is

$$\text{Flux} = 4\sqrt{(p_1 + p_2)^2} = 2s. \quad (4.23)$$

23. Show that the phase space for the two-body final state is Lorentz invariant.

24. Show that the phase space for the two-body final state in the center-of-momentum frame of the colliding pair becomes

$$d\Phi_2 = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \frac{d^3 q_1}{2E_1(2\pi)^3} \frac{d^3 q_2}{2E_2(2\pi)^3} \quad (4.24a)$$

$$= \frac{d^3 q^*}{4E_1^* E_2^* (2\pi)^2} \delta(\sqrt{s} - E_1^* - E_2^*), \quad (4.24b)$$

$$E_i^* = \sqrt{m_\mu^2 + \mathbf{q}^{*2}} = |\mathbf{q}^*|, \quad (4.24c)$$

$$\sqrt{s} = E_1^* + E_2^*, \quad (4.24d)$$

where  $\mathbf{q}^*$  is the momentum  $\mathbf{q}_1$  measured in the center-of-momentum frame.

25. Neglecting azimuthal (angle  $\phi$ ) dependence, we can write

$$d^3\mathbf{q}^* = |\mathbf{q}^*|^2 d|\mathbf{q}^*| d\Omega^* = 2\pi |\mathbf{q}^*|^2 d|\mathbf{q}^*| d\cos\theta^*. \quad (4.25)$$

Using the relation,

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}, \quad (4.26)$$

for a function made of single zeros at  $x = x_i$ , show that

$$\begin{aligned} \int d^3\mathbf{q}^* \delta(\sqrt{s} - E_1^* - E_2^*) &= \int |\mathbf{q}^*|^2 d|\mathbf{q}^*| d\Omega^* \delta(\sqrt{s} - E_1^* - E_2^*) \\ &= \frac{|\mathbf{q}^*|^2 d\Omega^*}{\frac{|\mathbf{q}^*|}{E_1^*} + \frac{|\mathbf{q}^*|}{E_2^*}} = E_1^* E_2^* \frac{|\mathbf{q}^*|}{\sqrt{s}} d\Omega^*. \end{aligned} \quad (4.27a)$$

26. Show finally that

$$d\Phi_2 = \frac{1}{8\pi} \times \frac{2|\mathbf{q}^*|}{\sqrt{s}} \times \frac{d\Omega^*}{4\pi}. \quad (4.28)$$

27. For massless final states, show that

$$d\Phi_2 = \frac{1}{8\pi} \times \frac{d\Omega^*}{4\pi} = \frac{d\cos\theta^*}{16\pi}. \quad (4.29)$$

28. If the amplitude is independent of  $\Omega^*$  (rotationally symmetric), show that

$$d\Phi_2 = \frac{1}{8\pi}. \quad (4.30)$$

29. Combining previous results, show that

$$\sigma = \left( \frac{1}{2s} \right)_{\text{Flux}} \int_{-1}^1 \frac{d\cos\theta^*}{16\pi} \left[ \frac{2(4\pi)^2 \alpha^2}{s^2} (t^2 + u^2) \right]_{\sum |\mathcal{M}|^2}. \quad (4.31a)$$

30. In the center-of-momentum frame, show that

$$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad (4.32a)$$

$$p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad (4.32b)$$

$$q_1 = \frac{\sqrt{s}}{2}(1, \sin\theta^* \cos\phi^*, \sin\theta^* \sin\phi^*, \cos\theta^*), \quad (4.32c)$$

$$q_2 = (1, -\sin\theta^* \cos\phi^*, -\sin\theta^* \sin\phi^*, \cos\theta^*). \quad (4.32d)$$

Therefore,

$$t = -2p_1 \cdot q_1 = -\frac{s}{2}(1 - \cos\theta^*), \quad (4.33a)$$

$$u = -2p_1 \cdot q_2 = -\frac{s}{2}(1 + \cos\theta^*), \quad (4.33b)$$

$$\frac{t^2 + u^2}{s^2} = \frac{1}{2}(1 + \cos^2\theta^*). \quad (4.33c)$$

31. Using the following definite integral

$$\int_{-1}^1 dx \frac{1}{2}(1+x^2) = \frac{4}{3}, \quad (4.34)$$

show that

$$\int_{-1}^1 d\cos\theta^* \frac{t^2 + u^2}{s^2} = \frac{4}{3}. \quad (4.35)$$

32. Substituting above results to the cross section formula, show that

$$\begin{aligned} \sigma(e^+e^- \rightarrow \mu^+\mu^-) &= \frac{1}{2s} \times \frac{2(4\pi)^2\alpha^2}{16\pi} \int_{-1}^1 d\cos\theta^* \frac{t^2 + u^2}{s^2} \\ &= \frac{4\pi\alpha^2}{3s}. \end{aligned} \quad (4.36a)$$

### 4.3 $e^+e^- \rightarrow q + \bar{q}$

33. Neglecting masses, show that

$$\sigma(e^+e^- \rightarrow q\bar{q}) = N_c \times e_q^2 \times \sigma(e^+e^- \rightarrow \mu^+\mu^-), \quad (4.37)$$

where  $e_q$  is the fractional charge of the quark.

### 4.4 $q\bar{q} \rightarrow g^* \rightarrow q' + \bar{q}'$

34. Neglecting masses and QED interaction, show that

$$\sigma(q\bar{q} \rightarrow q'\bar{q}') = \frac{2}{9} \left(\frac{\alpha_s}{\alpha}\right)^2 \times \sigma(e^+e^- \rightarrow \mu^+\mu^-), \quad (4.38)$$

where  $e_q$  is the fractional charge of the quark.

35. We first replace the coupling  $-e$  by  $g_s$  that is  $\alpha \rightarrow \alpha_s$ .

36. The color factor in the amplitude level is

$$T_{ji}^a T_{lk}^a, \quad (4.39)$$

where the color indices are for  $q_i\bar{q}_j \rightarrow q'_l + \bar{q}'_k$ .

37. Squaring the color factor, we get

$$\text{Tr}(T^a T^b) \text{Tr}(T^a T^b) = \frac{1}{4} \delta^{ab} \delta^{ab} = \frac{N_c^2 - 1}{4} = 2. \quad (4.40)$$

And we have to average over the initial color states by multiplying the factor

$$\left(\frac{1}{3^2}\right)_{\text{color-average}}. \quad (4.41)$$

38. Therefore,

$$\sigma(q\bar{q} \rightarrow q'\bar{q}') = \frac{2}{9} \left(\frac{\alpha_s}{\alpha}\right)^2 \times \sigma(e^+e^- \rightarrow \mu^+\mu^-), \quad (4.42)$$

# Chapter 5

## Higher Order Correction

### 5.1 Fermion Self Energy

Define  $-i\Sigma(\not{p})$  as sum of all the one particle irreducible diagrams.

$$\frac{i}{\not{p} - m} \times (-i\Sigma(\not{p})) \times \frac{i}{\not{p} - m} = \frac{i}{\not{p} - m} \times \frac{\Sigma(\not{p})}{\not{p} - m} \quad (5.1)$$

where  $\Sigma(\not{p})$  commutes with  $(\not{p} - m)^{-1}$ . Then full fermion propagator is given by an infinite geometric series

$$\frac{iZ_2}{\not{p} - m_R} = \frac{i}{\not{p} - m} \times \left[ 1 + \frac{\Sigma(\not{p})}{\not{p} - m} + \left( \frac{\Sigma(\not{p})}{\not{p} - m} \right)^2 + \dots \right] \quad (5.2)$$

$$= \frac{i}{(\not{p} - m) \left( 1 - \frac{\Sigma(\not{p})}{\not{p} - m} \right)} = \frac{i}{\not{p} - m - \Sigma(\not{p})} \quad (5.3)$$

where  $Z_2$  is the wavefunction renormalization constant and  $m_R$  is the renormalized mass. Expanding right hand side of above equation around the renormalized pole mass:

$$\Sigma(\not{p}) = \Sigma(m) + (\not{p} - m) \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m} + O((\not{p} - m)^2) \quad (5.4)$$

$$\frac{\not{p} - m_R}{Z_2} = \not{p} - m - \Sigma(\not{p}) \quad (5.5)$$

$$= \not{p} - m - \Sigma(m) - \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m} (\not{p} - m) \quad (5.6)$$

$$= \left( 1 - \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m} \right) \not{p} - \left( m - m \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m} + \Sigma(\not{p} = m) \right) \quad (5.7)$$

$$= \left( 1 - \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m} \right) \left( \not{p} - \left( m + \frac{\Sigma(\not{p} = m)}{1 - \left[ \frac{d\Sigma}{d\not{p}} \right]_{\not{p}=m}} \right) \right) \quad (5.8)$$

Here left hand side is renormalized one and the right hand side is bare perturbation result. RHS is expanded in terms of bare perturbation quantities(bare mass  $m, \dots$ ). we have

$$\boxed{\begin{aligned} Z_2^{-1} &= 1 - \left[ \frac{d\Sigma}{d\cancel{p}} \right]_{\cancel{p}=m} \\ m_R &= m + \frac{\Sigma(\cancel{p}=m)}{1 - \left[ \frac{d\Sigma}{d\cancel{p}} \right]_{\cancel{p}=m}} \end{aligned}}$$

If we define  $\Sigma^{(n)}$  as n-th order irreducible diagram

$$\Sigma = \Sigma^{(1)} + \Sigma^{(2)} + \dots \quad (5.9)$$

Therefore, if we need to calculate wavefunction renormalization factor with  $\alpha_s^n$  accuracy, we should calculate  $-d\Sigma_n/d\cancel{p}$  at  $\cancel{p}=m$  and add to  $n-1$ th order result:

$$Z_2^{-1} = 1 - \left[ \frac{d\Sigma^{(1)}}{d\cancel{p}} + \frac{d\Sigma^{(2)}}{d\cancel{p}} + \dots \right]_{\cancel{p}=m} \quad (5.10)$$

$$Z_2 = 1 + \sum_{k=1}^{\infty} \left[ \frac{d\Sigma^{(1)}}{d\cancel{p}} \right]_{\cancel{p}=m}^k \quad (5.11)$$

In our case : QCD  $N$ -loop correction to  $J/\psi \rightarrow \ell^+ \ell^-$ , Full amplitude  $\mathcal{M}_N$  valid to this order can be calculated by summing all the  $k$ -th( $k \leq N$ ) order vertex correction diagram  $\mathcal{M}_V^{(k)}$  obtained by bare perturbation theory multiplied by quark wavefunction renormalization factor  $Z_Q$ .

