Anomalies and SPT phases

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Based on

- A review of [1508.04715] by Witten
- [1607.01873] KY
- [1609.?????] Yuji Tachikawa and KY

Introduction

One of the motivations:

What is the most general ('t Hooft) anomaly of QFTs?

Anomaly can have many implications for

- High energy physics
- Condensed matter physics
- Mathematics
- …

Introduction

Perturbative anomalies (one-loop diagrams) are not enough!

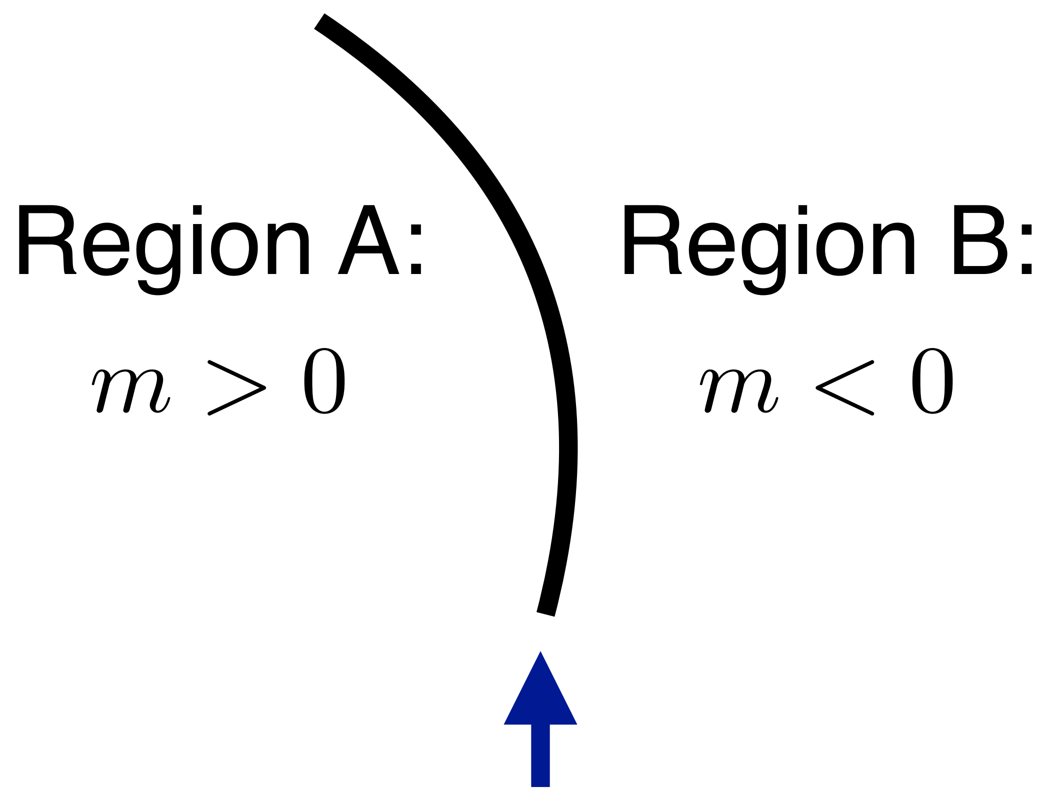
- Global anomalies
- Anomalies of discrete symmetries (e.g. time-reversal)
- Anomalies of TQFTs
- …

Just for chiral fermions, a general anomaly formula was proposed only recently. [Witten,2015]

