Quantum Metastability in Atomtronic Superfluid Circuits

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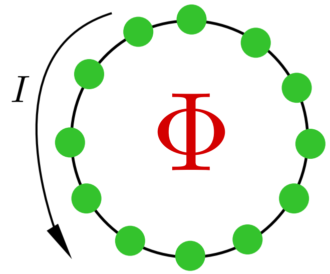
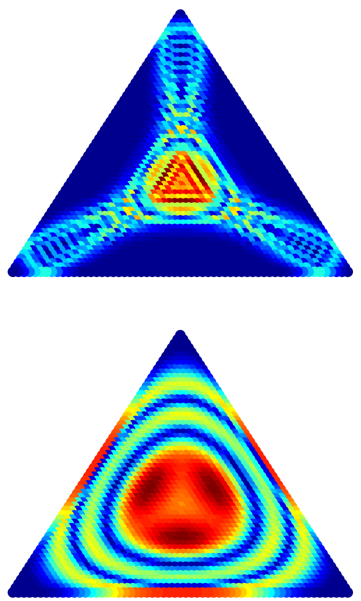
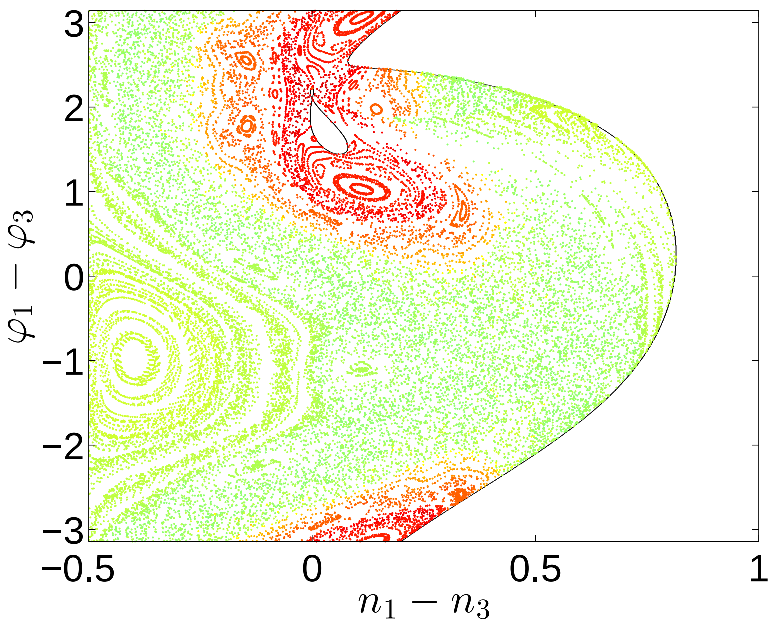
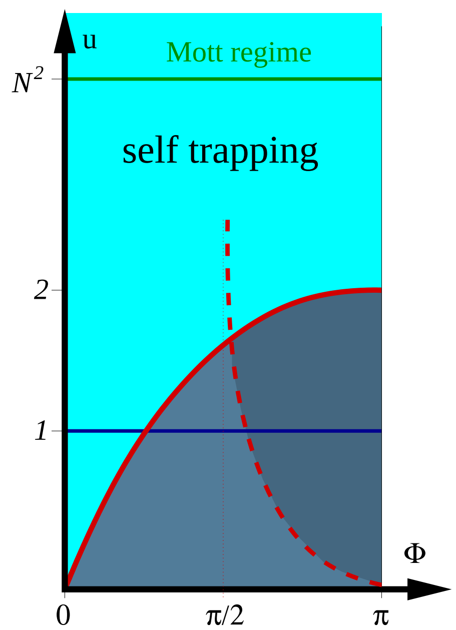
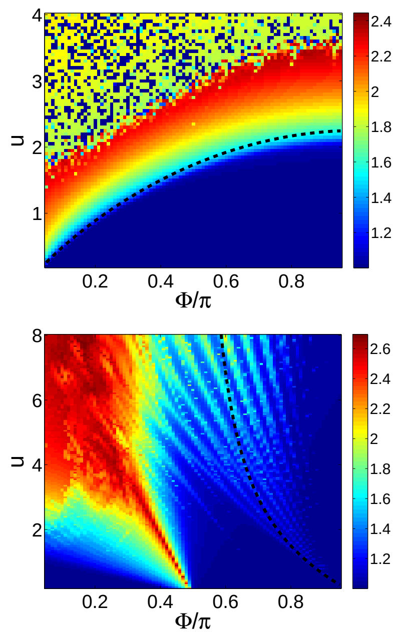
[**1**] Geva Arwas, Amichay Vardi, Doron Cohen [**PRA 2014**]
[**2**] GA, AV, DC [**Scientific Reports 2015**]
[**3**] Additional collaborations (see next page)