$$\mathcal{M} = Z_Q \mathcal{M}_V = Z_Q (1 + \lambda) \mathcal{M}^{(0)} = \left( 1 + \sum_{k=1}^{\infty} \left[ \frac{d\Sigma^{(1)}}{d\cancel{p}} \right]_{\cancel{p}=m}^k \right) (1 + \lambda) \mathcal{M}^{(0)} \quad (5.12)$$

where we use

$$\mathcal{M}_V = \mathcal{M}^{(0)} + \sum_{k=1}^{\infty} \mathcal{M}_V^{(k)} = \left( 1 + \sum_{k=1}^{\infty} \lambda^{(k)} \right) \mathcal{M}^{(0)} = (1 + \lambda) \mathcal{M}^{(0)} \quad (5.13)$$

$$\mathcal{M}_V^{(k)} = \lambda^{(k)} \mathcal{M}^{(0)}, \quad \lambda = \sum_{k=1}^{\infty} \lambda^{(k)} \quad (5.14)$$

If we consider one loop correction

$$\mathcal{M} = \left( 1 + \lambda^{(1)} + \left[ \frac{d\Sigma^{(1)}}{d\cancel{p}} \right]_{\cancel{p}=m} + O(g_s^2) \right) \mathcal{M}^{(0)} \quad (5.15)$$

In two loop correction,

$$\mathcal{M} = \left( 1 + \left[ \lambda^{(1)} + \left. \frac{d\Sigma^{(1)}}{d\cancel{p}} \right|_{\cancel{p}=m} \right] + \left[ \lambda^{(2)} + \left. \frac{d\Sigma^{(2)}}{d\cancel{p}} \right|_{\cancel{p}=m} + \lambda^{(1)} \left. \frac{d\Sigma^{(1)}}{d\cancel{p}} \right|_{\cancel{p}=m} + \left. \frac{d\Sigma^{(1)}}{d\cancel{p}} \right|_{\cancel{p}=m}^2 \right] + O(g_s^3) \right) \mathcal{M}^{(0)}$$

$$\Sigma = i(-ig\mu^{2-d/2})^2 \sum_{a,i} T_{ij}^a T_{jk}^a \int \frac{d^d k}{(2\pi)^d} \gamma_\alpha \frac{-i}{k^2} \frac{+i}{(\not{p} - \not{k} - m)} \gamma^\alpha \quad (5.17)$$

$$= -ig^2 \mu^{4-d} C_2(R) \delta_{ik} \int \frac{d^d k}{(2\pi)^d} \gamma_\alpha \frac{\not{p} - \not{k} + m}{k^2((p-k)^2 - m^2)} \gamma^\alpha \quad (5.18)$$

$$= C_s \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(\not{p} - \not{k}) + dm}{k^2((p-k)^2 - m^2)} \quad (5.19)$$

$$C_s = -ig^2 \mu^{4-d} C_2(R) \delta_{ik} \quad (5.20)$$

$$Z_Q = \left. \frac{d\Sigma}{d\not{p}} \right|_{\not{p}=m} = C_s \frac{p^\mu}{m} \frac{\partial}{\partial p^\mu} \Sigma(\not{p}, m) \Big|_{\not{p}=m} \quad (5.21)$$

$$= \frac{C_s}{2m^2} [-4m^2 I_1 - 2(d-2)I_0 - 8m^4 I'_1 - 2(d-4)m^2 I'_0 + (d-2)I'_{-1}] \quad (5.22)$$

$$= -\frac{C_s}{2m^2} \times \frac{(d-1)(d-2)}{(d-3)} I_0 \quad (5.23)$$

where we use

$$I'_n = (d-2n-3)I_{n+1} \quad (5.24)$$

$$I_n = \frac{d-n-1}{2m^2(d-2n-1)} I_{n-1} \quad (5.25)$$

$$I_0 = \frac{2m^2}{(d-2)} F_2(m^2) \quad (5.26)$$

$$\Sigma = i(-ig\mu^{2-d/2})^2 \sum_{a,i} T_{ij}^a T_{jk}^a \int \frac{d^d k}{(2\pi)^d} \gamma_\alpha \frac{-i}{k^2} \frac{+i}{(\not{p} - \not{k} - m)} \gamma^\alpha \quad (5.27)$$

$$= -ig^2 \mu^{4-d} C_2(R) \delta_{ik} \int \frac{d^d k}{(2\pi)^d} \gamma_\alpha \frac{\not{p} - \not{k} + m}{k^2((p-k)^2 - m^2)} \gamma^\alpha \quad (5.28)$$

$$= C_s \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(\not{p} - \not{k}) + dm}{k^2((p-k)^2 - m^2)} \quad (5.29)$$

$$= C_s \int \frac{d^d k}{(2\pi)^d} \left[ \frac{(2-d)}{2} \times \frac{\not{p} + \frac{m^2}{\not{p}}}{k^2((p+k)^2 - m^2)} + \frac{d-2}{2 \not{p}((p+k)^2 - m^2)} + \frac{dm}{k^2((p+k)^2 - m^2)} \right] \quad (5.30)$$

$$Z_Q = \left. \frac{d\Sigma(\not{p})}{d\not{p}} \right|_{\not{p}=m} = C_s \int \frac{d^d k}{(2\pi)^d} \left[ 2m \frac{d}{d\not{p}} \frac{1}{k^2((p+k)^2 - m^2)} + \frac{(2-d)}{2m^2} \times \frac{1}{(p+k)^2 - m^2} \right] \Big|_{\not{p}=m} \quad (5.31)$$

$$= C_s \frac{1-d}{d-3} F_2(m^2) \quad (5.32)$$

$$= \frac{4 \cdot 4\pi\alpha_s}{3} \times \frac{\Gamma(\frac{4-d}{2})}{(4\pi)^{d/2}} m^{d-4} \times \frac{1-d}{d-3} \quad (5.33)$$

$$= \frac{4\alpha_s}{3\pi} \left[ \frac{3}{2(d-4)} - 1 + \frac{3}{4} \left( \gamma_E + \log\left(\frac{m^2}{4\pi}\right) \right) \right] \quad (5.34)$$

so that one-loop correction factor is set as

$$\lambda + Z_Q = \lambda + \left. \frac{d\Sigma(\not{p})}{d\not{p}} \right|_{\not{p}=m} = -\frac{8\alpha_s}{3\pi} \quad (5.35)$$

```

C0=I/(4 Pi)^(d/2) Gamma[(4-d)/2]

I0[0]=2 C m^2/(d-2) m^(d-4)

Fn[n_]:=Product[(d-k-1)/((2m^2)*(d-2k-1)),{k,1,n}]/;(n>0)

I0[n_]:=I0[0]*Fn[n]
IP[n_]:=(d-2n-3) I0[n+1]

F=(d-2)/( 2m^2(d-1) )I0[0]+(d-2)(d-3)I0[1]/(d-1)-4m^2 I0[2]
CS=-I*4/3 * 4*Pi*As

V=CS*F
TST=Simplify[ Series[Expand[V/.C->C0],{d,4,0}]]
Unprotect[Log,Power]
Log[m^2]=2Log[m]
FSELF =(4 As)/(3 Pi)*(
 3/(2(4-d) ) -1 -3/4*(EulerGamma+Log[m^2/(4Pi)])
)
Simplify[Series[ Expand[TST-FSELF],{d,4,0}]]

(*****)

F=(1-d)/(d-3)IP[0]
V=CS*F
TST=Simplify[ Series[Expand[V/.C->C0],{d,4,0}]]
FVTX =(4 As)/(3 Pi)*(
 3/(2(d-4) ) -1 +3/4*(EulerGamma+Log[m^2/(4Pi)])
)
Simplify[Series[ Expand[TST-FVTX],{d,4,0}]]
Simplify[Series[ Expand[FSELF+FVTX],{d,4,0}]]

```

## 5.2 Gluon Self Energy

Define  $i\Pi(q)$  as sum of all the one particle irreducible diagrams.

$$\frac{-ig^{\mu\alpha}}{q^2} \times (i\Pi_{\alpha\beta}) \times \frac{-ig^{\beta\nu}}{q^2} = \frac{-i\Pi^{\mu\nu}}{q^4} \quad (5.36)$$

$$= \Pi(q^2) \times \frac{-i \left[ g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right]}{q^2} \quad (5.37)$$

$$\Pi^{\mu\nu} = \Pi(q^2)[q^2 g^{\mu\nu} - q^\mu q^\nu] \quad (5.38)$$

Then full gluon propagator is given by an infinite geometric series

$$\frac{-iZ_3g^{\mu\nu}}{q^2} = \frac{-ig^{\mu\nu}}{q^2} + \frac{-i\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]}{q^2} \times \Pi(q^2) + \dots \quad (5.39)$$

$$= \frac{-i\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]}{q^2} \times [\Pi(q^2) + \Pi^2(q^2) + \dots] - \frac{iq^\mu q^\nu}{q^4} \quad (5.40)$$

$$= \frac{-i\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]}{q^2[1 - \Pi(q^2)]} - \frac{iq^\mu q^\nu}{q^4} \quad (5.41)$$

$$= \frac{-ig^{\mu\nu}}{q^2[1 - \Pi(q^2)]} \quad (5.42)$$

$$\rightarrow \frac{-ig^{\mu\nu}}{q^2[1 - \Pi(q^2 = 0)]} \quad (5.43)$$

where  $Z_3$  is charge renormalization constant. If  $\Pi(q^2)$  is regular at  $q^2 = 0$ , pole structure remains and therefore gluon is massless for all order.

$Z_3 = \frac{1}{1 - \Pi(q^2 = 0)}$
$g_R = \sqrt{Z_3}g$

where  $g_R$  is renormalized color charge.

### 5.3 Scalar Integrals

$$I_n \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 - 2k \cdot p)} \quad (5.44)$$

$$I'_n \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 - 2k \cdot p)^2} \quad (5.45)$$

$$I_n^\mu \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^n (k^2 - 2k \cdot p)} \quad (5.46)$$

$$I'_n \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^n (k^2 - 2k \cdot p)^2} \quad (5.47)$$

Reduction formulae

$$I_n^\mu \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2)^n (k^2 - 2k \cdot p)} \quad (5.48)$$

$$I_n^\mu = Ap^\mu \quad (5.49)$$

$$Ap^2 = \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot p}{(k^2)^n (k^2 - 2k \cdot p)} \quad (5.50)$$

$$= \int \frac{d^d k}{(2\pi)^d} \left[ -\frac{1}{2} \frac{k^2 - 2k \cdot p}{(k^2)^n (k^2 - 2k \cdot p)} + \frac{1}{2} \frac{k^2}{(k^2)^n (k^2 - 2k \cdot p)} \right] \quad (5.51)$$

$$= \int \frac{d^d k}{(2\pi)^d} \left[ -\frac{1}{2} \frac{1}{(k^2)^n} + \frac{1}{2} \frac{1}{(k^2)^{n-1} (k^2 - 2k \cdot p)} \right] \quad (5.52)$$

$$I_n^\mu = \frac{p^\mu}{2p^2} \left[ I_{n-1} - \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n} \right] \quad : \quad \int \frac{dk^d}{(2\pi)^d} \frac{1}{(k^2)^n} = 0 \text{ if } n \leq 1 \quad (5.53)$$

$$I_n'^\mu \equiv \frac{p^\mu}{2p^2} [I'_{n-1} - I_n] \quad (5.54)$$

$$\frac{1}{(k^2 - 2k \cdot p)(k^2 + 2k \cdot p)} = \frac{1}{2k^2} \left[ \frac{1}{(k^2 - 2k \cdot p)} + \frac{1}{(k^2 + 2k \cdot p)} \right] \quad (5.55)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 - 2k \cdot p)} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 + 2k \cdot p)} \quad (5.56)$$