Introduction

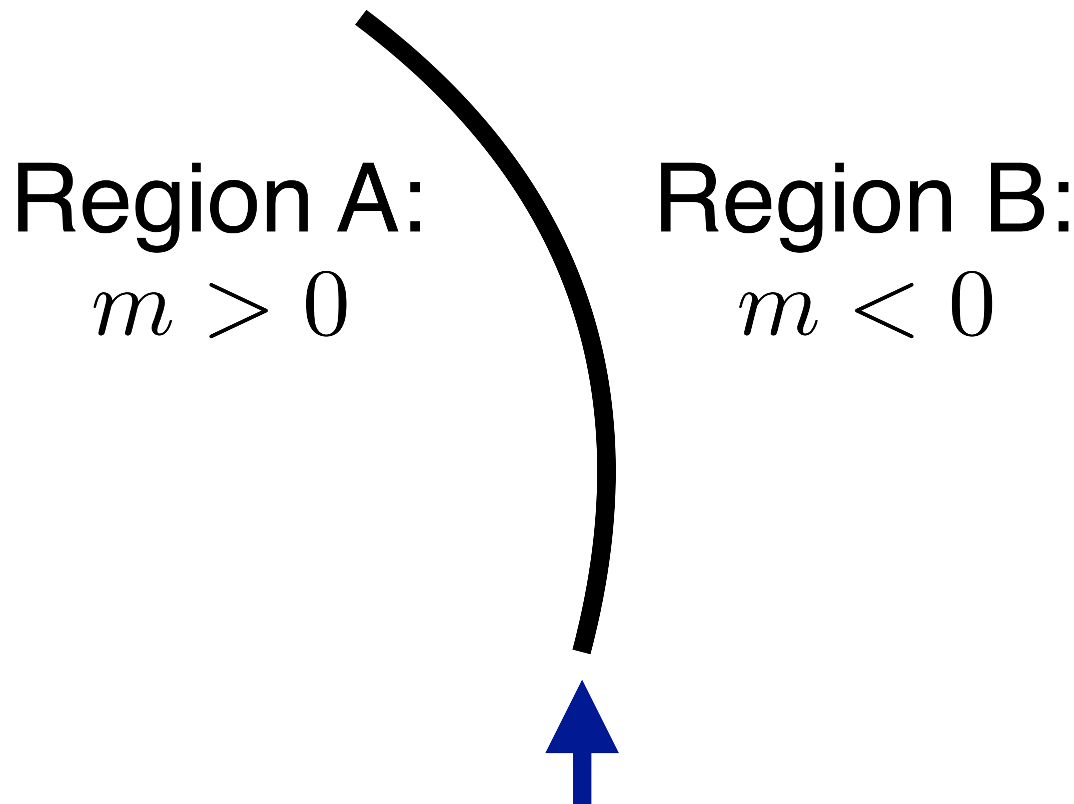
A chiral fermion in d-dimensions may be realized as a domain wall fermion in one higher dimensions.

$$\boxed{\begin{array}{l}\text{Massive fermion in (d+1)-dimensions} \\ \qquad \mathcal{L} = \bar{\Psi}(iD + m)\Psi \quad (\, D \text{ : Dirac operator})\end{array}}$$



Localized zero mode between two regions.

Introduction



Massless chiral fermion between two regions.

Introduction

Modern understanding:

Region A:
$m > 0$

→ SPT phase1

Region B:
$m < 0$

→ SPT phase2

Massless chiral fermion between two regions.

→ Anomalous quantum field theory must appear.

Short summary of the talk

Message 1

- **Anomalies = phase ambiguity of partition function are classified by SPT phases**

Message 2

- **TQFTs can have ('t Hooft) anomalies**

Contents

SPT phases

G-protected Symmetry Protected Topological (SPT) phase
(in high energy physics point of view:)

- It is a QFT with global symmetry G.
- The Hilbert spaces are always one-dimensional on any compact space.
- The partition functions are just a phase (up to continuous deformation).

$Z(X)$: partition function on a compact manifold X

$$|Z(X)| = 1$$

Also called invertible field theory by mathematicians.
[Freed,Hopkins,…]

SPT phases

<u>Example</u>: Integer quantum hall state

- 3=2+1 dimensions
- Global symmetry G=U(1)
- Partition function: Chern-Simons action of background

$$\log Z = \int \frac{ik}{4\pi} A dA$$

$A = A_\mu dx^\mu$: background field (connection) for G=U(1)

$k \in \mathbb{Z}$: integer

SPT phases

Example: Integer quantum hall state

SPT phase1

$$\log Z = \int \frac{ik_1}{4\pi} AdA$$

SPT phase2

$$\log Z = \int \frac{ik_2}{4\pi} AdA$$

$(k_1 - k_2)$ massless chiral fermions between two regions due to anomaly inflow.

Contents

SPT from massive fermions

A class of SPT phases is obtained from massive fermions.

Massive fermion in (d+1)-dimensions

$$S = \int d^{d+1}x \; \bar{\Psi}(iD + m)\Psi$$

$$D = i\gamma^\mu(\partial_\mu - iA_\mu + \cdots)$$

: Dirac operator coupled to background G-field A_μ .

SPT from massive fermions

A class of SPT phases is obtained from massive fermions.

