Non-Abelian Vortex in Four Dimensions as a Critical String on a Conifold

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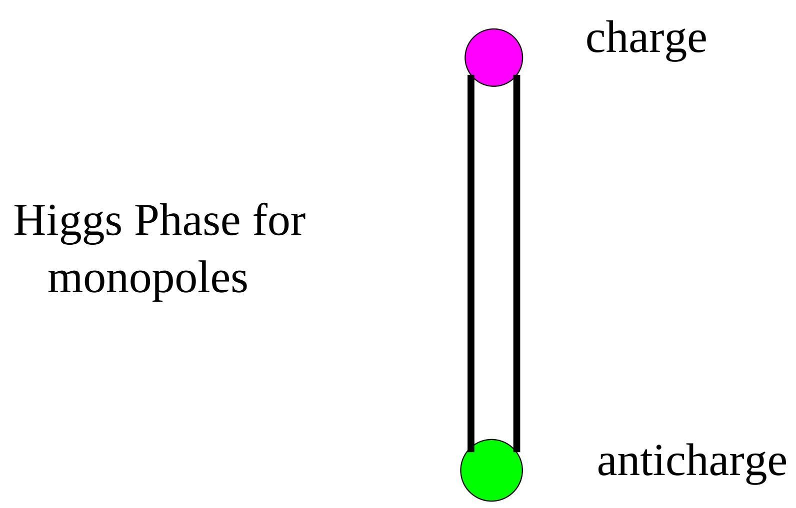
1 Introduction

It is believed that confinement in QCD is due to formation of confining vortex strings.

Nambu, Mandelstam and 't Hooft 1970's:

Confinement is a dual Meissner effect upon condensation of monopoles.

Monopoles condense $\rightarrow$ electric Abrikosov-Nielsen-Olesen flux tubes are formed $\rightarrow$ electric charges are confined



Hadron spectrum is well described by linear Regge trajectories.

However in all known examples the Regge trajectories show linear behavior only at asymptotically large spins.

Examples:

- Abrikosov-Nielsen-Olesen (ANO) vortex in weakly coupled Abelian-Higgs model
- Seiberg-Witten confinement in $\mathcal{N} = 2$ super-Yang-Mills theory

Length of the rotating string:

$$L^2 \sim \frac{J}{T}$$

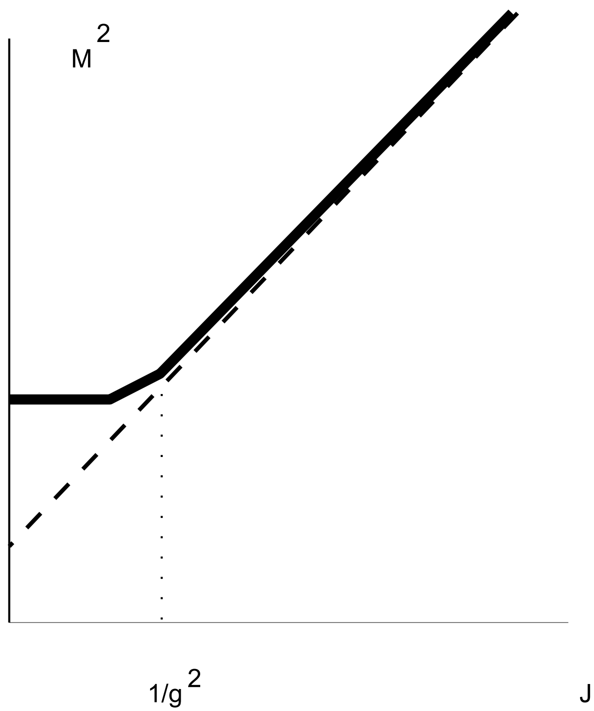
Transverse size of the string is given by the inverse mass of the bulk fields forming the string:

$$m \sim g\sqrt{T}$$

String length $\gg$ its transverse size:

$$mL \gg 1, \qquad J \gg \frac{1}{g^2}$$

We expect



In the real world Regge trajectories are linear at $J \sim 1$

Can we find any example of $4D$ theory
where confining string remains thin at $J \sim 1$?

Thin string condition:

$$T \ll m^2$$

2 Thin string regime

How the problem of "thick" string is seen in the world sheet effective theory?