## 5.4 Vertex Correction

Using

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k^2 + 2k \cdot p) (k^2 - 2k \cdot p)} = I_2 \quad (5.57)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{k^2 (k^2 + 2k \cdot p) (k^2 - 2k \cdot p)} = 0 \quad (5.58)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{k^2 (k^2 + 2k \cdot p) (k^2 - 2k \cdot p)} = I_1 \quad (5.59)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{k^2 (k^2 + 2k \cdot p) (k^2 - 2k \cdot p)} = \frac{1}{(d-1)} \left[ -g^{\mu\nu} \left( \frac{I_0}{4p^2} - I_1 \right) + \frac{p^\mu p^\nu}{p^2} \left( \frac{dI_0}{4p^2} - I_1 \right) \right] \quad (5.60)$$

$$\Lambda^\mu = C_s \int \frac{d^d k}{(2\pi)^d} \frac{-4mk^\mu I + \gamma^\mu ((d-2)k^2 - 4m^2) + 2(2-d)kk^\mu}{k^2 (k^2 + 2k \cdot p) (k^2 - 2k \cdot p)} \quad (5.61)$$

$$\Lambda^i = C_s \gamma^i \left[ \frac{(d-2)I_0}{2m^2(d-1)} + \frac{(d-2)(d-3)I_1}{(d-1)} - 4m^2 I_2 \right] \quad (5.62)$$

$$= \gamma^i C_2(R) 4\pi \alpha_s \frac{1}{(4\pi)^{d/2}} \left( \frac{d-7}{d-5} \right) m^{d-4} \quad (5.63)$$

$$= \gamma^i \times \frac{4\alpha_s}{3\pi} \left[ \frac{3}{2(4-d)} - 1 - \frac{3}{4} \left( \gamma_E + \log \left( \frac{m^2}{4\pi} \right) \right) \right] \quad (5.64)$$

where we neglected  $p^i = 0$ <sup>1</sup>

```
C0=I/(4 Pi)^(d/2) Gamma[(4-d)/2]
I0[0]=2 C m^2/(d-2) * m^(d-4)
Fn[n_]:=Product[(d-k-1)/((2m^2)*(d-2k-1)),{k,1,n}]/;(n>0)
```

Fn[1]

```
I0[n_]:=I0[0]*Fn[n]
IP[n_]:=(d-2n-3) I0[n+1]
```

```
F=(d-2)/(2m^2(d-1))I0[0]+(d-2)(d-3)I0[1]/(d-1)-4m^2 I0[2]
CS=-I*4/3 * 4*Pi*As
```

```
V=CS*F
TST=Simplify[
Series[Expand[V/.C->C0],{d,4,0}]]
Unprotect[Log,Power]
Log[m^2]=2Log[m]
FF =(4 As)/(3 Pi)*(
 3/(2(4-d)) -1 -3/4*(EulerGamma+Log[m^2/(4Pi)])
 )
Simplify[Series[ Expand[TST-FF],{d,4,0}]]
```

## 5.5 Gloun Self Energy

### 5.5.1 Scalar Integrals-Massless fermion loop

Let us define an integral

$$I \equiv \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 l^2} \quad : \quad q \equiv k + l \quad (5.65)$$

From now on, these are useful

$\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2} = 0$
$\int \frac{d^d l}{(2\pi)^d} = \int \frac{d^d l}{(2\pi)^d} l^2 = \int \frac{d^d l}{(2\pi)^d} l^4 = \dots = 0$
$k \cdot l = \frac{1}{2}(q^2 - l^2 - k^2) \leftarrow q^2 = (k + l)^2 = k^2 + l^2 + 2k \cdot l$

---

<sup>1</sup>See vtx1.red

Reduction formulae

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{q^2 l^2} = Ak^\mu \quad (5.66)$$

$$Ak^2 = \int \frac{d^d l}{(2\pi)^d} \frac{k \cdot l}{q^2 l^2} = \frac{1}{2} \int \frac{d^d l}{(2\pi)^d} \frac{q^2 - l^2 - k^2}{q^2 l^2} \quad (5.67)$$

$$= \frac{1}{2} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{1}{l^2} - \frac{1}{q^2} - k^2 \times \frac{1}{q^2 l^2} \right] = -\frac{k^2}{2} I \quad (5.68)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{q^2 l^2} = -\frac{k^\mu}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 l^2} = -\frac{k^\mu}{2} I \quad (5.69)$$

For tensor integral, we may use the same method

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{q^2 l^2} = -g^{\mu\nu} A + \frac{k^\mu k^\nu}{k^2} B \quad (5.70)$$

$$-dA + B = \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{q^2 l^2} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2} = 0 \quad (5.71)$$

$$k^2(-A + B) = \int \frac{d^d l}{(2\pi)^d} \frac{(k \cdot l)^2}{q^2 l^2} = \frac{1}{4} \int \frac{d^d l}{(2\pi)^d} \frac{(q^2 - l^2 - k^2)^2}{q^2 l^2} = \frac{k^4}{4} \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 l^2} \quad (5.72)$$

where all the other terms vanish. Solving

$$\begin{aligned} -dA + B &= 0 \\ -A + B &= \frac{k^2}{4} I \end{aligned} \rightarrow \left\{ \begin{array}{l} A = \frac{k^2}{4(d-1)} I \\ B = dA \end{array} \right. \quad (5.73)$$

And

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu q^\nu}{q^2 l^2} = \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu (k+l)^\nu}{q^2 l^2} = -\frac{k^\mu k^\nu}{2} I + \frac{I}{4(d-1)} [-k^2 g^{\mu\nu} + dk^\mu k^\nu] \quad (5.74)$$

$$= \frac{I}{4(d-1)} [-k^2 g^{\mu\nu} + (2-d)k^\mu k^\nu] \quad (5.75)$$

<sup>2</sup>

$I = \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 l^2}$	: $q = k + l$
$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{q^2 l^2} = -\frac{k^\mu}{2} \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 l^2} = -\frac{k^\mu}{2} I$	
$\int \frac{d^d l}{(2\pi)^d} \frac{q^\mu}{q^2 l^2} = +\frac{k^\mu}{2} I$	
$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{q^2 l^2} = \int \frac{d^d l}{(2\pi)^d} \frac{q^\mu q^\nu}{q^2 l^2} = \frac{I}{4(d-1)} [-k^2 g^{\mu\nu} + dk^\mu k^\nu]$	
$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu q^\nu}{q^2 l^2} = \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu (k+l)^\nu}{q^2 l^2} = \frac{I}{4(d-1)} [-k^2 g^{\mu\nu} + (2-d)k^\mu k^\nu]$	

<sup>2</sup>See gluon1.red

### 5.5.2 Scalar Integrals-Massive fermion loop

Let us define an integral

$$f_1 \equiv \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{q^2 - m^2} \quad (5.76)$$

$$f_2 \equiv \int \frac{d^d l}{(2\pi)^d} \frac{1}{(q^2 - m^2)(l^2 - m^2)} \quad : \quad q \equiv k + l \quad (5.77)$$

From now on, these are useful

	$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{l^2 - m^2} = 0$	
	$\int \frac{d^d l}{(2\pi)^d} l^2 = \int \frac{d^d l}{(2\pi)^d} l^4 = \dots = 0$	
	$k \cdot l = \frac{1}{2}(q^2 - l^2 - k^2) = \frac{1}{2}[(q^2 - m^2) - (l^2 - m^2) - k^2]$	
	$\leftarrow q^2 = (k + l)^2 = k^2 + l^2 + 2k \cdot l$	
	$\int \frac{d^d l}{(2\pi)^d} \frac{q^2 - m^2}{l^2 - m^2} = \int \frac{d^d l}{(2\pi)^d} \frac{l^2 - m^2}{q^2 - m^2} = k^2 f_1$	

Reduction formulae

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{l^2 - m^2} = \int \frac{d^d l}{(2\pi)^d} \left[ 1 + \frac{m^2}{l^2 - m^2} \right] = m^2 f_1 \quad (5.78)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2 - m^2}{q^2 - m^2} = \int \frac{d^d l}{(2\pi)^d} \frac{(l - k)^2 - m^2}{l^2 - m^2} \quad \leftarrow \text{ translation} \quad (5.79)$$

$$= \int \frac{d^d l}{(2\pi)^d} \frac{(l + k)^2 - m^2}{l^2 - m^2} \quad \leftarrow \text{ function of } k^2 \quad (5.80)$$

$$= \int \frac{d^d l}{(2\pi)^d} \frac{q^2 - m^2}{l^2 - m^2} \quad (5.81)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{q^2 - m^2}{l^2 - m^2} = \int \frac{d^d l}{(2\pi)^d} \frac{l^2 - m^2 + 2k \cdot l + k^2}{l^2 - m^2} \quad (5.82)$$

$$= \int \frac{d^d l}{(2\pi)^d} \left[ 1 + 2k_\alpha \frac{l^\alpha}{l^2 - m^2} + k^2 \frac{1}{l^2 - m^2} \right] \quad (5.83)$$

$$= k^2 \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - m^2} = k^2 f_1 \quad (5.84)$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{(q^2 - m^2)(l^2 - m^2)} = Ak^\mu \quad (5.85)$$

$$\begin{aligned} Ak^2 &= \int \frac{d^d l}{(2\pi)^d} \frac{k \cdot l}{(q^2 - m^2)(l^2 - m^2)} = \frac{1}{2} \int \frac{d^d l}{(2\pi)^d} \frac{(q^2 - m^2) - (l^2 - m^2)}{(q^2 - m^2)(l^2 - m^2)} \frac{-k^2}{(5.86)} \\ &= \frac{1}{2} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{1}{l^2 - m^2} - \frac{1}{q^2 - m^2} - k^2 \times \frac{1}{(q^2 - m^2)(l^2 - m^2)} \right] = -\frac{k^2}{2} f_2 \end{aligned}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{(q^2 - m^2)(l^2 - m^2)} = -\frac{k^\mu}{2} f_2 \quad (5.88)$$