Massive fermion in (d+1)-dimensions

$$S = \int d^{d+1}x \; \bar{\Psi}(iD + m)\Psi$$

$\Downarrow$ Low energy limit (RG flow)

SPT phase (invertible field theory)

SPT from massive fermions

Partition function:
Euclidean path integral on a compact manifold X

$$
\begin{aligned}
Z &= \int \mathfrak{D}\Psi e^{-S} \\
&= \frac{\det(iD+m)}{\det(iD+\Lambda)} \\
&= \prod_{\lambda \in \mathrm{Spec}(D)} \frac{(i\lambda+m)}{(i\lambda+\Lambda)}
\end{aligned}
$$

Λ : Pauli-Villars regulator mass

λ : eigenvalues of the Dirac operator D

SPT from massive fermions

Low energy limit $\longleftrightarrow$ Taking the mass m very large

$$m \to \pm\Lambda$$

$$\begin{aligned} Z &= \prod_{\lambda \in \mathrm{Spec}(D)} \frac{(i\lambda \pm \Lambda)}{(i\lambda + \Lambda)} \\ &= \begin{cases} 1 & m = +\Lambda \\ \exp(-2\pi i \cdot \eta) & m = -\Lambda \end{cases} \qquad (\Lambda \to \infty) \end{aligned}$$

η : Atiyah-Patodi-Singer eta-invariant.

[Atiyah-Patodi-Singer, 1975]

SPT from massive fermions

SPT phase 1

$$m > 0$$

$$Z = 1$$

: trivial phase

SPT phase 2

$$m < 0$$

$$Z = \exp(-2\pi i \cdot \eta)$$

: nontrivial phase

The SPT phases realized by fermions are completely characterized by the Atiyah-Patodi-Singer eta-invariant!

SPT from massive fermions

Example 1: Integer quantum hall state

Suppose there are k copies of massive fermion coupled to

$A = A_\mu dx^\mu$: background field (connection) for G=U(1)

$$Z = \exp(-2\pi i \cdot \eta)$$
$$= \exp(\int \frac{ik}{4\pi} AdA)$$

: Atiyah-Patodi-Singer index theorem

We recover $\log Z = \int \frac{ik}{4\pi} AdA$

SPT from massive fermions

<u>Example 2</u>: 3+1 dim. topological superconductors

- Low energy limit of ν copies of massive Majorana fermion

- Symmetry G = time-reversal
  (or equivalently spatial reflection by CPT theorem)

- Background field for the time-reversal symmetry
  = Putting the theory on unorientable (pin^+) manifold.

SPT from massive fermions

Example 2: 3+1 dim. topological superconductors

- Fact 1: $Z(X)$ is always a 16-th root of unity.

- Fact 2: $\mathbb{RP}^4$: real projective space (generator of cobordism group)

$$
\begin{aligned}
Z(\mathbb{RP}^4) &= \exp(-\pi i \cdot \eta(\mathbb{RP}^4)) \quad \text{(for Majorana)} \\
&= \exp(\frac{2\pi i \nu}{16})
\end{aligned}
$$

[Witten,2015]