ANO string: Nambu-Goto action

$$S_{\rm NG} = T \int d^2\sigma \left\{ \sqrt{h} + O\left(\frac{\partial^n}{m^n} \right) \right\}$$

where

$$h = det(\partial_\alpha x^\mu \, \partial_\beta x_\mu)$$

Polchinski-Strominger, 1991: Without higher derivative terms the world sheet theory is not UV complete

Higher derivative terms at weak coupling, $g \ll 1$

$$O\left(\frac{\partial^n}{m^n}\right), \qquad m \sim g\sqrt{T}$$

At $J \sim 1 \qquad \partial \to \sqrt{T}$

Thus higher derivative terms

$$\to \left(\frac{T}{m^2}\right)^n$$

blow up at weak coupling!

Polyakov: string surface become "crumpled".

We want to find a regime in which the string remains thin. This means that the higher derivative corrections should be parametrically small.

The low-energy world-sheet theory should be UV complete. This leads us to the following necessary conditions to have such a regime:

(i) The low-energy world-sheet theory on the string must be conformally invariant;

(ii) It must have the critical value of the Virasoro central charge.

- Bosonic string D=26
  Superstring D=10
- ANO string in D=4 is not critical

$\boxed{\text{Non-Abelian vortex strings}}$

Non-Abelian strings were suggested in $\mathcal{N} = 2$ U(N) QCD

Hanany, Tong 2003

Auzzi, Bolognesi, Evslin, Konishi, Yung 2003

Shifman Yung 2004

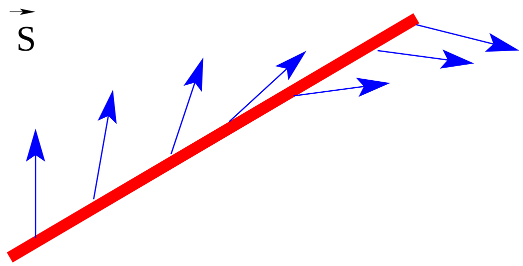
Hanany Tong 2004

Z_N Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/Z_2$

$$flux \sim \tau_3$$

Non-Abelian string : $\boxed{\text{Orientational zero modes}}$

Rotation of color flux inside SU(N).



Idea:

Non-Abelian string has more moduli then ANO string.

It has translational + orientaional moduli

Shifman and Yung, 2015:

Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring

We can fulfill the criticality condition: The solitonic non-Abelian vortex must have six orientational moduli, which, together with four translational moduli, will form a ten-dimensional space.

3 Non-Abelian vortex strings

For $U(N)$ gauge group in the bulk we have 2D $CP(N-1)$ model on the string

CP$(N-1)$ == U(1) gauge theory in the strong coupling limit

$$S_{\mathrm{CP}(N-1)} = \int d^2x \left\{ \left| \nabla_\alpha n^P \right|^2 + \frac{e^2}{2} \left(|n^P|^2 - 2\beta \right)^2 \right\} ,$$

where n^P are complex fields $P = 1, ..., N$,

Condition

$$|n^P|^2 = 2\beta \; = \frac{4\pi}{g^2},$$

imposed in the limit $e^2 \to \infty$

More flavors $\Rightarrow$ semilocal non-Abelian string

The orientational moduli described by a complex vector n^P (here $P = 1, ..., N$),

$\tilde{N} = (N_f - N)$ size moduli are parametrized by a complex vector ρ^K $(K = N + 1, ..., N_f)$.

The effective two-dimensional theory is sigma model with the target space $\mathcal{O}(-1)^{\oplus(N_f-N)}_{CP^1}$ ($\mathcal{N} = (2,2)$ weighted CP model)

$$S_{\rm WCP} = \int d^2x \left\{ \left|\nabla_\alpha n^P\right|^2 + \left|\tilde{\nabla}_\alpha \rho^K\right|^2 + \frac{e^2}{2}\left(|n^P|^2 - |\rho^K|^2 - 2\beta\right)^2 \right\},$$

$$P = 1, ..., N\,, \qquad K = N + 1, ..., N_f\,.$$

The fields n^P and ρ^K have charges $+1$ and -1 with respect to the auxiliary U(1) gauge field

$e^2 \to \infty$

4 From non-Abelian vortices to critical strings

String theory

$$
\begin{aligned}
S &= \frac{T}{2}\int d^2\sigma\sqrt{h}\,h^{\alpha\beta}\partial_\alpha x^\mu\,\partial_\beta x_\mu \\
&+ \int d^2\sigma\sqrt{h}\left\{h^{\alpha\beta}\left(\tilde{\nabla}_\alpha\bar{n}_P\,\nabla_\beta\,n^P + \nabla_\alpha\bar{\rho}_K\,\tilde{\nabla}_\beta\,\rho^K\right)\right. \\
&+ \left.\frac{e^2}{2}\left(|n^P|^2 - |\rho^K|^2 - 2\beta\right)^2\right\} + \text{fermions}\,,
\end{aligned}
$$

where $h^{\alpha\beta}$ is the world sheet metric. It is independent variable in the Polyakov formulation.