For tensor integral, we may use the same method

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(q^2 - m^2)(l^2 - m^2)} = -g^{\mu\nu} A + \frac{k^\mu k^\nu}{k^2} B \quad (5.89)$$

$$-dA + B = \int \frac{d^d l}{(2\pi)^d} \frac{l^2 - m^2 + m^2}{(q^2 - m^2)(l^2 - m^2)} = f_1 + m^2 f_2 \quad (5.90)$$

$$k^2(-A + B) = \int \frac{d^d l}{(2\pi)^d} \frac{(k \cdot l)^2}{(q^2 - m^2)(l^2 - m^2)} \quad (5.91)$$

$$= \frac{1}{4} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{[(q^2 - m^2) - (l^2 - m^2)]^2}{(q^2 - m^2)(l^2 - m^2)} - 2k^2 \frac{(q^2 - m^2) - (l^2 - m^2)}{(q^2 - m^2)(l^2 - m^2)} \right. \\ \left. + \frac{k^4}{(q^2 - m^2)(l^2 - m^2)} \right] \quad (5.93)$$

$$= \frac{1}{4} \int \frac{d^d l}{(2\pi)^d} \left[ \frac{q^2 - m^2}{l^2 - m^2} + \frac{q^2 - m^2}{l^2 - m^2} \right] + \frac{k^2}{4} \int \frac{d^d l}{(2\pi)^d} \frac{1}{(q^2 - m^2)(l^2 - m^2)} \quad (5.94)$$

$$= \frac{k^2}{2} f_1 + \frac{k^4}{4} f_2 \quad (5.95)$$

where all the other terms vanish. Solving

$$\begin{aligned} -dA + B &= f_1 + m^2 f_2 \\ -A + B &= \frac{1}{2} f_1 + \frac{k^2}{4} f_2 \end{aligned} \quad \left. \right\} \rightarrow \begin{cases} A &= \frac{1}{4(d-1)} [2f_1 + (4m^2 - k^2)f_2] \\ B &= \frac{1}{4(d-1)} [2(d-2)f_1 + (dk^2 - 4m^2)f_2] \end{cases} \quad (5.96)$$

3

### 5.5.3 Sign Convention of $\Pi$

Sign of  $\Pi^{\mu\nu}$  is chosen so that

$$\text{Amplitude} = i\Pi \rightarrow \frac{-ig^{\mu\alpha}}{k^2} \times i\Pi^{\alpha\beta} \times \frac{-ig^{\beta\nu}}{k^2} = \frac{-ig^{\mu\alpha}}{k^2} \times \frac{\Pi^{\mu\nu}}{k^2} \quad (5.97)$$

whereas Sign of  $\Sigma$  is chosen so that

$$\text{Amplitude} = -i\Sigma \rightarrow \frac{i}{\not{p} - m} \times (-i\Sigma) \times \frac{i}{\not{p} - m} = \frac{i}{\not{p} - m} \times \left( \Sigma \frac{1}{\not{p} - m} \right) \quad (5.98)$$

### 5.5.4 Fermion Loop

$$\text{Factor} = (-i)_{\text{convention}} \times (-1)_{\text{fermion-loop}} (-i)_{\text{vertex}}^2 (+i)_{\text{propagator}}^2 \text{Tr} [T^a T^b] = i \frac{\delta^{ab}}{2} \quad (5.99)$$

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<sup>3</sup>See gm1.red

$$\Pi^{\mu\nu}(\text{quark}) = i \frac{\delta^{ab}}{2} \sum_q \int \frac{d^d l}{(2\pi)^d} \frac{\text{Tr} [\gamma^\mu (\not{q} + m_q) \gamma^\nu (\not{l} + m_q)]}{(q^2 - m_q^2)(l^2 - m_q^2)} \quad (5.100)$$

$$= i 2 \delta^{ab} \sum_q \int \frac{d^d l}{(2\pi)^d} \frac{q^\mu l^\nu + q^\nu l^\mu + (m_q^2 - q \cdot l) g^{\mu\nu}}{(q^2 - m_q^2)(l^2 - m_q^2)} \quad (5.101)$$

$$= i \delta^{ab} \frac{(d-2)}{(d-1)} \sum_q \left[ \left( 1 + \frac{4m_q^2}{(d-2)k^2} \right) f_2 - 2f_1 \right] \times [k^2 g^{\mu\nu} - k^\mu k^\nu] \quad (5.102)$$

$$= i 2 N_f \delta^{ab} \int \frac{d^d l}{(2\pi)^d} \frac{q^\mu l^\nu + q^\nu l^\mu - q \cdot l g^{\mu\nu}}{q^2 l^2} \quad (5.103)$$

$$= i I(k^2) N_f \delta^{ab} \frac{(d-2)}{(d-1)} [k^2 g^{\mu\nu} - k^\mu k^\nu] \quad (5.104)$$

where we neglected light quark mass  $m_q$  where  $q = u, d, s$  ( $N_f = 3$ ).

### 5.5.5 Ghost Loop

$$\text{Factor} = (-i)_{\text{convention}} \times (-1)_{\text{ghost-loop}} \left[ (-f^{yxa})(-f^{xyb}) \right]_{\text{vertex}} (+i)_{\text{propagator}}^2 = 3i \delta^{ab} \quad (5.105)$$

$$\Pi^{\mu\nu}(\text{ghost}) = 3i \delta^{ab} \int \frac{d^d l}{(2\pi)^d} \frac{q^\mu l^\nu}{q^2 l^2} \quad (5.106)$$

$$= -3i \delta^{ab} \frac{I(k^2)}{4(d-1)} [k^2 g^{\mu\nu} + (d-2) k^\mu k^\nu] \quad (5.107)$$

### 5.5.6 Gluon Loop

$$\text{Factor} = (-i)_{\text{convention}} \times \left( \frac{1}{2} \right)_{\text{gluon-loop}} \left[ (-f^{yxa})(-f^{xyb}) \right]_{\text{vertex}} (+i)_{\text{propagator}}^2 = -i \frac{3}{2} \delta^{ab} \quad (5.108)$$

$$\Pi^{\mu\nu}(\text{gluon}) = -i \frac{3}{2} \delta^{ab} \int \frac{d^d l}{(2\pi)^d} \frac{G^{yx\mu}(-l, q, -k) G_{xy\nu}(-q, l, k)}{q^2 l^2} \quad (5.109)$$

$$= -3i \delta^{ab} \frac{I(k^2)}{4(d-1)} [(6d-5)k^2 g^{\mu\nu} + (6-7d)k^\mu k^\nu] \quad (5.110)$$

### 5.5.7 Gluon + Ghost

$$\Pi^{\mu\nu}(\text{gluon}) + \Pi^{\mu\nu}(\text{ghost}) = -3i \delta^{ab} I(k^2) \frac{3d-2}{2(d-1)} [k^2 g^{\mu\nu} - k^\mu k^\nu] \quad (5.111)$$

### 5.5.8 Gluon Self Energy

$$\Pi^{\mu\nu}(k^2) = \Pi^{\mu\nu}(\text{quark}) + \Pi^{\mu\nu}(\text{gluon}) + \Pi^{\mu\nu}(\text{ghost}) \quad (5.112)$$

$$= \Pi(k^2) [k^2 g^{\mu\nu} - k^\mu k^\nu] \quad (5.113)$$

$$\Pi(k^2) = i\delta^{ab} I(k^2) \frac{2N_f(d-2) + 3(2-3d)}{2(d-1)} \quad (5.114)$$

$$= i\delta^{ab} \frac{2N_f(d-2) + 3(2-3d)}{2(d-1)} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(k+l)^2} \quad (5.115)$$

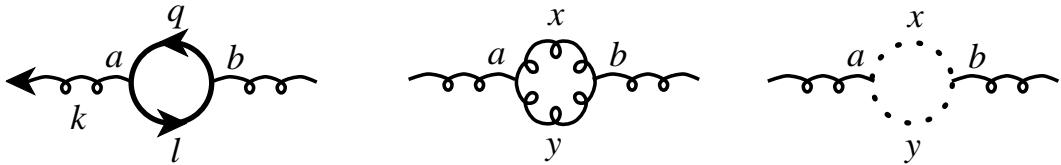
$$= i\delta^{ab} I(k^2) \left( \frac{2N_f}{3} - 5 \right) \quad \leftarrow d = 4 \quad (5.116)$$

Considering

$$I(k^2) = \frac{i}{(4\pi)^2 \epsilon} + \dots \quad (5.117)$$

$$\Pi^{\mu\nu}(k^2) = \frac{1}{16\pi^2 \epsilon} \left( 5 - \frac{2N_f}{3} \right) [k^2 g^{\mu\nu} - k^\mu k^\nu] + \dots \quad (5.118)$$

Therefore one loop gluon propagator



ONE PARTICLE IRREDUCIBLE DIAGRAM

$$\text{---} \bullet \text{---} = \text{---} + \text{1-LOOP} + \text{2-LOOP}$$

$$\text{---} \bullet \text{---} = \text{---} + \text{1-LOOP}$$

is given by

$$\frac{-ig^{\mu\nu}}{k^2 [1 - \Pi(k^2)]} \quad (5.119)$$

and 1-loop correction term is

$$-i \frac{g^{\mu\nu}}{k^2} \Pi(k^2) = C_\Pi \int \frac{d^d l}{(2\pi)^d} \frac{-ig^{\mu\nu}}{k^2 l^2 q^2} \quad \leftarrow q = -(k+l) \quad (5.120)$$

$$C_\Pi = i\delta^{ab} \frac{2N_f(d-2) + 3(2-3d)}{2(d-1)} \quad (5.121)$$

## 5.6 $Z_Q$ Contribution

Sign of  $\Sigma$  is chosen so that

$$\text{Amplitude} = -i\Sigma \rightarrow \frac{i}{\not{p}-m} \times (-i\Sigma) \times \frac{i}{\not{p}-m} = \frac{i}{\not{p}-m} \times \left( \Sigma \frac{1}{\not{p}-m} \right) \quad (5.122)$$

$$\Sigma = -ig^2 \mu^{4-d} C_2(R) \int \frac{d^d k}{(2\pi)^d} \frac{(2-d)(\not{p}-\not{k}) + dm}{k^2((p-k)^2 - m^2)} \quad (5.123)$$

$$Z_Q - 1 = \frac{d\Sigma}{d\not{p}} \Big|_{\not{p}=m} - 1 = \frac{\alpha_s}{\pi} \left[ -\frac{1}{3} \left( \frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} \right) - \frac{4}{3} + \gamma_E - \log \left( \frac{4\pi\mu^2}{m_Q^2} \right) \right] + O(\epsilon) \quad (5.124)$$

After renormalizing heavy quark wavefunction as

$$\bar{u}_R = \sqrt{Z_Q} \bar{u}_B \quad \text{and} \quad v_R = \sqrt{Z_Q} v_B \quad (5.125)$$

The correction is

$$2 \times (\sqrt{Z_Q} - 1) = Z_Q - 1 \quad (5.126)$$

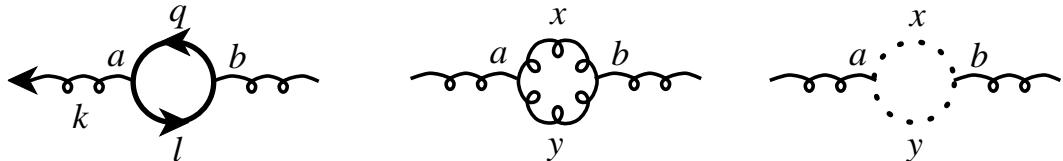
where factor 2 comes from heavy quark and antiquark one by one.