3+1 dim. topological superconductors are classified by

$$\nu \in \mathbb{Z}_{16}$$

Berry phase of SPT phase

Berry phase of SPT phase

$$S = \int d^{d+1}x \ \bar{\Psi}(iD + m)\Psi$$

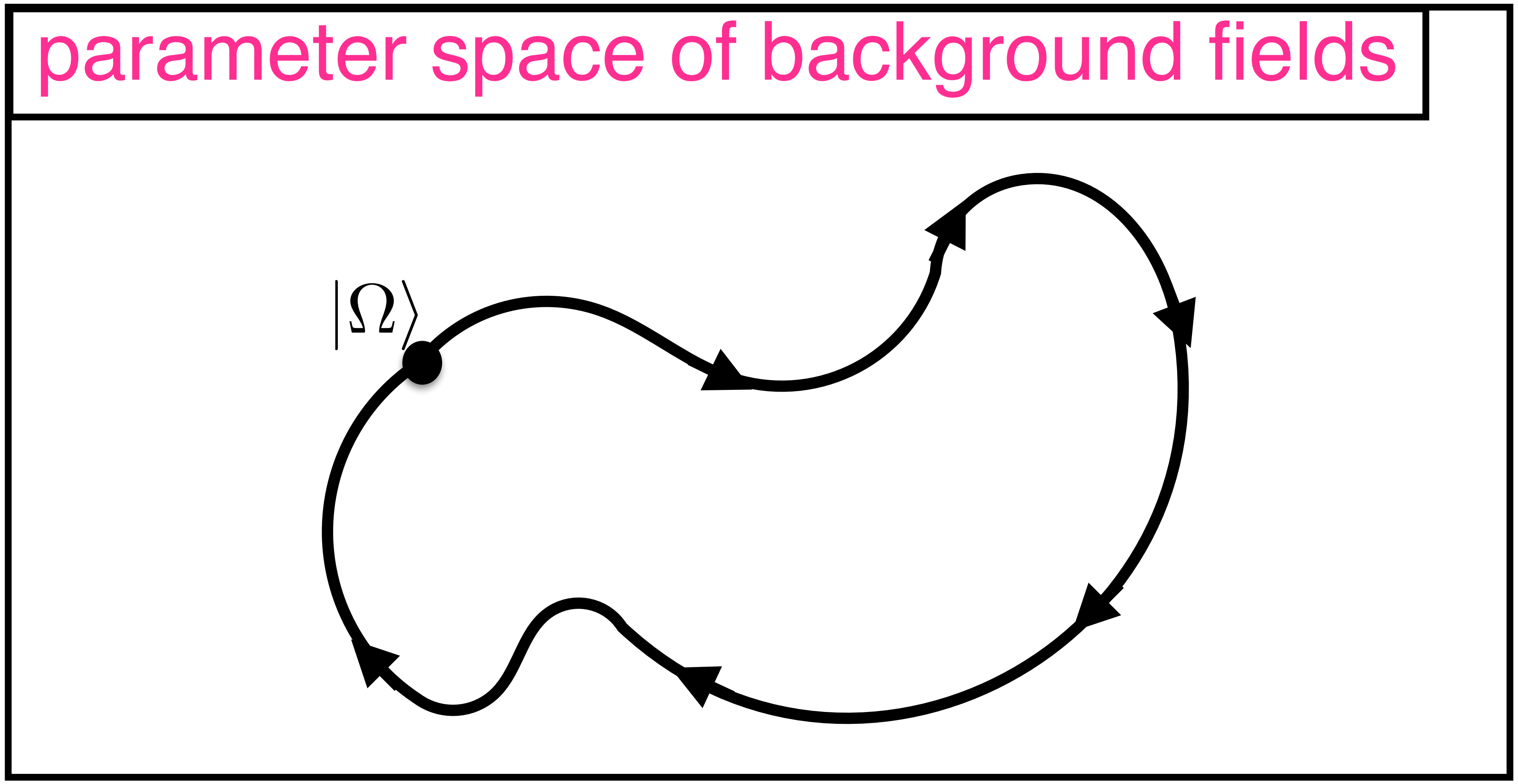
The ground state is one-dimensional in the Hilbert space.

= The unique state of the SPT phase (invertible field theory)

$|\Omega\rangle$ on a spacial manifold Y

(d-dimensional)

Berry phase of SPT phase



$|\Omega\rangle \to e^{iB}|\Omega\rangle$: Berry phase

(Mathematically, it is a holonomy of a line bundle.)

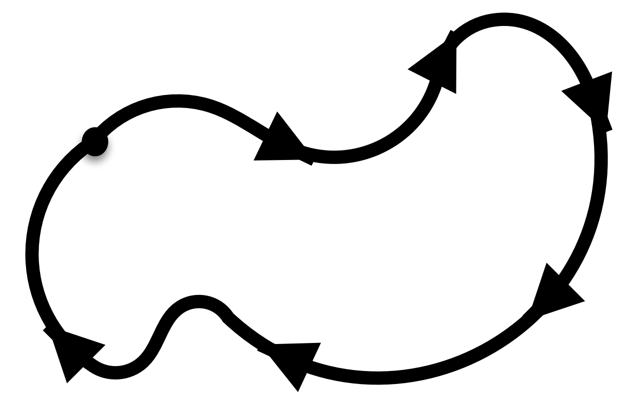
Berry phase of SPT phase

Change parameters as (Euclidean) time τ evolves.