BHH - dimers and trimers

The Bose-Hubbard Hamiltonian (BHH):

$$\mathcal{H}_{\text{BHH}} \;=\; \frac{U}{2}\sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j \;-\; \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right) \qquad\qquad u \equiv \frac{NU}{K}$$

Dimer (M=2): minimal BHH; Bosonic Josephson junction; Pendulum physics [1,5].

Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].

Trimer (M=3): minimal model for chaos; Coupled pendula physics.

Triangular trimer: minimal model with topology, Superfluidity [6], Stirring [7].

Coupled trimers: minimal model for mesoscopic thermalization [8,9].

[1] Chuchem, Smith-Mannschott, Hiller, Kottos, Vardi, Cohen (PRA 2010).

[2] Smith-Mannschott, Chuchem, Hiller, Kottos, Cohen (PRL 2009).

[3] Boukobza, Moore, Cohen, Vardi (PRL 2010).

[4] Khripkov, Vardi, Cohen (PRA 2012)

[5] Khripkov, Cohen, Vardi (JPA 2013, PRE 2013).

[6] Geva Arwas, Vardi, Cohen (PRA 2014).

[7] Hiller, Kottos, Cohen (EPL 2008, PRA 2008).

[8] Tikhonenkov, Vardi, Anglin, Cohen, PRL (2013).

[9] Christine Khripkov, Vardi, Cohen, NJP (2015).

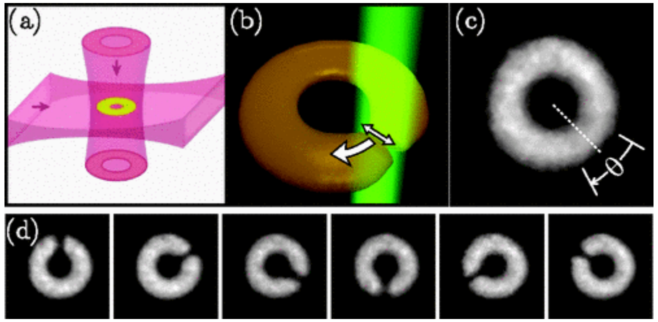
Scope

- The recent experimental realization of confining potentials with toroidal shapes [1] has opened a new arena of studying superfluidity in low dimensional circuits.

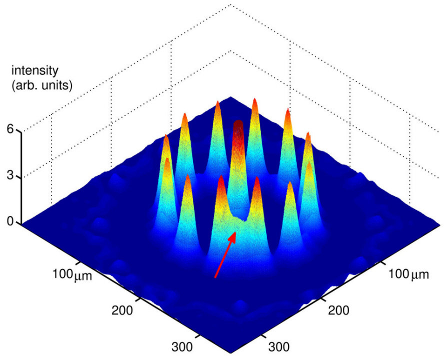
$$\mathcal{H}_{\mathrm{BHH}} \;=\; \frac{U}{2}\sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j \;-\; \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)$$

- The hallmark of superfluidity is a metastable non-equilibrium steady-state current.
- The traditional paradigm is based on the Landau criterion and the BdG stability analysis [3-5].
- We challenge the traditional paradigm and highlight the role of chaos in the analysis.

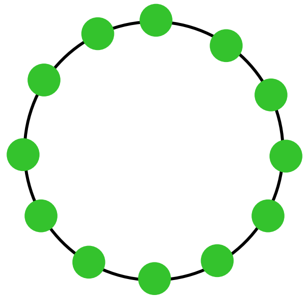
Torodial Ring [1]



Discrete ring [2]