## 5.7 $Z_g$ Contribution

Sign of  $\Pi^{\mu\nu}$  is chosen so that

$$\text{Amplitude} = i\Pi \rightarrow \frac{-ig^{\mu\alpha}}{k^2} \times i\Pi^{\alpha\beta} \times \frac{-ig^{\beta\nu}}{k^2} = \frac{-ig^{\mu\nu}}{k^2} \times \frac{\Pi^{\mu\nu}}{k^2}$$

One loop gluon propagator



ONE PARTICLE IRREDUCIBLE DIAGRAM

$$\text{---} \bullet \text{---} = \text{---} + \text{1-LOOP} + \text{2-LOOP}$$

$$\sim \bullet \sim = \sim + \text{1-LOOP}$$

is given by

$$\frac{-ig^{\mu\nu}}{k^2 [1 - \Pi(k^2)]}$$

and 1-loop correction factor  $\Pi$  is

$$\begin{aligned} \Pi(k^2 = 4m_Q^2) &= i4\pi\alpha_s \frac{2N_f(d-2) + 3(2-3d)}{2(d-1)} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 q^2} \leftarrow q = -(k+l) \\ &= \frac{2\alpha_s}{\pi} \left[ \frac{1}{4} + \frac{1}{8} \left( 5 - \frac{2}{3}n_f \right) \left( \frac{1}{\epsilon_{UV}} + \frac{5}{3} - \gamma_E + \log \left( \frac{\pi\mu^2}{m_Q^2} \right) + i\pi \right) \right] + O(\epsilon) \\ &= \frac{\alpha_s}{\pi} \left[ \frac{1}{2} + \frac{1}{12} (15 - 2n_f) \left( \frac{1}{\epsilon_{UV}} + \frac{5}{3} - \gamma_E + \log \left( \frac{\pi\mu^2}{m_Q^2} \right) + i\pi \right) \right] + O(\epsilon) \end{aligned} \quad (5.127)$$

Since dimensional regularization guarantees

$$\begin{aligned} \lim_{k^2 \rightarrow 0} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(l+k)^2} &= \frac{i}{(4\pi)^2} \frac{\Gamma(\epsilon_{UV})}{2-2\epsilon} \lim_{k^2 \rightarrow 0} \left( \frac{k^2}{4} \right)^{-\epsilon} = 0, \quad \epsilon < 0 \\ &\rightarrow \frac{i}{2(4\pi)^2} \left[ \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right] \end{aligned} \quad (5.128)$$

we have

$$Z_g \equiv 1 + \Pi(k^2 = 0)|_{UV} = 1 + \frac{\alpha_s}{12\pi} (15 - 2n_f) \left[ \frac{1}{\epsilon_{UV}} \right] \quad (5.129)$$

Note that UV divergence should be the same for any values of  $k^2$ , since  $\lim l \rightarrow \infty$  is not changed by  $k$ . This will be used to check the charge renormalization process. If the gluon is on-shell, we could have renormalized the gluon wavefunction as

$$\epsilon_R^\mu = \sqrt{Z_g} \epsilon_B^\mu \quad (5.130)$$

and the correction term is

$$\sqrt{Z_g} - 1 \quad (5.131)$$

In our case to consider the gluon fragmentation, the gluon is off-shell and it is created by the operator explicitly shown in the gauge invariant definition of fragmentation. Therefore, we should renormalize this gluon operator instead of on-shell gluon.

## 5.8 1-gluon Exchange Vertex Contribution $\lambda_1$

$$\begin{aligned} \lambda_1 &= i \frac{2\pi\alpha_s\mu^{2\epsilon}}{N_c} \int \frac{d^d k}{(2\pi)^d} \frac{k^2 \frac{(d-2)(d-3)}{(d-1)} + 2(k \cdot p)^2 \frac{(d-2)}{(d-1)m_Q^2} - 4m_Q^2}{k^2(k^2 + k \cdot p)(k^2 - k \cdot p)} \\ &= i \frac{2\pi\alpha_s\mu^{2\epsilon}}{N_c} \left[ \frac{(d-2)(d-3)}{(d-1)} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + k \cdot p)} + \frac{(d-2)}{2(d-1)m_Q^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + k \cdot p)} \right. \\ &\quad \left. - 4m_Q^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2(k^2 + k \cdot p)} \right] \end{aligned} \quad (5.132)$$

$$\lambda_1 = \frac{\alpha_s}{24\pi} \left[ - \left( \frac{1}{\epsilon_{UV}} + \frac{2}{\epsilon_{IR}} \right) + 4 + 3 \left( \gamma_E - \log \left( \frac{4\pi\mu^2}{m_Q^2} \right) \right) \right] + O(\epsilon) \quad (5.133)$$

## 5.9 tri-gluon Vertex Contribution $\lambda_2$

$$\begin{aligned} \lambda_2 = & -i2\pi\alpha_s\mu^{2\epsilon}N_c \times \left[ \frac{d-2}{d-1} \frac{1}{2m_Q^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m_Q^2} - \frac{d}{(d-1)} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k+p)^2} \right. \\ & \left. + 4 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + k \cdot p)} \right] \end{aligned} \quad (5.134)$$

$$\lambda_2 = \frac{\alpha_s}{\pi} \left[ \frac{9}{8} \left( \frac{1}{\epsilon_{UV}} - \gamma_E + \log \left( \frac{4\pi\mu^2}{m_Q^2} \right) \right) + 2 + \log 2 - \frac{i\pi}{2} \right] + O(\epsilon) \quad (5.135)$$

## 5.10 Sum of vertex corrections

$$\begin{aligned} \lambda &= \lambda_1 + \lambda_2 \\ &= \frac{\alpha_s}{\pi} \left[ \frac{1}{12} \left( \frac{13}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) - \gamma_E + \log \left( \frac{4\pi\mu^2}{m_Q^2} \right) + \frac{13}{6} + \log 2 - \frac{i\pi}{2} \right] + O(\epsilon) \end{aligned} \quad (5.136)$$

$$(2a) = \Delta_{(2a)} \times (1) \quad (5.137)$$

$$\begin{aligned} \Delta_{(2a)} &= 2\text{Re} [\lambda + (Z_Q - 1) + \Pi(k^2 = 4m_Q^2)] \\ &= \frac{\alpha_s}{\pi} \left[ \frac{12 - n_f}{3} \frac{1}{\epsilon_{UV}} - \frac{3}{2} \frac{1}{\epsilon_{IR}} \right. \\ &\quad \left. + \frac{1}{2} \left( 5 - \frac{2n_f}{3} \right) \left( -\gamma_E + \log \left( \frac{\pi\mu^2}{m_Q^2} \right) \right) \right. \\ &\quad \left. + \frac{1}{3} \left( \frac{41}{2} - \frac{5n_f}{3} \right) + 2\log 2 \right] + O(\epsilon) \end{aligned} \quad (5.138)$$

Note that there is an IR divergence and it should be exactly cancelled by another IR divergence from the rest of the contributions.

## 5.11 Charge renormalization

Before doing final step calculation, let us consider the renormalization process. Charge renormalization concerns only on the UV divergent term.

$$\sqrt{Z_g} - 1 = \frac{\alpha_{sB}}{24\pi} \frac{15 - 2n_f}{\epsilon_{UV}} \quad (5.139)$$

$$Z_Q - 1 = -\frac{\alpha_{sB}}{24\pi} \frac{8}{\epsilon_{UV}} \quad (5.140)$$

$$\lambda = \frac{\alpha_{sB}}{24\pi} \frac{26}{\epsilon_{UV}} \quad (5.141)$$

Therefore

$$\begin{aligned} g_{sR} &= g_{sB} \left( (\sqrt{Z_g} - 1) + (Z_Q - 1) + \lambda \right) \\ &= g_{sB} \left( 1 + \frac{\alpha_{sB}}{\pi} \frac{33 - 2n_f}{24} \left[ \frac{1}{\epsilon_{UV}} - \gamma_E + \log \left( \frac{4\pi\mu^2}{\mu_R^2} \right) \right] \right) \end{aligned} \quad (5.142)$$

In  $\overline{\text{MS}}$  scheme,

$$\alpha_{sR} = \alpha_{sB} \left( 1 + \frac{\alpha_{sB}}{\pi} \frac{33 - 2n_f}{12} \left[ \frac{1}{\epsilon_{UV}} - \gamma_E + \log \left( \frac{4\pi\mu^2}{\mu_R^2} \right) \right] \right) \quad (5.143)$$

where  $\mu$  and  $\mu_R$  are dimensional regularization scale and charge renormalization scale, respectively. After summing over the diagrams, we can simply add

$$-\frac{\alpha_s}{\pi} \frac{33 - 2n_f}{12} \left[ \frac{1}{\epsilon_{UV}} - \gamma_E + \log \left( \frac{4\pi\mu^2}{\mu_R^2} \right) \right] \quad (5.144)$$

to the correction factor to get the charge renormalized result.

## 5.12 UV/IR-finite final result

- This is the results for the NLO corrections to  $g^* \rightarrow Q\bar{Q}$  process from the paper E. Braaten and Jungil Lee, Nucl. Phys. B **586**, 427 (2000), hep-ph/0004228.

If we write  $(3\delta)$  as the  $\delta(1-z)$  contribution in  $Q\bar{Q}$  final state

$$(2b) + (2c) + (2d) + (2e) + (3\delta) = \Delta_{\sim a} \times (1) \quad (5.145)$$

Though we don't know the origin of the divergence clearly,

$$\Delta_{\sim a} = \frac{3\alpha_s}{\pi} \left[ \frac{1}{\epsilon} + 1 - \gamma_E + \log \left( \frac{4\pi\mu^2}{m_Q^2} \right) + 2\log^2 2 - \frac{\pi^2}{6} \right] + O(\epsilon) \quad (5.146)$$

now we can find the IR divergence part to cancell that of diagram 2(a) and we also obtain the UV part simultaneously.

$$\frac{3\alpha_s}{\pi} \left[ \frac{1}{\epsilon} \right] = \frac{\alpha_s}{\pi} \left[ \frac{3}{2} \frac{1}{\epsilon_{UV}} + \frac{3}{2} \frac{1}{\epsilon_{IR}} \right] \quad (5.147)$$

And

$$\begin{aligned} \Delta_{(2)} &= \Delta_a + \Delta_{\sim a} \\ &= \frac{\alpha_s}{\pi} \left[ \frac{33 - 2n_f}{6} \left( \frac{1}{\epsilon_{UV}} + \frac{5}{3} - \gamma_E + \log \left( \frac{\pi\mu^2}{m_Q^2} \right) \right) \right. \\ &\quad \left. + \frac{2}{3} + 8\log 2 + 6\log^2 2 - \frac{\pi^2}{2} \right] + O(\epsilon) \end{aligned} \quad (5.148)$$

The UV divergent term in above equation ( $\delta(1-z)$  contribution) is proportional to that of the gluon splitting function

$$P_{gg}(z) = 6 \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left( \frac{11}{12} - \frac{n_f}{18} \right) \delta(1-z) \right] \quad (5.149)$$



# Chapter 6

## Dimensional Regularization

### 6.1 Minkowski space volume integral

- If we calculate next-to-leading or higher-order contributions in perturbation theory, we find the amplitude involves integrals over intermediate loop momenta in the form such as

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m^2 + i\epsilon)^\nu}. \quad (6.1)$$

Our original relativistic notation is written in the Minkowski space, where the scalar product of any two vectors are defined by  $A \cdot B = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$ .