$g_{ij}(\tau, \vec{x}),\ \vec{A}(\tau, \vec{x})$: slowly change as τ evolves

Returning to the same point:

$$\tau \in S^1$$

$S^1 =$ 

Berry phase = partition function on

$$S^1 \times Y$$

Euclidean time

d-dim. space

Berry phase of SPT phase

Berry phase formula:

$$
\begin{aligned}
e^{iB} &= Z(S^1 \times Y) \\
&= \exp(-2\pi i \cdot \eta(S^1 \times Y)) \qquad (m < 0)
\end{aligned}
$$

Remark: More generally, one can replace $S^1 \times Y$ by a fiber bundle with base S^1 and fiber Y

We may call $Z = \exp(-2\pi i \cdot \eta(X))$: generalized Berry phase (only in this talk)

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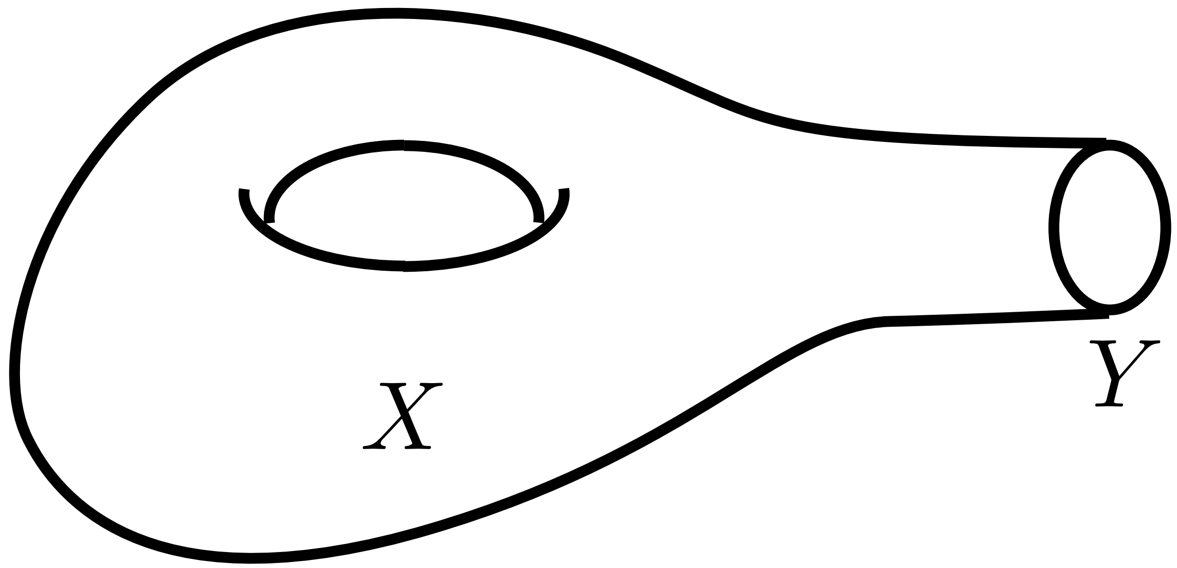
Manifolds with boundary



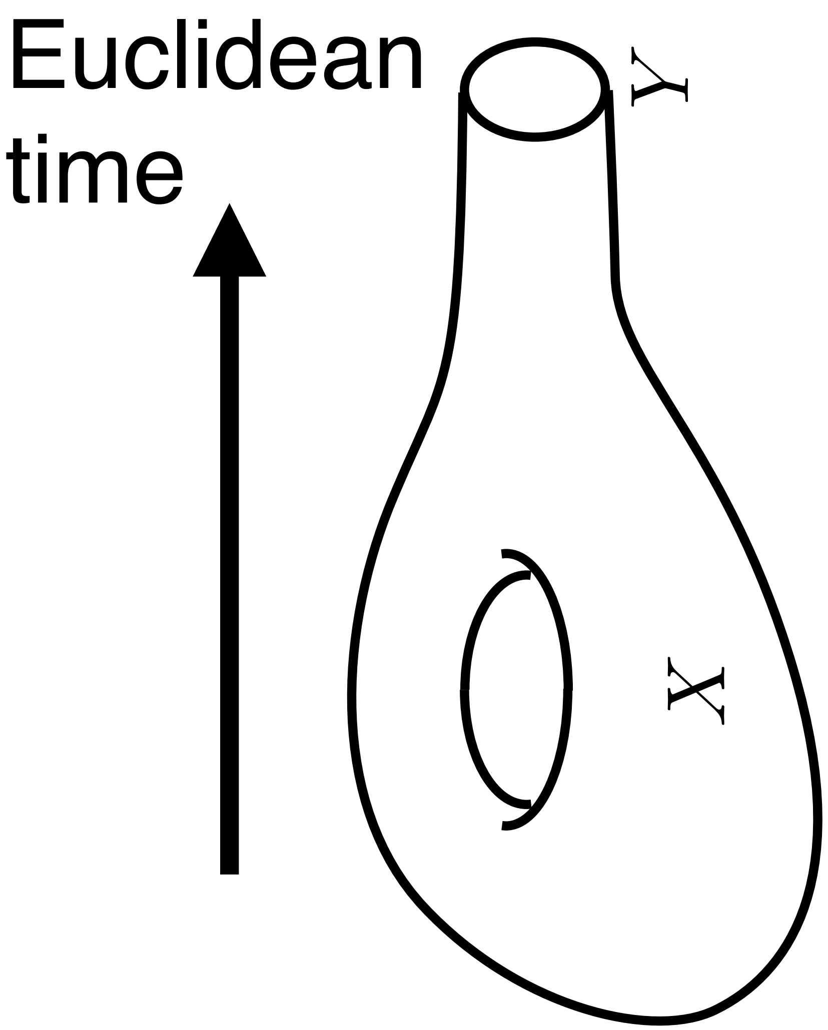
Mathematical fact:

$Z_{\text{bulk}} = \exp(-2\pi i \cdot \eta)$ is ambiguous in the presence of the boundary Y.

(But this "ambiguity" is very well controlled, in terms of what is called determinant line bundle.) [Dai-Freed,1994]

Physics of ambiguity 1

When Y is a time-slice



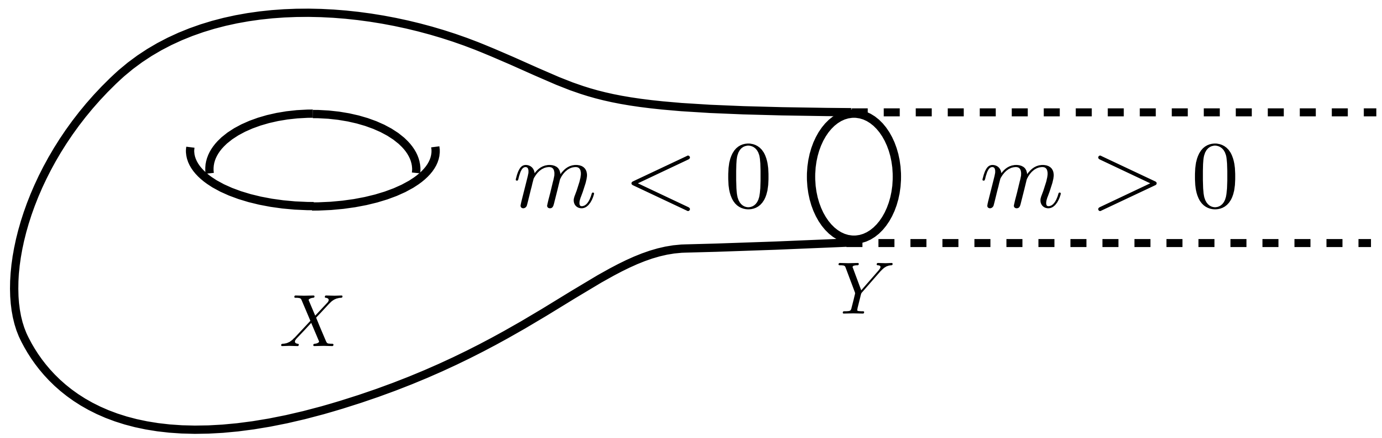
- The path integral creates a state $|X\rangle$ on the manifold Y

$Z_{\mathrm{bulk}} = \exp(-2\pi i \cdot \eta)$ is related to the amplitude $\langle\Omega|X\rangle$

But the phase of $|\Omega\rangle$ is ambiguous due to (generalized) Berry phase.