Bode-Hubbard Model

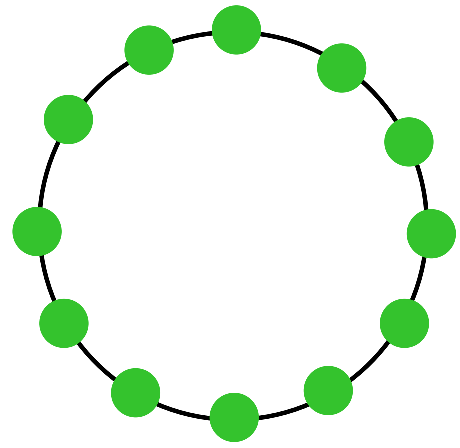


[1] Wright, Blakestad, Lobb, Phillips, Campbell, PRL (2013).
[1] Ekel, Lee, Jendrzeejewski, Murray, Clark, Lobb, Phillips, Edwards, Campbell, Nature (2014).
[2] Amico, Aghamalyan, Auksztol, Crepaz, Dumke, Kwek, Sci. Rep. (2014).
[3] Wu, Niu, NJP (2003).
[4] Smerzi, Trombettoni, Kevrekidis, Bishop, PRL (2002).
[5] Cataliotti, Fallani, Ferlaino, Fort, Maddaloni, Inguscio, NJP (2003).

The Model (non-rotating ring)

A Bose-Hubbard system with M sites and N bosons:

$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^{\dagger}a_j^{\dagger}a_j a_j - \frac{K}{2}\left(a_{j+1}^{\dagger}a_j + a_j^{\dagger}a_{j+1}\right)\right]$$



In a semi-classical framework:

$$a_j \;=\; \sqrt{\boldsymbol{n}_j}\,\mathrm{e}^{i\boldsymbol{\varphi}_j} \quad , \quad [\boldsymbol{\varphi}_j, \boldsymbol{n}_i] \;=\; i\delta_{ij}$$

$$z \;=\; (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_M, \;\; \boldsymbol{n}_1, \cdots, \boldsymbol{n}_M)$$

This is like M coupled oscillators with $\mathcal{H} = H(z)$

$$H(z) \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}\boldsymbol{n}_j^2 - K\sqrt{\boldsymbol{n}_{j+1}\boldsymbol{n}_j}\cos\left(\boldsymbol{\varphi}_{j+1} - \boldsymbol{\varphi}_j\right)\right]$$

The dynamics is generated by the Hamilton equation:

$$\dot{z} \;=\; \mathbb{J}\boldsymbol{\partial} H \quad , \quad \mathbb{J} \;=\; \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

(DNLS)

Classically there is a single dimensionless parameter:

$$u \;=\; \frac{NU}{K}$$

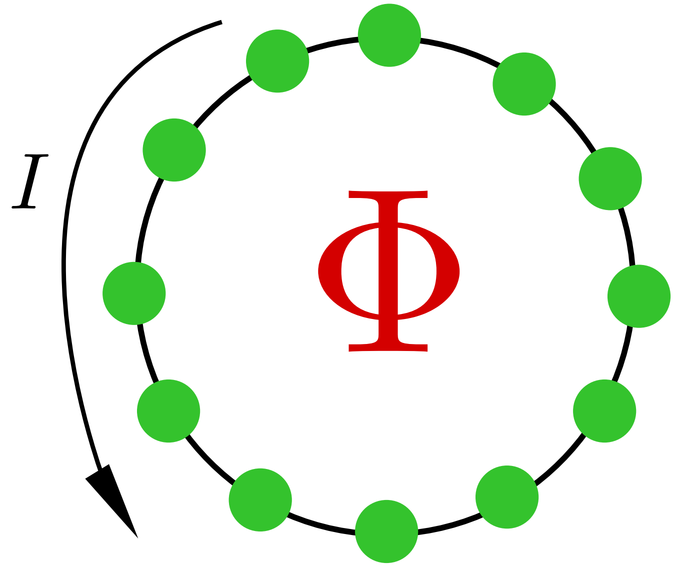
Rescaling coordinates:

$$\tilde{\boldsymbol{n}} \;=\; \boldsymbol{n}/N$$

$$[\boldsymbol{\varphi}_j, \tilde{\boldsymbol{n}}_i] \;=\; i\hbar\delta_{ij}$$

$$\hbar \;=\; \frac{1}{N}$$

The model (rotating ring)



In the rotating reference frame we have a Coriolis force,
which is like magntic field $\mathcal{B} = 2\mathsf{m}\Omega$. Hence is is like having flux

$$\Phi \;=\; 2\pi R^2 \mathsf{m}\,\Omega \;=\; \frac{M^2}{2\pi}\left(\frac{\mathsf{m}}{\mathsf{m}_{\text{eff}}}\right)\frac{\Omega}{K}$$

Note: there are optional experimental realizations.

$$\mathcal{H} \;=\; \sum_{j=1}^{M}\left[\frac{U}{2}a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2}\left(\mathrm{e}^{i(\Phi/M)}a_{j+1}^\dagger a_j + \mathrm{e}^{-i(\Phi/M)}a_j^\dagger a_{j+1}\right)\right]$$

Summary of model parameters:

The "classical" dimensionless parameters of the DNLS are u and Φ.
The mumber of particles N is the "quantum" paramater.
The system has effectively $d = M-1$ degrees of freedom.