What about necessary conditions for thin string?

- Conformal invariance

$$b_{WCP} = N - \tilde{N} = 0 \Rightarrow N = \tilde{N}, \quad N_f = 2N$$

- Critical dimension =10

  Number of orientational + size degrees of freedom
  $= 2(N + \tilde{N} - 1) = 2(2N - 1)$

$$4 + 2(2N - 1) = 4 + 6 = 10, \qquad \text{for } N = 2$$

Our string is BPS so we have $\mathcal{N} = (2,2)$ supersymmetry on the world sheet.

For these values of N and $\tilde{N}$ the target space of the weighted $CP(2,2)$ model is a non-compact Calabi-Yau manifold studied by Witten and Vafa, namely conifold.

Given that for non-Abelian vortex low energy world sheet theory is critical

Conjecture:

Thin string regime

$$T \ll m^2$$

is actually satisfied at strong coupling $g_c^2 \sim 1$.

$$m(g) \to \infty, \qquad g^2 \to g_c^2$$

Higher derivative corrections can be ignored

Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

For closed string moving on Calabi-Yau manifold $\mathcal{N} = (2, 2)$ world sheet supersymmetry ensures $\mathcal{N} = 2$ supersymmetry in 4D.

This is expected since we started with 4D QCD with $\mathcal{N} = 2$ supersymmetry.

Type IIB string is a chiral theory and breaks parity while Type IIA string theory is left-right symmetric and conserves parity.

Our bulk theory conserves parity $\Rightarrow$ we have Type IIA superstring

There is self-duality in 4D bulk theory

$$\tau \to \tau_D = -\frac{1}{\tau}, \qquad \tau = \frac{4\pi i}{g^2} + \frac{\theta_{4D}}{2\pi},$$

We conjectured that the string becomes thin at $g^2 \to g_c^2 \sim 1$.

It is natural to expect that $g_c^2 = 4\pi =$ self-dual point.

$$m^2 \to T \times \begin{cases} g^2, & g^2 \ll 1 \\ \infty, & g^2 \to 4\pi \\ 16\pi^2/g^2, & g^2 \gg 1 \end{cases},$$

In 2D theory on the string self-dual point is $\beta = 0$

Conifold develops conical singularity.

5 4D Graviton

Our goal:

Study massless states of closed string propagating on

$$R_4 \times Y_6, \qquad Y_6 = \text{conifold}$$

and interpret them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Massless 10D graviton

$$\delta G_{\mu\nu} = \delta g_{\mu\nu}(x)\, g_6(y), \qquad \delta G_{\mu i} = B_\mu(x)\, g_i(y), \qquad \delta G_{ij} = g_4(x)\, \delta g_{ij}(y)$$

Lichnerowicz equation

$$(\partial_\mu \partial^\mu + \Delta_6)\, g_4(x) g_6(y) = 0$$

Expand $g_6(y)$ in eighenfunctions

$$-\Delta_6 g_6(y) = \lambda_6 g_6(y), \qquad \lambda_6 = \text{mass}$$

Consider massless states $\lambda_6 = 0$

$$-\Delta_6 g_6(y) = 0.$$

Solutions of this equation for Calabi-Yau manifolds are given by elements of Dolbeault cohomology $H^{(p,q)}(Y_6)$, where (p,q) denotes numbers of holomorphic and anti-holomorphic indices in the form. The dimensions of these spaces $h^{(p,q)}$ are called Hodge numbers for a given Y_6.

For 4D graviton $g_6(y)$ is scalar

$$-D_i \partial^i g_6 = 0$$

The only solution is

$$g_6(y) = \mathrm{const}$$

Non-normalizable on non-compact conifold Y_6.

No 4D graviton == good news!

