ON BRANES & (non-linear) INSTANTONS

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SUPERSYMMETRIC BRANES:

$$\left.\begin{array}{rl} \text{susy}: & \delta_\epsilon\theta = \epsilon \\ \kappa\text{-symmetry}: & \delta_\kappa\theta = (1+\Gamma)\kappa(\sigma) \end{array}\right\} \Rightarrow \text{BPS}: \quad (\delta_\epsilon+\delta_\kappa)\theta = 0 = \epsilon+(1+\Gamma)\kappa(\sigma) \quad \Rightarrow$$

◇ $$\underline{\Gamma\epsilon = \epsilon}$$

◇ Dp-brane:

$$\Gamma = (-\det(g+\mathcal{F}))^{-1/2} \sum_{2l+s=p+1} \tfrac{1}{l!s!2^l}\epsilon^{n_1\ldots n_{2l}m_1\ldots m_s}\mathcal{F}_{n_1n_2}\ldots\mathcal{F}_{n_{2l-1}n_{2l}}\gamma_{m_1\ldots m_s}\mathcal{P}_{s+1} \;.$$

with $\mathcal{P}_n = \begin{pmatrix} 0 & 1 \\ (-1)^{[(n+1)/2]} & 0 \end{pmatrix}$ and $\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

◇ without WW excitations ($\mathcal{F}=0$):

$$\Gamma_0 = (-\det(g))^{-1/2}\tfrac{1}{(p+1)!}\epsilon^{m_1\ldots m_{p+1}}\gamma_{m_1\ldots m_{p+1}}\mathcal{P}_{p+2}$$

◇ “linear”” and “non-linear” components

$$\epsilon = \epsilon_+ + \epsilon_- = \frac{1}{2}(1+\Gamma_0)\epsilon + \frac{1}{2}(1-\Gamma_0)\epsilon$$

BULK SUPERSYMMETRY (type II):

Democracy $\Leftrightarrow$ $$\begin{cases} D_M = \nabla_M + \frac{1}{8} H_{MNP}\Gamma^{NP}\mathcal{P} + \frac{1}{16} e^{\phi} \sum_n \frac{1}{(2n)!} F_{M_1\ldots M_{2n}} \Gamma^{M_1\ldots M_{2n}} \Gamma_M \mathcal{P}_{2n} \\ \not{D} = \not{\nabla} + \frac{1}{24} \not{H} \mathcal{P} - \not{\partial}\phi \qquad \textit{modified}\text{ dilatino} \end{cases}$$

- Lichnerowicz theorem (for Levi-Clvita): $\left(\nabla^a \nabla_a - (\not{\nabla})^2\right) \epsilon = \frac{1}{4} R \epsilon$
- Bismut-LT:

$$\left((\nabla_a + \tfrac{1}{8} H_{abc}\Gamma^{bc})(\nabla^a + \tfrac{1}{8} H^a_{bc}\Gamma^{bc}) - (\not{\nabla} + \frac{1}{24}\not{H})^2\right) \epsilon = \left[\tfrac{1}{4}(R - \tfrac{1}{12} H^2) + \gamma^{abcd}\nabla_a H_{bcd}\right]\epsilon$$

- gen. LBT: $\left(D_a^{\text{(NS)}})^2 - (\not{D})^2\right) \epsilon = \left[\frac{1}{4} S^{\text{(NS)}} + \gamma^{abcd} I_{abcd}\right]\epsilon$
  - ◇ $S^{\text{(NS)}} = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12} H^2$
  - ◇ $I_{abcd} = \frac{1}{6} \nabla_{[a} H_{bcd]} = 0$
- More gLBT: $\left(D^A D_A - D^2\right) \epsilon = \left[\frac{1}{4} S + \gamma^{(n)} I_{(n)}\right]\epsilon$ tensorial S & BI/EOM $I_{(n)} = 0$
  - ◇ e.g. for Het. strings: $D_A = (D_m^{\text{(NS)}}, D_\alpha = -\frac{1}{8}\sqrt{2\alpha'} \hat{\mathcal{F}}_{ab\alpha}\gamma^{ab})$
  - ◇ Effective action: $S = R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{12} H^2 - \frac{\alpha'}{4} \,\mathrm{tr}\, \hat{\mathcal{F}}^2$
  - ◇ $I_{abcd} = \frac{1}{6} \nabla_{[a} H_{bcd]} - \frac{\alpha'}{8} \,\mathrm{tr}\, \hat{\mathcal{F}}_{[ab} \hat{\mathcal{F}}_{cd]} = 0$

REWIRING BRANE SUPERSYMMETRY:

- $$\Gamma = e^{-a/2}\Gamma_0' e^{a/2} \quad \text{with} \quad \Gamma_0' \sim \begin{cases} (\Gamma_{11})^{\frac{p-2}{2}} \mathcal{P}_{p+2}\Gamma_0 \\ (\sigma_3)^{\frac{p-3}{2}} \sigma_2 \otimes (\mathcal{P}_{p+2}\Gamma_0) \end{cases} ; \; a = \begin{cases} -\frac{1}{2} Y_{ij}\gamma^{ij}\Gamma_{11} \\ \frac{1}{2} Y_{ij}\sigma_3 \otimes \gamma^{ij} \end{cases}$$

$$\mathcal{F} = \text{tahn}(Y)$$

- $$\Gamma\epsilon = \epsilon \quad \Rightarrow \quad \tfrac{1}{2}\begin{pmatrix} \{\Gamma,\Gamma_0\} & -[\Gamma,\Gamma_0] \\ [\Gamma,\Gamma_0] & -\{\Gamma,\Gamma_0\} \end{pmatrix} \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$$

Further rewrite:

$$\sqrt{\frac{\det(g)}{\det(g+\mathcal{F})}} \begin{pmatrix} \not{\cosh(\mathcal{F})} & \not{\sinh(\mathcal{F})}\mathcal{P} \\ -\not{\sinh(\mathcal{F})}\mathcal{P} & -\not{\cosh(\mathcal{F})} \end{pmatrix} \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$$

BULK + BRANE SUPERSYMMETRY:

$$\left.\begin{array}{c} \Gamma\epsilon = \epsilon \;\&\; D_m\epsilon = 0 \\ \epsilon = \epsilon_+ + \epsilon_- \end{array}\right\} \quad \Rightarrow \quad \begin{pmatrix} \{D_m,\Gamma_0\} & -[D_m,\Gamma_0] \\ -[D_m,\Gamma_0] & \{D_m,\Gamma_0\} \end{pmatrix} \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} = 0$$

similar for dilatino variation $\not{D}$

BRANES IN SUPERSYMMETRIC BACKGROUNDS

OPEN $\rightarrow$ – – – – – – – $\downarrow$ CLOSED	A –	B $\Gamma_0\epsilon = \epsilon$	C $\Gamma_0\epsilon = \Gamma\epsilon = \epsilon$	D $\Gamma\epsilon = \epsilon$
1: $\nabla_m\epsilon = 0$	special holonomy $Ricci = 0$	calibrations $\overline{\Phi}\vert_Y = 0$	+ HYM $(\not{\mathcal{F}}\epsilon = 0)$	NL instantons [MMMS] $\mathrm{Im}\left(e^{i\alpha}\overline{\Phi}\vert_Y e^{\mathcal{F}}\right) = 0$
2: $D_m\epsilon_+ = 0$	–	Global SYM on curved spaces	"2 B + 1C"	$\supset$ "landscape of HYM" $O(F^{RR}, H)\not{\mathcal{F}}\epsilon = 0$
3: $D_m\epsilon = 0$	Gen. CY $\mathrm{d}\Phi = 0 \ + \ \ldots$	generalised calibrations	–	✓

SUPERSYMMETRIC BACKGROUNDS

10D string theory with

- the metric: $ds_{10}^2 = e^{2A(y)} \mathrm{d}s_4^2(M_4) + \mathrm{d}s_6^2$
- the fluxes: $F^{(10)} = F + \mathrm{vol}_4 \wedge \lambda(*F)$ (where $\lambda(F_n) = (-1)^{\mathrm{Int}[n/2]} F_n$)

Equations of motion $\Leftrightarrow$

- pure spinor equations

  $$\mathrm{d}\left(e^{3A}\Phi_1\right) = 0 \qquad \Leftrightarrow \qquad \text{Gen. CY structure}$$

  $$\mathrm{d}\left(e^{2A}\mathrm{Re}\Phi_2\right) = 0$$

  $$\mathrm{d}\left(e^{4A}\mathrm{Im}\Phi_2\right) = e^{4A} e^{-B} * \lambda(F)$$

- Bianchi identities

  $$(\mathrm{d} - H\wedge)F = \delta(\text{source})$$

Φ_1 and Φ_2 are even/odd poly-forms for IIA/B: IIA $\rightarrow$ $\Phi_1 = \Phi_+$, $\Phi_2 = \Phi_-$ IIB $\rightarrow$ $\Phi_1 = \Phi_-$, $\Phi_2 = \Phi_+$

PURE SPINORS and GCG

$$\Phi_+ = 8\, e^{-\phi} e^{-B} |\eta^1_+ \otimes \eta^{2\,\dagger}_+|_{\text{norm}}$$
$$\Phi_- = 8\, e^{-\phi} e^{-B} |\eta^1_+ \otimes \eta^{2\,\dagger}_-|_{\text{norm}}$$