$M = 2$	Bosonic Josephson junction (Integrable)
$M = 3$	Minimal circuit (mixed chaotic phase-space)
$M > 3$	High dimensional chaos (Arnold diffusion)
$M \to \infty$	Continuous ring (Integrable)

Types of meta-stability

- (traditional) Energetic metastability, aka Landau criterion.
- (traditional) Dynamical metastability via linear stability analysis, aka BdG.
- Strict dynamical metastability (KAM, applies if $d = 2$)
- Quasi dynamical metastability (might be the case for $d > 2$)

In the absence of constants of motion, a generic system with $d > 2$ degrees-of-freedom is always ergodic. But the equilibration might be an extremely slow process.

Quasi stability might become Quantum stability due to dynamical localization.
The breaktime is determined from the breakdown of the QCC requirement:

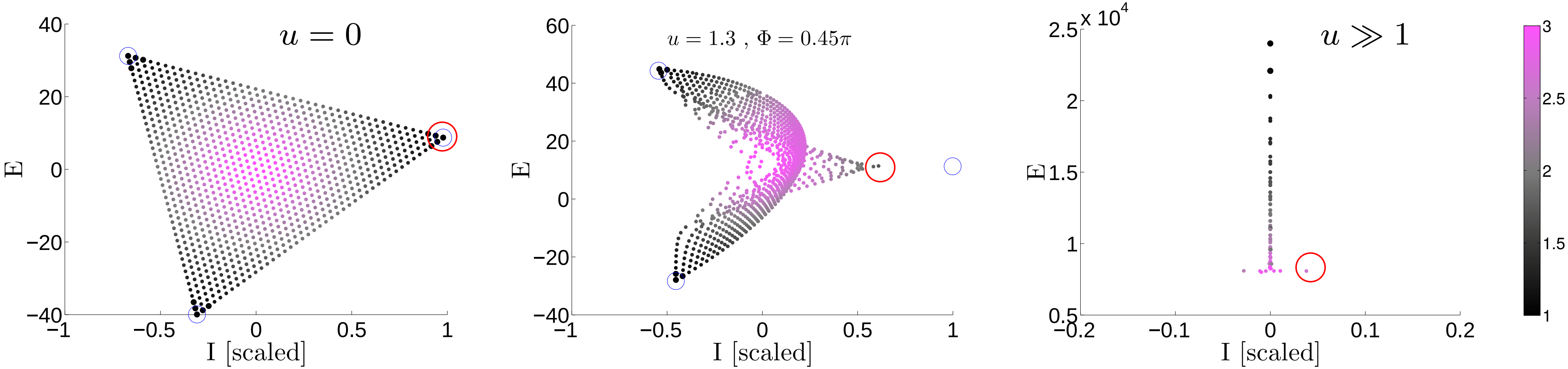
$$t \quad \ll \quad t_H[\Omega(t)] \qquad \rightsquigarrow \qquad t^*$$

Implication: violation of the Eigenstate Thermalization Hypothesis.

The many-body spectrum

We characterize each eigenstate $|\alpha\rangle$ of the BHH by $(\mathcal{I}_\alpha, E_\alpha)$ and colorcode by $\mathcal{M}_\alpha$

The expected location of a vortex state, and the maximum current state, are encircled by ◯ and ◯



$$|m\rangle \;=\; \left(\tilde{a}_m^\dagger\right)^N |0\rangle \qquad m = 1...M$$

$$\mathcal{I}_m \;=\; N \times \left(\frac{K}{M}\right) \sin\left(\frac{1}{M}(2\pi m - \Phi)\right)$$

$$\mathcal{I}_\alpha \;\equiv\; -\left\langle \frac{\partial \mathcal{H}}{\partial \Phi} \right\rangle_\alpha$$

$$\rho_{ij} \;\equiv\; \frac{1}{N}\langle a_j^\dagger a_i \rangle_\alpha \;=\; \text{reduced probability matrix}$$