Physics of ambiguity 2

When Y is a spatial boundary

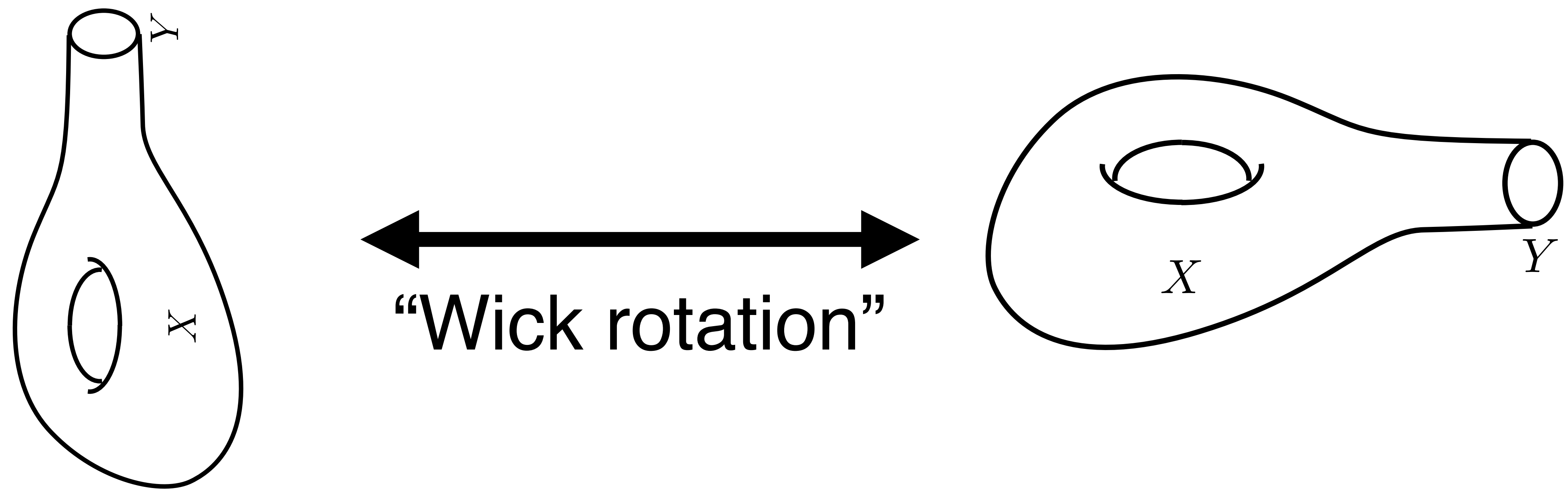


Ambiguity of $Z_{\text{bulk}} = \exp(-2\pi i \cdot \eta)$ must be compensated by another ambiguity from the **boundary theory anomaly**.

$$\left. \begin{array}{l} Z_{\text{bulk}} = \exp(-2\pi i \cdot \eta) \\ Z_{\text{boundary}} \end{array} \right\}$$:ambiguous (section of a line bundle)

$$Z = Z_{\text{bulk}} \cdot Z_{\text{boundary}}$$:not ambiguous (function)

Bulk/Boundary relation



Anomaly of boundary theory

= (Generalized) Berry phase of bulk SPT phase

(Mathematically they are sections of the determinant line bundle and its dual.)

For more details and mathematical structure, see [Witten,2015] [KY,2016]

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3. A class of SPT phases from massive fermions
4. Manifolds with boundary and fermion anomalies
5. Anomalies of topological quantum field theory
6. Summary

’t Hooft anomaly matching

UV anomaly = IR anomaly

UV theory

$\Big\downarrow$ RG flow

IR theory

't Hooft anomaly matching

Sometimes it is possible that

$$\begin{tikzcd} \text{UV fermion theory} \arrow[d, "\text{RG flow}"] \\ \text{IR TQFT} \end{tikzcd}$$

IR TQFT must have anomaly!

See e.g. [Seiberg-Witten,2016]
[Witten,2016]

Topological superconductor

I only discuss topological superconductor

- 3-dim. theory on boundary (Chern-Simons theory, etc.)

- Symmetry G = time-reversal → unorientable manifold

- Classified in bulk by

$$\nu \in \mathbb{Z}_{16}$$

Framing anomaly

Before going to the time-reversal anomaly:

In Chern-Simons theories, a well-known anomaly is the framing anomaly.