(use: $\eta^1_+ \otimes \eta^{2\,\dagger}_\pm = \frac{1}{8} \sum_{k=0}^{6} \frac{1}{k!} \left(\eta^{2\,\dagger}_\pm \gamma_{m_k \dots m_1} \eta^1_+ \right) \gamma^{m_1 \dots m_k}$ for spinors

IIA $\rightarrow$ $\epsilon_1 = \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-$, $\epsilon_2 = \zeta_+ \otimes \eta^2_- + \zeta_- \otimes \eta^2_+$

IIB $\rightarrow$ $\epsilon_1 = \zeta_+ \otimes \eta^1_+ + \zeta_- \otimes \eta^1_-$, $\epsilon_2 = \zeta_+ \otimes \eta^2_+ + \zeta_- \otimes \eta^2_-$)

On Generalized tangent bundle: $0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0\,,$

$$\Phi_\pm \in L \otimes \Lambda^{\text{even/odd}} T^*M\,.$$

EXAMPLES:

SU(3) structure (1A)	SU(3) holonomy (3A)
$\Phi_+ = e^{-\phi} e^{-B-i\omega}$	$\overline{\Phi}_+ = e^{-i\omega}$ symplectic
$\Phi_- = e^{-\phi} e^{-B} \Omega$	$\overline{\Phi}_- = \Omega$ complex

1B: calibrations - susy cycles in manifolds of irreducible non-trivial holonomy

p+1	$SU(2)$	$SU(3)$	G_2	$SU(4)$	$Spin(7)$
2	divisor/SLag	holomorphic	–	holomorphic	–
3	–	SLag	associative	–	–
4	X	divisor	coassociative	Cayley	Cayley
5	–	–	–	–	–
6	–	X	–	divisor	–
7	–	–	X	–	–
8	–	–	–	X	X

EXAMPLES:

(SU(N) - holonomy)

SLag (A)	holomorphic (B)
$\mathrm{Im}(e^{i\vartheta}\Omega_N\|_Y) = 0$	$\omega^n\|_Y \sim \mathrm{vol}(Y)$
$\mathcal{F} = 0$	$\Omega_N\|_Y = 0$

1C:

$\forall$ two statements, the third follows:

for product spinor anstaz

$$\begin{cases} \Gamma\epsilon = \epsilon \\ \Gamma_0\epsilon = \epsilon \\ \not{\mathcal{F}}\epsilon = 0 \end{cases}$$

NOT very good.... for e.g. mirror symmetry

$\mathrm{Im}(e^{i\vartheta}\Omega_N\|_Y) = 0$	$\omega^n\|_Y \sim \mathrm{vol}(Y)$
	$\Omega_N\|_Y = 0$
$\mathcal{F} = 0$	$\mathcal{F}^{(2,0)} = \mathcal{F}^{(0,2)} = 0; \ \mathcal{F} \wedge \omega^{n-1} = 0$
	ϑ MISSING!!!

1D: NonLinear Instantons

$\mathrm{Im}(e^{i\vartheta}\Omega_N\|_Y e^{\mathcal{F}}) = 0$	hol. cycle
flat connection	$\mathrm{Im}(e^{i\vartheta}e^{-i\omega}\|_Y e^{\mathcal{F}}) = 0$

- slope(SLag) $\Leftrightarrow$ slope(deformed HYM)
- type B branes on holomorphic cycles:
  - ◇ EX (4d): $\omega|_Y \wedge \mathcal{F} = \tan\vartheta\,(\text{vol}_4 - \mathcal{F}\wedge\mathcal{F})$
  - ◇ Euclidean space: $g_{\mu\nu} = \varepsilon\delta_{\mu\nu}$ and introduce $\xi = \varepsilon/2\pi\alpha'$
    - ○ zero-slope limit ($\alpha' \to 0,\ \varepsilon$ fixed; $\xi \to \infty$): $\mathcal{F}\wedge\omega^{n-1} = \mu\,\omega^n$
      ▷ $\mu = deg/rk$ where $deg = c_1[\omega]^{n-1}$
    - ○ SW limit ($\xi \to 0,\ \alpha' \sim \varepsilon^{\frac{1}{2}} \to 0$; $\xi \to 0$)) - NC instantons:
      ▷ $\hat{\mathcal{F}}\wedge\omega^{n-1} = 0;\ \hat{\mathcal{F}}^{(2,0)} = 0 \quad (\hat{\mathcal{F}} = \frac{1}{1+F\theta}F)$
  - ◇ slope $\vartheta = \arg(\int e^{\mathcal{F}-i\omega})(+ik\pi)$
  - ◇ in general (?) $\text{ch}(\mathcal{F}) \to \sqrt{\hat{A}(TY)/\hat{A}(N)}e^{-c_1(N)/2}\text{ch}(\mathcal{F})$ and $\vartheta = \arg(Z)$