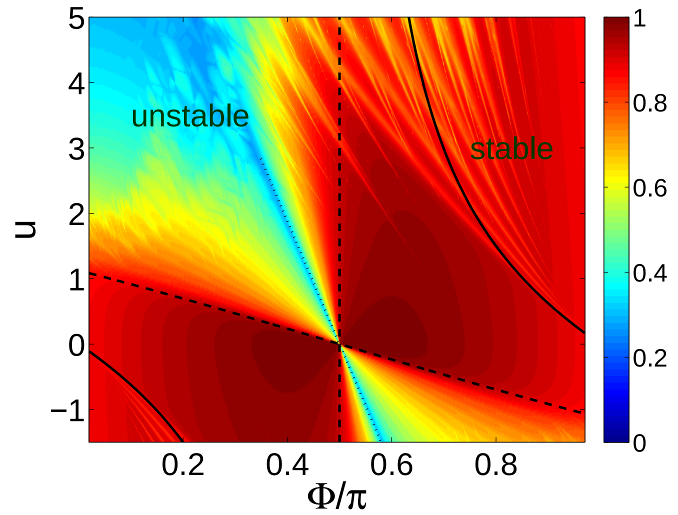
$$\mathcal{M}_\alpha \;\equiv\; [\mathrm{trace}(\rho^2)]^{-1} \;\in\; [1, M]$$

$\mathcal{M}_\alpha = 1$ for coherent state (condensation).

$\mathcal{M}_\alpha \sim M$ for maximally fragmented or chaotic state.

Constructing the regime diagram:

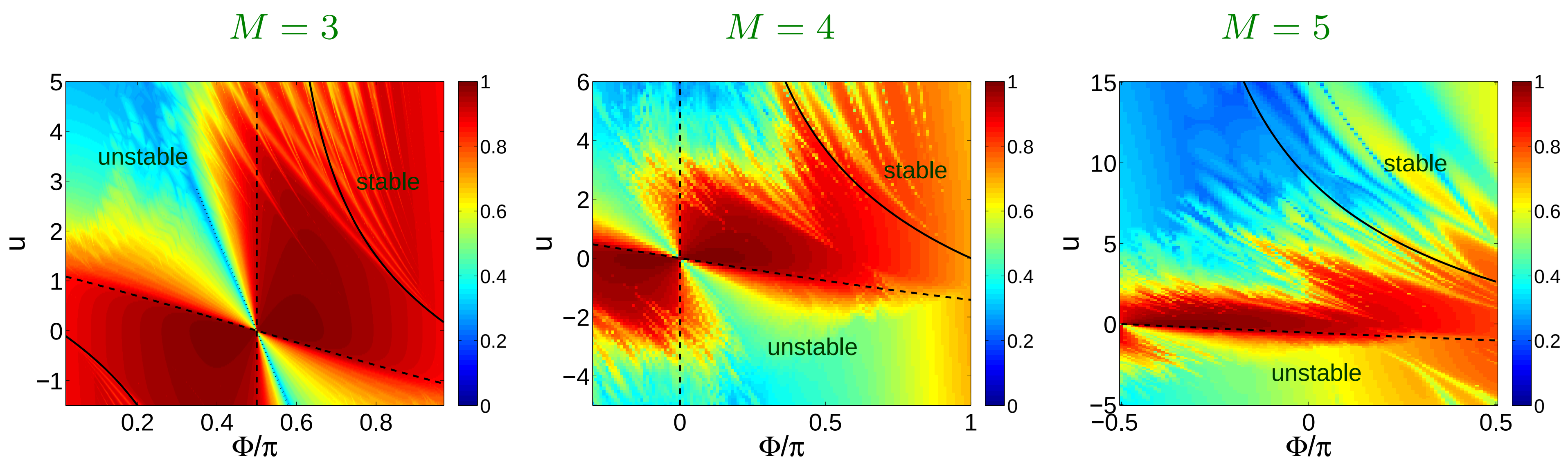
For every (Φ, u) value we plot $\max\{I_\alpha\}$



Regime diagram

The I of the maximum current state is imaged as a function of (Φ, u)

solid lines = energetic stability borders (Landau); dashed lines = dynamical stability borders (BdG)



The traditional paradigm associates vortex states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov de Gennes linear-stability-analysis, are conventionally used to determine the viability of superfluidity.