- Find all the zeros of the denominator of the integrand.
- Discuss physical meaning of the zeros.
- Find all the residues of the integrand.
- By an appropriate change of variables, we can rewrite the integral in the following form.

$$\int \frac{d^D x}{(2\pi)^D} \frac{1}{(x^2 - z_0)^\nu}, \quad (6.2)$$

where  $z_0$  includes the infinitesimal imaginary part  $i\epsilon$ .

- Evaluate the following integral.

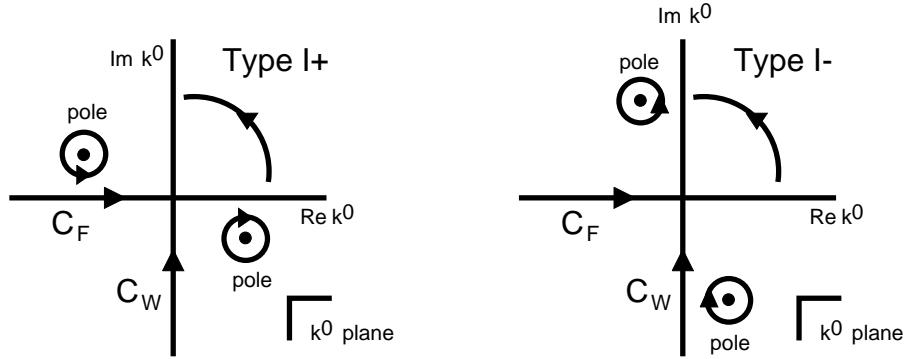
$$I(\nu, z_0) \equiv \int \frac{d^D x}{(2\pi)^D} \frac{1}{(x^2 - z_0)^\nu} = \frac{i}{(-1)^\nu (4\pi)^{\frac{D}{2}}} \times \frac{\Gamma(\nu - \frac{D}{2})}{\Gamma(\nu)} \times z_0^{\frac{D}{2} - \nu} \quad : \quad t, x \in R \text{ and } z_0 \in C - R$$

$$\begin{aligned}
I(\nu, z_0) &\equiv \int \frac{d^D x}{(2\pi)^D} \frac{1}{(x^2 - z_0)^\nu} \quad \leftarrow \quad x^2 = t^2 - r^2, \quad d^D x = dt d^{D-1} x = \Omega_{D-1} dt \ r^{D-2} dr \\
&= \frac{\Omega_{D-1}}{(2\pi)^D} \int_{-\infty}^{\infty} dt \int_0^{\infty} r^{D-2} dr \frac{1}{(-1)^\nu (r^2 + (z_0 - t^2))^\nu} \\
&\quad \leftarrow \frac{B(p, q)}{2(z_0 - t^2)^q} : p = \frac{D-1}{2}, \quad q = \nu - \frac{D-1}{2} \\
&= \frac{\Omega_{D-1} B(p, q)}{(2\pi)^D} \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{(-1)^\nu (z_0 - t^2)^q} \\
&= \frac{\Omega_{D-1} B(p, q)}{(2\pi)^D} \frac{1}{2} \int_{-\infty}^{\infty} \frac{t^{1-1} dt}{(-1)^{\nu+q} (t^2 + (-z_0))^q} \\
&= \frac{\Omega_{D-1} B(p, q) B(\frac{1}{2}, q - \frac{1}{2})}{(2\pi)^D 2(-1)^{\nu+q} (-z_0)^{q-\frac{1}{2}}} = \frac{\Omega_{D-1} B(p, q) B(\frac{1}{2}, \nu - \frac{D}{2})}{(2\pi)^D 2(-1)^{\nu+2q-\frac{1}{2}} z_0^{\nu-\frac{D}{2}}} \\
&\quad \leftarrow q - \frac{1}{2} = \nu - \frac{D}{2}, \quad (-1)^{2q} = [(-1)^2]^q = 1 \\
&= \frac{(-1)^{-\nu+\frac{1}{2}}}{4\pi} \times \frac{\Omega_{D-1}}{(2\pi)^{D-1}} \times \frac{\Gamma(p)\Gamma(q)}{\Gamma(\nu)} \times \frac{\sqrt{\pi}\Gamma(\nu - \frac{D}{2})}{\Gamma(q)} \times z_0^{\frac{D}{2}-\nu} \\
&= \frac{i(-1)^{-\nu}}{4\pi} \times \frac{2}{(4\pi)^p \Gamma(p)} \times \frac{\sqrt{\pi}\Gamma(p)\Gamma(\nu - \frac{D}{2})}{\Gamma(\nu)} \times z_0^{\frac{D}{2}-\nu} \\
&\quad \leftarrow p + \frac{1}{2} = \frac{D}{2} \\
&= \frac{i(-1)^{-\nu}}{4\pi} \times \frac{1}{(4\pi)^{p+\frac{1}{2}}} \times \frac{\Gamma(\nu - \frac{D}{2})}{\Gamma(\nu)} \times z_0^{\frac{D}{2}-\nu} \\
&= i(-1)^{-\nu} \times \frac{1}{(4\pi)^{\frac{D}{2}}} \times \frac{\Gamma(\nu - \frac{D}{2})}{\Gamma(\nu)} \times z_0^{\frac{D}{2}-\nu}
\end{aligned} \tag{6.3}$$

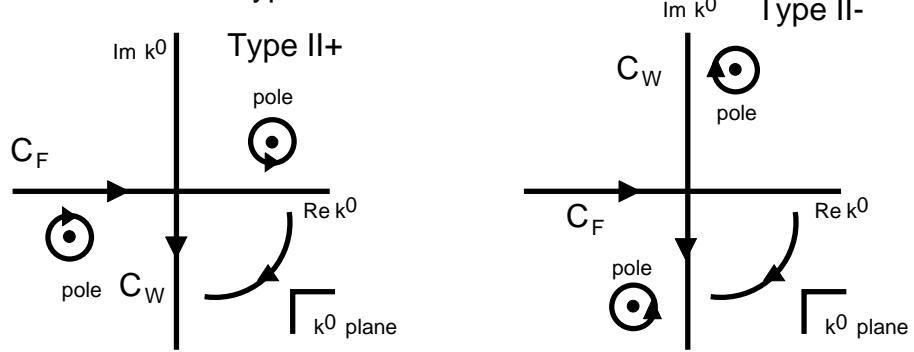
## 6.2 Wick Rotation

7. Our amplitude is written in Lorentz scalars. In the Minkowski space the metric tensor is not symmetric for space and time. Using Cauchy integral formula, we can deform the contour as far as the contour do not pass the poles.

### Wick Rotation Type I



### Wick Rotation Type II



$$I = \int_{-\infty}^{+\infty} \frac{f(t^2)dt}{(t^2 - z_0)^\nu} \quad (6.4)$$

$$z_0 = m^2 \mp i\epsilon : \int_{-\infty}^{+\infty} \frac{f(t^2)dt}{(t^2 - z_0)^\nu} = \pm i \int_{-\infty}^{+\infty} \frac{f(-u^2)du}{(-1)^\nu(u^2 + z_0)^\nu}, \quad m^2 \in R^+ \cup R^-, \epsilon \in R^+$$

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \frac{f(t^2)dt}{(t^2 - z_0)^\nu} = \int_{\mp i\infty}^{\pm i\infty} \frac{f(t^2)dt}{(t^2 - z_0)^\nu} \\ &= \int_{+\infty}^{-\infty} \frac{f((\pm iu)^2)(\pm i)du}{((\pm iu)^2 - z_0)^\nu} = \pm i \int_{-\infty}^{+\infty} \frac{f(-u^2)du}{(-u^2 - z_0)^\nu} \quad \leftarrow t = \pm iu \end{aligned} \quad (6.5)$$

<b>Type I<sup>+</sup></b>	$z_0 = m^2 - i\epsilon = (\pm  m e^{-i\epsilon})^2$	$m^2 > 0$
<b>Type I<sup>-</sup></b>	$z_0 = m^2 - i\epsilon = (\pm i m e^{+i\epsilon})^2$	$m^2 < 0$
<b>Type II<sup>+</sup></b>	$z_0 = m^2 + i\epsilon = (\pm  m e^{+i\epsilon})^2$	$m^2 > 0$
<b>Type II<sup>-</sup></b>	$z_0 = m^2 + i\epsilon = (\pm i m e^{-i\epsilon})^2$	$m^2 < 0$

Type I and II have different sign since the orientation of the contour enclosing the pole is opposite direction. By using Cauchy integral formula

$$f(a) = \oint_{C_F} dz_1 \frac{f(z_1)}{z_1 - a} = \oint_{C_W} dz_2 \frac{f(z_2)}{z_2 - a} \quad (6.6)$$

if the contours  $C_F$  and  $C_W$  have the same poles inside them. If there are poles as shown in the above figure, and the integration over the large circle is negligible,

$$k^0 = x + iy, \quad z_1 = \operatorname{Re} k^0 = x, \quad z_2 = i \operatorname{Im} k^0 = i y \quad (6.7)$$

$$f(a) = \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - a} = i \int_{-\infty}^{+\infty} dy \frac{f(iy)}{iy - a} \quad (6.8)$$

For example,

$$i\Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{k^2 - m^2 + i\varepsilon} \quad (6.9)$$

where  $p^0$  integration is along the real axis. Since the pole with respect to the complex variable  $p^0$  is just like that of the above figure,

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \times \frac{e^{-ik \cdot x}}{k^2 - m^2 + i\varepsilon} &= i \int \frac{d^4 k_E}{(2\pi)^4} \times \frac{e^{+ik_E \cdot x_E}}{-k_E^2 - m^2 + i\varepsilon} \\ &= -i \int \frac{d^4 k_E}{(2\pi)^4} \times \frac{e^{+ik_E \cdot x_E}}{k_E^2 + m^2 - i\varepsilon} \end{aligned} \quad (6.10)$$

If we define  $k_E^2 \equiv (k^0)^2 + \mathbf{k}^2$ ,

$$k_E = (k^1, k^2, k^3, k^4) = (k^1, k^2, k^3, k^0) \quad (6.11)$$

$$a_E \cdot b_E = \mathbf{a} \cdot \mathbf{b} + a^4 b^4 = \mathbf{a} \cdot \mathbf{b} - a^0 b^0 = -a \cdot b \quad (6.12)$$

If a complex function  $f(k^0)$  of a complex variable  $k^0$  has poles only in the

$$I = \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^r}{[k^2 - a^2 + i\varepsilon]^m}, \quad \text{with} \quad \begin{array}{lcl} d^D k & = & dk^0 dk^1 dk^2 \dots dk^{D-1} \\ k^2 & = & (k^0)^2 - (k^1)^2 - (k^2)^2 \dots - (k^{D-1})^2 \\ \epsilon, a^2 & > & 0 \end{array}$$