Setup: $Y = D^2 \times S^1$ ($D^2 =$ disk)

$$\partial Y = S^1 \times S^1 = T^2$$

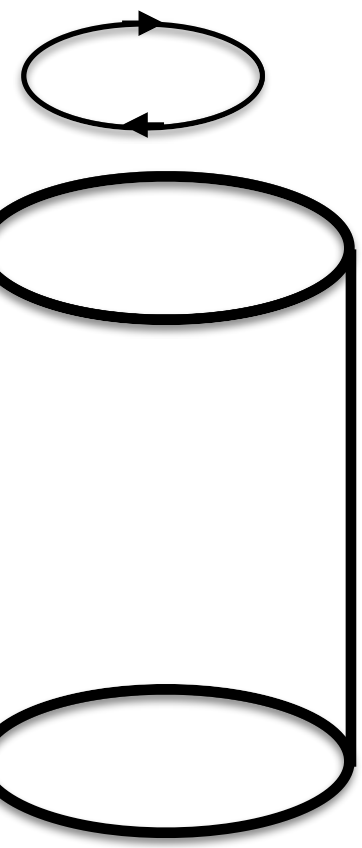
$|Y\rangle$: state on the torus $\partial Y = T^2$ obtained by the path integral on Y

$T \in SL(2, Z)$ an element of modular transformation of the torus.

Framing anomaly

Point : $T \in SL(2, Z)$ does not change the topology of the manifold $Y = D^2 \times S^1$

$T :$



Twist of $D^2 \times [0, 1]$ where $D^2 \times \{0\}$ and $D^2 \times \{1\}$ are glued.

Framing anomaly

Point : $T \in SL(2,Z)$ does not change the topology of the manifold $Y = D^2 \times S^1$

So naively $T|Y\rangle = |Y\rangle$, but actually

$$T|Y\rangle = e^{2\pi i c}|Y\rangle$$

$e^{2\pi i c}$: framing anomaly

In surgery, this gives an ambiguity of partition function.

[Witten,1988]

New anomaly

A time-reversal invariant theory has

$$c = 0$$

because it is a “parity odd” quantity.

However, now we encounter a new anomaly
which I ‘m going to discuss.

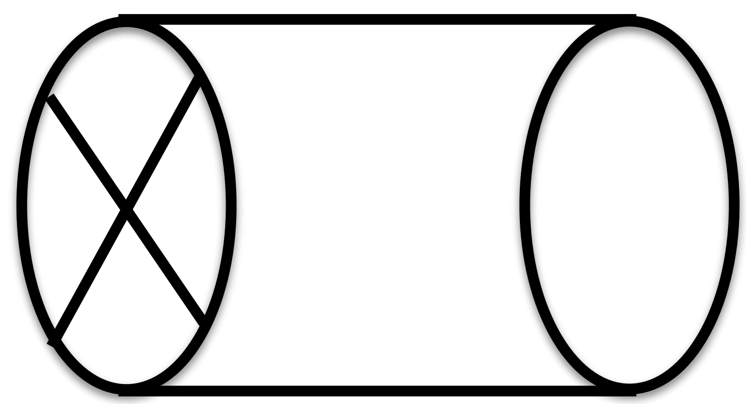
Time reversal anomaly

Setup:

$$Y' = CC^2 \times S^1$$
$$\partial Y' = S^1 \times S^1 = T^2$$

$CC^2 =$ crosscap =

: unoriented



$|Y'\rangle$: state on the torus $\partial Y' = T^2$ obtained by the path integral on Y'

$T \in SL(2, Z)$ an element of modular transformation of the torus.

Time reversal anomaly

Point : $T \in SL(2, Z)$ does not change the topology of the manifold $Y' = CC^2 \times S^1$

So naively $T|Y'\rangle = |Y'\rangle$, but actually

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle$$

$\exp(\frac{2\pi i\nu}{16})$: 't Hooft anomaly of time reversal.

[Tachikawa-KY,2016]

In "surgery", this gives an ambiguity of partition function.

Remark

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle$$

By anomaly matching with fermions, it is known that

$$\nu \in \mathbb{Z}_{16}$$

For some examples, please see our paper

[Tachikawa-KY, to appear]

But a systematic understanding of anomalies in TQFTs at the same level as the case of fermions in

[Dai-Freed 1994; Witten 2015; KY 2016]

is still lacking.

Contents

Summary

- Ambiguity of partition functions on manifolds with boundary gives an interesting relation

Anomaly of boundary theory

= (Generalized) Berry phase of bulk SPT phase

- TQFTs can have ambiguities of partition functions which satisfy anomaly matching.

$$T|Y'\rangle = \exp(\frac{2\pi i\nu}{16})|Y'\rangle \qquad \begin{matrix} Y' = CC^2 \times S^1 \\ T \in SL(2, Z) \end{matrix}$$

Future direction

- Systematically understand anomalies of TQFTs.

$\Updownarrow$ Anomaly matching

- Deeper understanding of $Z = \exp(-2\pi i \cdot \eta)$