- We challenge the application of the traditional paradigm to low-dimensional circuits.
- We highlight the role of chaos in the "stability analysis".
- We identify novel types of states that can support superfluidity.

Stability analysis of the excited vortex state

The dynamics is generated by the Hamilton equation: $\dot{z} = \mathbb{J}\boldsymbol{\partial}H(z)$ (DNLS)

Coherent states are supported by fixed-points of the classical Hamiltonian: $\boldsymbol{\partial}H(z) = 0$

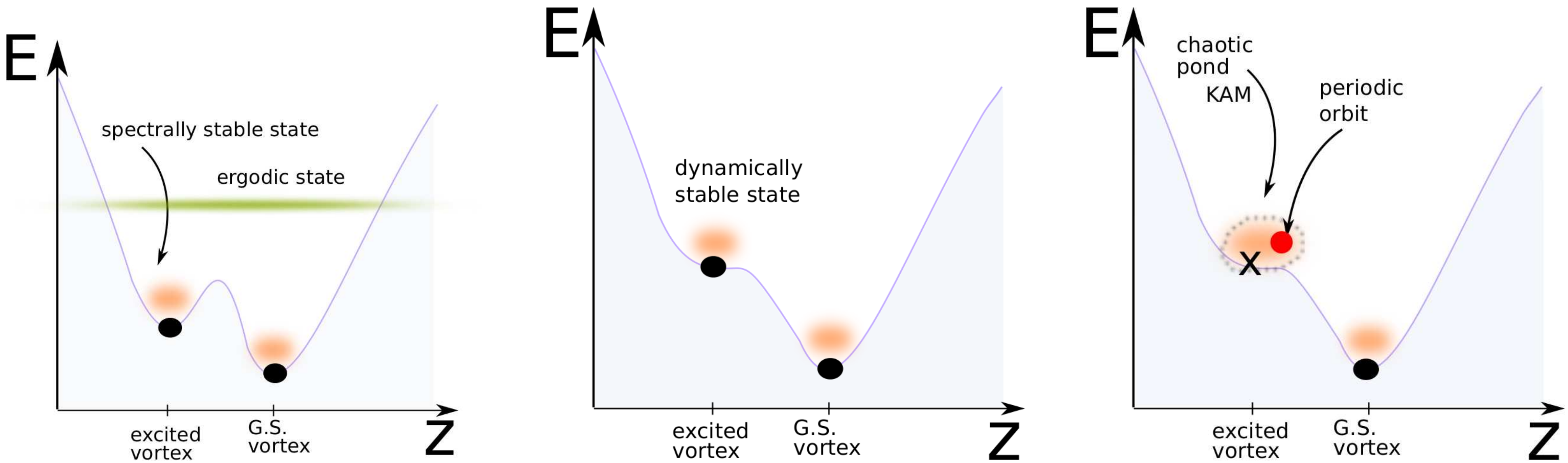
Technical note: The cyclic degree of freedom has to be separated (N is constant of motion).

Linear stability analysis (Bogoliubov de Gennes): $\dot{z} = \mathbb{J}\mathcal{A}z$ where $\mathcal{A}_{\nu,\mu} = \partial_\nu \partial_\mu H$

Energetic stability: Energy local extremal points (Landau criterion) – based on $\mathcal{A}$ diagonalization

Dynamical stability: Zero Lyapunov exponents (real BdG frequencies) – based on $\mathbb{J}\mathcal{A}$ diagonalization

Schematic illustration of the energy landscape $E = H(z)$



Stability of the "ground" vortex state

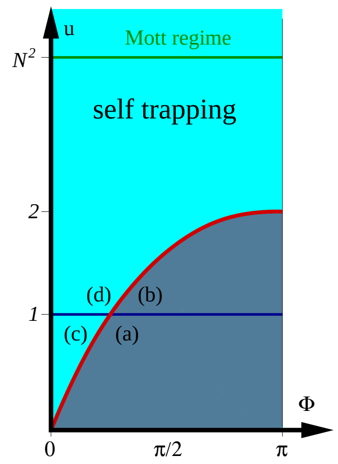
(digression)