Wick rotation

- the pole is  $k^0 = \omega - i\varepsilon, -\omega + i\varepsilon$
- the first and the third quarter plane is free from pole
- rotate the integration from  $(-\infty, \infty) \rightarrow (-i\infty, i\infty)$

$$\begin{array}{ll} k^0 & \rightarrow ik^D \\ \int_{-\infty}^{+\infty} dk^0 & \rightarrow i \int_{-\infty}^{+\infty} dk^D \\ k^2 & \rightarrow -k_E^2 \equiv -[(k^1)^2 + (k^2)^2 \dots + (k^D)^2] \\ d^D k & = dk^0 dk^1 dk^2 \dots dk^{D-1} \\ d^D k_E & = dk^1 dk^2 dk^3 \dots dk^D \\ \int d^D k & \rightarrow i \int dk_E^D \end{array}$$

### 6.3 Useful Integrals in the Euclidian Space

$$\begin{aligned} \int d^D k_E &= \int |k_E|^{D-1} d|k_E| d\Omega_D = \frac{1}{2} \int |k_E|^{D-2} d|k_E|^2 d\Omega_D \\ &= \frac{1}{2} \int u^{\frac{D}{2}-1} du \Big|_{u=|k_E|^2} d\Omega_D = \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int u^{\frac{D}{2}-1} du \Big|_{u=|k_E|^2} \end{aligned} \quad (6.13)$$

$$\int \frac{d^D k_E}{(2\pi)^D} = \frac{1}{(2\pi)^D} \times \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int u^{\frac{D}{2}-1} du \Big|_{u=|k_E|^2} = \frac{1}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{D}{2})} \int u^{\frac{D}{2}-1} du \Big|_{u=|k_E|^2} \quad (6.14)$$

$$\begin{aligned} \int d\Omega_D &= \int_0^{2\pi} d\theta_1 \int_0^\pi d\theta_2 \sin \theta_2 \int_0^\pi d\theta_3 \sin^2 \theta_3 \dots \int_0^\pi d\theta_{D-1} \sin^{D-2} \theta_{D-1} \\ &= 2\pi \times (\sqrt{\pi})^{D-2} \frac{\Gamma(1)}{\Gamma(3/2)} \cdot \frac{\Gamma(3/2)}{\Gamma(5/2)} \dots \frac{\Gamma((D-3)/2)}{\Gamma((D-1)/2)} \cdot \frac{\Gamma((D-1)/2)}{\Gamma(D/2)} \\ &= \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}, \quad \text{since } \int_0^\pi \sin^m \theta d\theta = \frac{\sqrt{\pi} \Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \end{aligned} \quad (6.15)$$

### 6.4 Feynman Parametrization

8. The following integral formulas are useful in Feynman parametrization, which simplifies the integrals with multiple factors in the denominator.

$$\int_0^\infty \frac{dx}{(\alpha + \beta x)^2} = \frac{1}{\alpha^2} \int_0^\infty \frac{\frac{\alpha}{\beta} d\left(\frac{\beta}{\alpha}x\right)}{\left(1 + \frac{\beta}{\alpha}x\right)^2} = \frac{1}{\alpha\beta} \int_0^\infty \frac{dt}{t^2} = \frac{1}{\alpha\beta} \quad (6.16)$$

$$\begin{aligned} \frac{x^n}{C^m} \Big|_{C=\alpha+\beta x} &= \frac{1}{(m-1)!} \left(-\frac{\partial}{\partial C}\right)^{m-2} \left(\frac{x^n}{C^2}\right) \\ &= \frac{1}{(m-1)!} \left(-\frac{\partial}{\partial C}\right)^{m-n-2} \left(-\frac{\partial}{\partial C}\right)^n \left(\frac{x^n}{C^2}\right) \\ &= \frac{1}{(m-1)!} \left(-\frac{\partial}{\partial \alpha}\right)^{m-n-2} \left(-\frac{\partial}{\partial \beta}\right)^n \left(\frac{x^n}{(\alpha + \beta x)^2}\right) \end{aligned} \quad (6.17)$$

$$\left(-\frac{\partial}{\partial x}\right)^n \left(\frac{1}{x}\right) = \frac{n!}{x^{n+1}} = \frac{\Gamma(n+1)}{x^{n+1}} \quad \leftarrow \quad \Gamma(n+1) = n! \quad (6.18)$$

$$\begin{aligned} \int_0^\infty \frac{x^n dx}{(\alpha + \beta x)^m} &= \frac{1}{(m-1)!} \left(-\frac{\partial}{\partial \alpha}\right)^{m-n-2} \left(-\frac{\partial}{\partial \beta}\right)^n \int_0^\infty \frac{dx}{(\alpha + \beta x)^2} \\ &= \frac{1}{(m-1)!} \left(-\frac{\partial}{\partial \alpha}\right)^{m-n-2} \left(\frac{1}{\alpha}\right) \left(-\frac{\partial}{\partial \beta}\right)^n \left(\frac{1}{\beta}\right) \\ &= \frac{1}{(m-1)!} \times \frac{(m-(n+2))!}{\alpha^{m-(n+1)}} \times \frac{n!}{\beta^{n+1}} \\ &= \frac{\Gamma(m-(n+1))\Gamma(n+1)}{\Gamma(m)} \times \frac{\alpha^{(n+1)-m}}{\beta^{n+1}} \end{aligned} \quad (6.19)$$

## 6.5 $\Gamma$ function characteristics

$z\Gamma(z)$	$= \Gamma(z+1)$
$\Gamma(n+\epsilon)$	$= \begin{cases} (n-1)! & : n \geq 1 \\ \frac{(-1)^{ n }}{ n !} \left( \frac{1}{\epsilon} - (1 + \frac{1}{2} + \dots + \frac{1}{ n }) - \gamma_E + O(\epsilon) \right) & : n \leq 0 \end{cases}$
$\Gamma(\epsilon)A^{-\epsilon}$	$= \frac{1}{\epsilon} - (1 + \frac{1}{2} + \dots + \frac{1}{ n }) - \gamma_E - \log A + O(\epsilon)$
$\Gamma(n+\epsilon)A^{-\epsilon}$	$= \begin{cases} (n-1)! & : n \geq 1 \\ \frac{(-1)^{ n }}{ n !} \left( \frac{1}{\epsilon} - (1 + \frac{1}{2} + \dots + \frac{1}{ n }) - \gamma_E - \log A + O(\epsilon) \right) & : n \leq 0 \end{cases}$
$\gamma_E$	$= \lim_{n \rightarrow \infty} 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n = 0.5772157$

$$(m-1)!, \quad \Gamma(2 - \frac{D}{2}) = \frac{2}{\eta} - \gamma + O(\eta)$$

$$\Gamma(m - \frac{D}{2}) = \Gamma(m-2) + O(\eta) = (m-3)! + O(\eta) \rightarrow \frac{\Gamma(m - \frac{D}{2})}{\Gamma(m)} = \frac{1}{(m-1)(m-2)}, \quad \text{for } m > 2.$$

$$\Gamma(n) = (n-1)! \quad : \quad n > 1 \quad (6.20)$$

$$\lim_{n \rightarrow n_+ \pm 0} \Gamma(n) \rightarrow \pm\infty \quad : \quad (n_+ = 0, -2, -4, \dots) \quad (6.21)$$

$$\lim_{n \rightarrow n_- \pm 0} \Gamma(n) \rightarrow \mp\infty \quad : \quad (n_- = -1, -3, -5, \dots) \quad (6.22)$$

$$\lim_{\epsilon \rightarrow +0} \Gamma(\epsilon)A^{-\epsilon} = \lim_{\epsilon \rightarrow +0} \left[ \frac{1}{\epsilon} - \gamma_E - \log \left( \frac{A}{4\pi} \right) + O(\epsilon) \right] \quad (6.23)$$

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \quad (\gamma = 0.577216\dots : \text{Euler - Mascheroni constant})$$

$$\prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \rightarrow (1 + z \sum_{k=1}^{\infty} \frac{1}{k} + O(z^2)) e^{-z \sum_{k=1}^{\infty} \frac{1}{k}} = 1 + O(z^2), \quad \text{as } z \rightarrow 0$$

$$\frac{1}{\Gamma(z)} \rightarrow z(1 + \gamma z + O(z^2)), \quad \text{as } z \rightarrow 0$$

$$\Gamma(bz) \rightarrow \left( \frac{1}{bz} - \gamma + O(z) \right), \quad \text{as } z \rightarrow 0$$

$$A^{az} = \exp[\log A^{az}] = e^{az \log A} \rightarrow 1 + az \log A + O(z^2), \quad \text{as } z \rightarrow 0$$

$$\lim_{z \rightarrow 0} A^{az} \Gamma(bz) \rightarrow \frac{1}{bz} - \gamma + \frac{a}{b} \log A + O(z)$$

$$\lim_{\eta \rightarrow 0} A^{-\frac{\eta}{2}} \Gamma(\frac{\eta}{2}) \rightarrow \frac{2}{\eta} - \gamma - \log A + O(\eta)$$

$$D = 4 - \eta, \quad \frac{D}{2} = 2 - \frac{\eta}{2}, \quad \frac{D}{2} - 2 = -\frac{\eta}{2}, \quad 2 - \frac{D}{2} = \frac{\eta}{2}.$$

## Chapter 7

# One Loop Recurrence Relations

### 7.1 Basic Scalar Integral Evaluation

$$\begin{aligned} F_n(D) &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - D)^n} = \frac{i}{(-1)^n (4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} D^{\frac{d}{2} - n} \\ F_2(D) &= CD^{\frac{d}{2} - 2} \\ C &= \frac{i}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) = \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \\ F_n(D) &= 0 \quad \text{if } n \leq 0 \\ F_1(D) &= \frac{2}{d-2} CD^{\frac{d-2}{2}} \\ F_2(D) &= CD^{\frac{d-4}{2}} \\ F_3(D) &= \frac{d-4}{4} CD^{\frac{d-6}{2}} \\ F_4(D) &= \frac{(d-4)(d-6)}{4 \cdot 6} CD^{\frac{d-8}{2}} \\ F_5(D) &= \frac{(d-4)(d-6)(d-8)}{4 \cdot 6 \cdot 8} CD^{\frac{d-10}{2}} \\ F_n(D) &= \frac{(d-4)(d-6)(d-8) \cdots (d-2(n-1))}{4 \cdot 6 \cdot 8 \cdots 2(n-1)} CD^{\frac{d-2n}{2}} \quad \text{if } n \geq 3 \\ \frac{\partial}{\partial D} F_n(D) &= n F_{n+1}(D) \end{aligned}$$