Aside: gLT and (not-so) linear supersymmetry

- D=4: $\frac{1}{p+1}\,(1-\not{\mathcal{F}}\mathcal{P}+q\gamma_4)\,\Gamma_0(\epsilon_+ + \epsilon_-) = \Gamma\epsilon = \epsilon = \Gamma_0(\epsilon_+ - \epsilon_-) \quad \Rightarrow$

$$\epsilon_\pm = (\epsilon^1_\pm, \mp i\gamma_4\epsilon^1_\pm) \qquad \Rightarrow \qquad \begin{cases} \not{\mathcal{F}}\epsilon^1_+ = -(2+p+q\gamma_4)\epsilon^1_- \\ (p-q\gamma_4)\epsilon^1_+ = \not{\mathcal{F}}\epsilon^1_- \end{cases}$$

  - ⋆ ϵ_+/ϵ_- generate linearly/non-linearly realised susy (cf [SW])
  - ⋆ with $\quad p = \sqrt{\det(\delta + g^{-1}\mathcal{F})} - 1 \ (\varepsilon = 1); \quad q = \frac{1}{4}(\star\mathcal{F})_{mn}\mathcal{F}^{mn}$
  - ⋆ cf $\quad (\epsilon^1_+)^\dagger\,(\not{\mathcal{F}} - r)^2\,\epsilon^1_+$ to $\mathcal{L}_{\mathrm{YM}}$

- D=6: "linear susy" $\quad \Rightarrow \quad \begin{cases} (\not{\mathcal{F}} - r)\epsilon^1_+ = 0 \\ (p - q\gamma_6)\epsilon^1_+ = 0 \end{cases}$

  - ⋆ $q = -\frac{1}{4}(\star\mathcal{F})_{mnpq}\mathcal{F}^{mn}\gamma^{pq}; \quad r = -\frac{1}{3!}\frac{1}{2^3}\epsilon^{mnpqrs}\mathcal{F}_{mn}\mathcal{F}_{pq}\mathcal{F}_{rs}$
  - ⋆ for normalised ϵ^1_+ :

$$(\epsilon^1_+)^\dagger\,(\not{\mathcal{F}} - r)^2\,\epsilon^1_+ = \frac{1}{2}\mathcal{F}_{mn}\mathcal{F}^{mn} - (1 - \frac{1}{2}\mathcal{F}_{mn}\mathcal{F}^{mn})p - \frac{1}{3!}(\frac{1}{2}\mathcal{F}_{mn}\mathcal{F}^{mn})^3$$

2B ($\mathcal{F}=0$) :

type II + susy D-branes : $\{D_M, \Gamma_0\}\epsilon_+ = 0, ...$ $\Rightarrow$

- Killing spinors $\Rightarrow$ Riem. manifolds for $\mathcal{N}=1$ 4D SYM
- auxiliary sugra fields $\Leftrightarrow$ NS and RR fluxes

$(M,g)\in\mathcal{S}$ (the set of all $d=4$ Riemannian manifolds admitting $\mathcal{N}=1$ susy theories) if and only if the manifold M admits a pair of Killing spinors $\eta^\pm$ of opposite chirality:

$$(\nabla_m + iA_m + iV_n\gamma^n\gamma_m)\,\eta^+ - M^+\eta^- = 0$$
$$(\nabla_m - iA_m - iV_n\gamma^n\gamma_m)\,\eta^- - M^-\eta^+ = 0$$

$A, V, M^\pm$ $\Leftarrow$

- background profiles of the auxiliary fields of $\mathcal{N}=1$, $d=4$ sugra
- made of fluxes of type II theoreis

3D ($\mathcal{F}\neq 0$)

$$\left.\begin{array}{c}\Gamma\epsilon=\epsilon \;\&\; D_m\epsilon=0\\ \epsilon=\epsilon_++\epsilon_-\end{array}\right\} \Rightarrow \begin{pmatrix}\{D_m,\Gamma_0\} & -[D_m,\Gamma_0]\\ -[D_m,\Gamma_0] & \{D_m,\Gamma_0\}\end{pmatrix}\begin{pmatrix}\epsilon_+\\ \epsilon_-\end{pmatrix}=0$$

General (non-linear) BPS equations (for different amounts of susy) for SYM on curved spaces....

- $\mathcal{N}=4$ solutions for (M,g) + non-linear instantons
- $\mathcal{N}=1$
  - ◇ on top of D3 add D7 susy projections
  - ◇ choice of projections (D7)
  - ◇ solutions for (M,g) + (non-linear) instantons

Questions/Future:

- Beyond BPS equations
- Computable α' expansion of SYM in curved spaces
- SYM + curious notion of “linear” susy in $D \geq 4$
- Landscape of field theories (?)
- ...