We do not have gravity in our 4D $\mathcal{N} = 2$ QCD

6 Kahler form deformations

Consider 4D scalar fields

Lichnerowicz equation on Y_6

$$D_k D^k \delta g_{ij} + 2R_{ikjl}\delta g^{kl} = 0.$$

Solutions = Kahler form deformations or complex structure deformations.

Kahler form deformations = variations of 2D coupling β

D-term condition in weighted CP(2,2) model

$$|n^P|^2 - |\rho^K|^2 = \beta, \qquad P = 1, 2, \qquad K = 1, 2$$

Resolved conifold

The effective action for $\beta(x)$ is

$$S(\beta) = T \int d^4x \, h_\beta (\partial_\mu \beta)^2,$$

where

$$h_\beta = \int d^6 y \sqrt{g} g^{li} \left(\frac{\partial}{\partial \beta} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \beta} g_{kl} \right)$$

Using explicit Calabi-Yau metric on resolved conifold we get

$$h_\beta = (4\pi)^3 \, \frac{5}{6} \int dr \, r = \infty$$

β - non-normalizable mode

Physical nature of non-normalizable modes

Gukov, Vafa, Witten 1999: Non-normalizable moduli = coupling constants in 4D

- 4D metric do not fluctuate. It is fixed to be flat. "Coupling constants."
- 2D coupling β is related to 4D coupling g^2. Fixed. Non-dynamical.

Another option:

Large $y_i \Rightarrow$ large n^P and ρ^K

Non-normalizable modes are not localized on the string.

Unstable states. Decay into massless perturbative states.

Higgs branch: $\dim\mathcal{H} = 4N\tilde{N} = 16.$

7 Deformation of the complex structure

D-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \qquad P = 1, 2, \qquad K = 1, 2$$

Construct U(1) gauge invariant "mesonic " variables"

$$w^{PK} = n^P \rho^K.$$

$$\det w^{PK} = 0$$

$\boxed{\text{Take } \beta = 0}$

Complex structure deformation ⇒ Deformed conifold

$$\det w^{PK} = b$$

b – complex modulos

The effective action for $b(x)$ is

$$S(\beta) = T \int d^4x\, h_b (\partial_\mu b)^2,$$

where

$$h_b = \int d^6y \sqrt{g} g^{li} \left(\frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left(\frac{\partial}{\partial \bar{b}} g_{kl} \right)$$

Using explicit Calabi-Yau metric on deformed conifold we get

$$h_b = (4\pi)^3 \, \frac{4}{3} \, \log \frac{T^2 L^4}{|b|}$$

For Type IIA string b is a part of hypermultiplet.

Another complex scalar $\tilde{b}$ comes from 10D 3-form.

$$S(b) = T \int d^4x \left\{ |\partial_\mu b|^2 + |\partial_\mu \widetilde{b}|^2 \right\} \log \frac{T^4 L^8}{|b|^2 + |\widetilde{b}|^2}$$

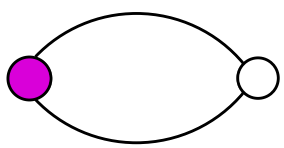
8 Monopole-monopole baryon

Weak coupling

Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

Quarks are condensed in the bulk theory. Therefore, monopoles are confined.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.



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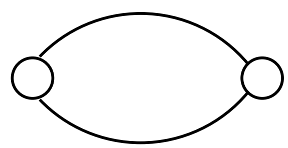

b

Monopole-antimonopole meson Monopole-monopole baryon

Stringy states are massive, with mass $\sim \sqrt{T}$.

Strong coupling

Global group of the 4D QCD:

$$SU(2) \times SU(2) \times U(1)$$

U(1) - "baryonic" symmetry.

b-hypermultiplet: (1, 1, 2)

Logarithmically divergent norm == Marginal stability at $\beta = 0$

b-state can decay into massless bi-fundamental (screened) quarks living on the Higgs branch.

9 Conclusions

- In $\mathcal{N}=2$ supersymmetric QCD with gauge group U(2) and $N_f=4$ quark flavors non-Abelian BPS vortex behaves as a critical fundamental superstring.
- Massless closed string state b associated with deformations of the complex structure of the conifold == monopole-monopole baryon.
- Successful tests of our gauge-string duality:
  - $\mathcal{N}=2$ supersymmetry in 4D QCD
  - Absence of graviton and unwanted vector fields.
  - Massless monopole-monopole baryon is present only at $\beta=0$ and cannot be continued to weak coupling.