## 7.2 Feynman Trick

$$\begin{aligned}
\frac{1}{AB} &= \int_0^1 dx \frac{1}{[(1-x)A + xB]^2} \\
\frac{1}{A_1 A_2 \cdots A_n} &= \Gamma(n) \int_0^1 dx_1 dx_2 \cdots dx_n \delta(\sum_i x_i - 1) \frac{1}{[\sum_i x_i A_i]^n} \\
\frac{1}{A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n}} &= \frac{\Gamma(\sum_i k_i)}{\Pi_i \Gamma(k_i)} \int_0^1 dx_1 dx_2 \cdots dx_n \delta(\sum_i x_i - 1) \frac{\Pi_i x_i^{k_i-1}}{[\sum_i x_i A_i]^{\sum_i k_i}} \\
\left(-\frac{\partial}{\partial A}\right)^n \frac{1}{A} &= \frac{\Gamma(n+1)}{A^{n+1}}, \quad \frac{1}{A^n} = \frac{1}{\Gamma(n)} \left(-\frac{\partial}{\partial A}\right)^{n-1} \frac{1}{A} \\
\frac{1}{A^m B^n} &= \frac{1}{\Gamma(m)\Gamma(n)} \left(-\frac{\partial}{\partial A}\right)^{m-1} \left(-\frac{\partial}{\partial B}\right)^{n-1} \frac{1}{AB} \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \left(-\frac{\partial}{\partial A}\right)^{m-1} \left(-\frac{\partial}{\partial B}\right)^{n-1} \int_0^1 dx \frac{1}{[(1-x)A + xB]^2} \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \left(-\frac{\partial}{\partial C}\right)^{m+n-2} \int_0^1 dx \frac{(1-x)^{m-1} x^{n-1}}{[(1-x)A + xB]^2} \\
&= \frac{1}{\Gamma(m)\Gamma(n)} \left(-\frac{\partial}{\partial C}\right)^{m+n-1} \int_0^1 dx \frac{(1-x)^{m-1} x^{n-1}}{[(1-x)A + xB]} \\
&= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 dx \frac{(1-x)^{m-1} x^{n-1}}{[(1-x)A + xB]^{m+n}}
\end{aligned}$$

## 7.3 primitive recurrence relation

$$\begin{aligned}
I_n &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 + 2k \cdot p)} \\
I'_n &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 + 2k \cdot p)^2}
\end{aligned}$$

$$\begin{aligned}
F_2(D) &= \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) D^{\frac{d-4}{2}} = CD^{\frac{d-4}{2}} \\
I_0 &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + 2k \cdot p)} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} = F_1(m^2) = \frac{2m^2}{d-2} F_2(m^2) \\
I'_0 &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + 2k \cdot p)^2} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)^2} = F_2(m^2) \\
I_1 &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + 2k \cdot p)} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+xp)^2 - x^2 m^2]^2} \\
&= \int_0^1 dx F_2(x^2 m^2) = F_2(m^2) \int_0^1 dx x^{d-4} = \frac{F_2(m^2)}{d-3} \\
I'_1 &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2 + 2k \cdot p)^2} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{2x}{[(k+xp)^2 - x^2 m^2]^3} \\
&= 2 \int_0^1 dx x F_3(x^2 m^2) = 2 F_3(m^2) \int_0^1 dx x^{d-5} = 2 \frac{d-4}{4m^2} F_2(m^2) \frac{1}{d-4} = \frac{1}{2m^2} F_2(m^2) \\
I_2 &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2(k^2 + 2k \cdot p)} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{2(1-x)}{[(k+xp)^2 - x^2 m^2]^3} \\
&= 2 \int_0^1 dx (1-x) F_3(x^2 m^2) = 2 \frac{d-4}{4m^2} F_2(m^2) \int_0^1 dx (1-x) x^{d-6} \\
&= \frac{d-4}{2m^2} \left( \int_0^1 dx x^{d-6} - \int_0^1 dx x^{d-5} \right) = \frac{d-4}{2m^2} \left( \frac{1}{d-5} - \frac{1}{d-4} \right) F_2(m^2) \\
&= \frac{1}{2m^2(d-5)} F_2(m^2)
\end{aligned}$$

where we neglect the divergent term  $\lim_{x \rightarrow 0} 1/x^{(d-5)}$  since it is Coulomb divergence term and it will be absorbed to the long-range matrix element.

## 7.4 Integration by parts

$$\begin{aligned}
I_n &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 - 2k \cdot p)} \\
I'_n &\equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^n (k^2 - 2k \cdot p)^2} \\
&\quad \int \frac{d^d k}{(2\pi)^d} f(k) = I \rightarrow \text{if convergent} \\
&\quad \rightarrow \int \frac{d^d k}{(2\pi)^d} f(k + \Delta) = \int \frac{d^d k}{(2\pi)^d} f(k) \\
&\quad \rightarrow \int \frac{d^d k}{(2\pi)^d} [f(k + \Delta) - f(k)] = 0 \\
&\quad \rightarrow \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} f(k) = 0
\end{aligned}$$

By the same way,

$$\text{If } \int \frac{d^d k}{(2\pi)^d} f^\mu(k) \text{ is convergent Then } \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k^\mu} f^\mu(k) = 0$$

By use of this method, we can obtain the relations among scalar integrals. For example,

$$\begin{aligned} f &= \frac{1}{(k^2)^n (k^2 + 2k \cdot p)} \\ f_1^\mu &= p^\mu f \\ f_2^\mu &= k^\mu f \\ \frac{\partial}{\partial k^\mu} f &= \frac{-2nk^\mu}{(k^2)^{n+1} (k^2 + 2k \cdot p)} + \frac{-2(k+p)^\mu}{(k^2)^n (k^2 + 2k \cdot p)^2} \\ \frac{\partial}{\partial k^\mu} k^\mu &= g_\mu^\mu = d, \quad \frac{\partial}{\partial k^\mu} p^\mu = 0 \\ \frac{\partial}{\partial k^\mu} (k^\mu f) &= df + \frac{-2nk^2}{(k^2)^{n+1} D} + \frac{-2k^2 - 2k \cdot p}{(k^2)^n D^2} \rightarrow D = (k^2 + 2k \cdot p) \\ &= +\frac{d}{(k^2)^n D} + \frac{-2n}{(k^2)^n D} + \frac{-(D+k^2)}{(k^2)^n D^2} \\ &= +\frac{d-2n-1}{(k^2)^n D} + \frac{-1}{(k^2)^{n-1} D^2} \\ \rightarrow I'_{n-1} &= (d-2n-1)I_n \\ \frac{\partial}{\partial k^\mu} (p^\mu f) &= \frac{-2nk \cdot p}{(k^2)^{n+1} (k^2 + 2k \cdot p)} + \frac{-2k \cdot p - 2m^2}{(k^2)^n (k^2 + 2k \cdot p)} \rightarrow D = (k^2 + 2k \cdot p) \\ &= \frac{-n(D-k^2)}{(k^2)^{n+1} D} + \frac{-(D-k^2) - 2m^2}{(k^2)^n D^2} \\ &= \frac{n}{(k^2)^n D} + \frac{-1}{(k^2)^n D} + \frac{1}{(k^2)^{n-1} D^2} + \frac{-2m^2}{(k^2)^n D^2} \\ \rightarrow 0 &= (n-1)I_n + I'_{n-1} - 2m^2 I'_n \\ I_{n+1} &= \frac{d-n-2}{2m^2(d-2n-3)} I_n \end{aligned}$$

The two recurrence relation

$$\begin{aligned} I'_{n-1} &= (d-2n-1)I_n \\ I_{n+1} &= \frac{d-n-2}{2m^2(d-2n-3)} I_n \end{aligned}$$

or

$$\begin{aligned} I'_n &= (d-2n-3)I_{n+1} \\ I_n &= \frac{d-n-1}{2m^2(d-2n-1)} I_{n-1} \end{aligned}$$

generates all the relations given in the previous section. Furthermore, now we should integrate only one scalar integral such as

$$I_0 = \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \times \frac{2m^2}{d-2}$$

So that

$$\begin{aligned} I_n &= \frac{i}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \times \frac{2m^2}{(d-2)} \times \prod_{k=1}^n \frac{(d-k-1)}{2m^2(d-2k-1)} \\ I'_n &= (d-2n-3)I_{n+1} \end{aligned}$$

Here, we only considered the convergent part from the beginning, we did not faced the divergences come from Coulomb interaction. Therefore the method **integration by parts** is powerful in the loop-integral evaluation.

## 7.5 $I_n$ for non-integer $n$

With the aid of integration by parts we have obtained the recurrence relation. In order to get  $I_x$  for any real number, we should first evaluate the integral  $I_x$  where  $0 < x < 1$  and extend by using the recurrence relation. However, there is no more difficulties obtaining the integral  $I_x$  for any real number  $x$  since the Feynman parametrization is not more complicated than the case  $0 < x < 1$ .

$$\frac{1}{A^n B} = n \int_0^1 dx \frac{x^{n-1}}{[xA + (1-x)B]^{n+1}}$$

Above formula is valid for any real number  $n$ . Using the Feynman parametrization, the integral  $I_n$  is evaluated as

$$\begin{aligned} I_n &= n \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{x^{n-1}}{[xA + (1-x)B]^{n+1}} \\ &= n \int_0^1 dx x^{n-1} \int \frac{d^d k}{(2\pi)^d} \frac{1}{([k + (1-x)p]^2 - (1-x)^2 m^2 + i\epsilon)^{n+1}} \\ &= \frac{ni}{(-1)^{n+1} (4\pi)^{d/2}} \frac{\Gamma(n+1-\frac{d}{2})}{\Gamma(n+1)} (m^2 - i\epsilon)^{\frac{d}{2}-(n+1)} \int_0^1 dx x^{n-1} (1-x)^{d-2n-2} \\ &= \frac{ni}{(-1)^{n+1} (4\pi)^{d/2}} \frac{\Gamma(n+1-\frac{d}{2})}{\Gamma(n+1)} (m^2 - i\epsilon)^{\frac{d}{2}-(n+1)} \frac{\Gamma(n)\Gamma(d-2n-1)}{\Gamma(d-n-1)} \\ &= \frac{i}{(-1)^{n+1} (4\pi)^{d/2}} \frac{\Gamma(n+1-\frac{d}{2}) \Gamma(d-2n-1)}{\Gamma(d-n-1)} (m^2 - i\epsilon)^{\frac{d}{2}-(n+1)} \\ &= \frac{i}{(4\pi)^2} \frac{\Gamma(n+1-\frac{d}{2}) \Gamma(d-2n-1)}{\Gamma(d-n-1)} \left(\frac{4\pi\mu^2}{m^2 - i\epsilon}\right)^{\frac{4-d}{2}} (-m^2 + i\epsilon)^{1-n} \end{aligned}$$

where  $A = k^2 + i\epsilon$  and  $B = k^2 + 2k \cdot p$ .

$$\begin{aligned} J_n &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2 - i\epsilon)^n (k^2 + 2k \cdot p)} \\ &= (-1)^n I_n = \frac{-i}{(4\pi)^2} \frac{\Gamma(n+1-\frac{d}{2}) \Gamma(d-2n-1)}{\Gamma(d-n-1)} \left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{4-d}{2}} (m^2)^{1-n} \end{aligned}$$