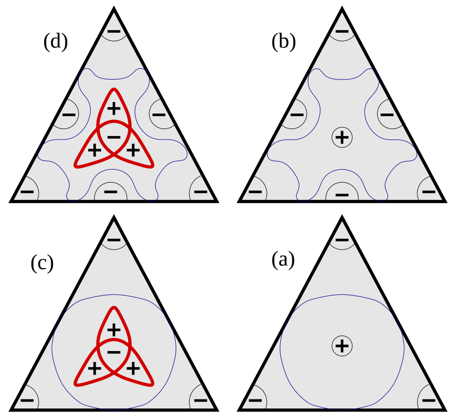
The ground-state vortex can destabilize as well:

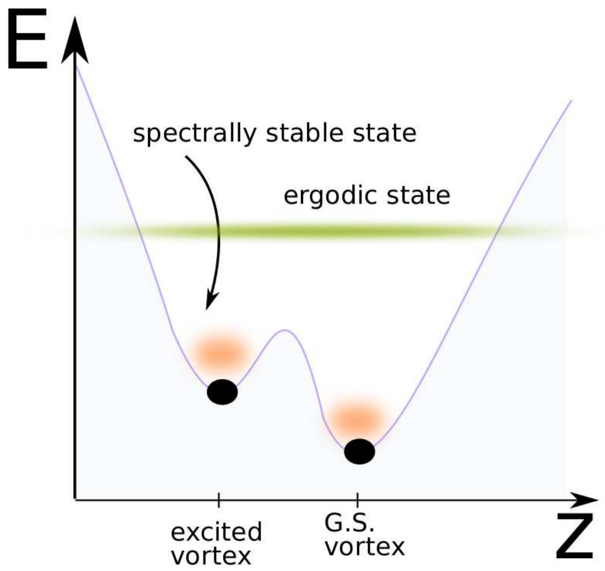
Quantum transition: Mott transition for $u > N^2/M$

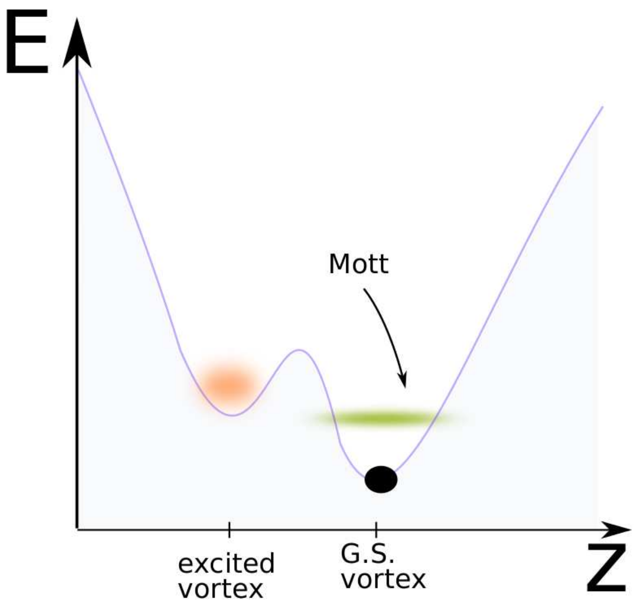
Classical transition: Self-trapping for $u >$ something

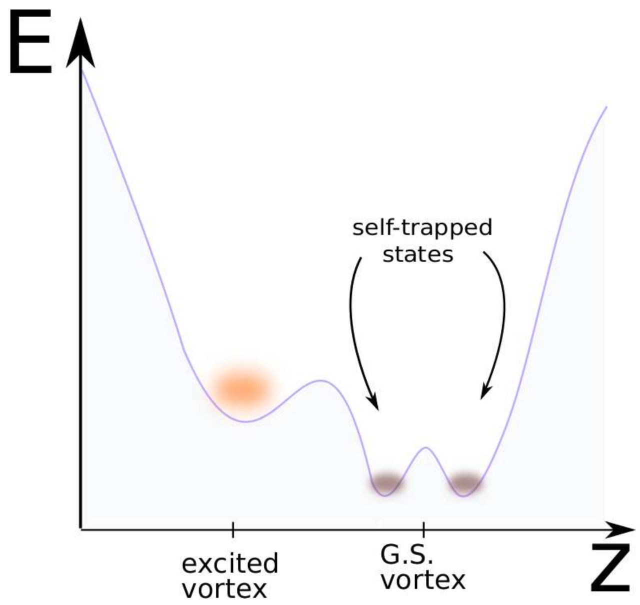
Note: upper state is like ground state for $U \mapsto -U$





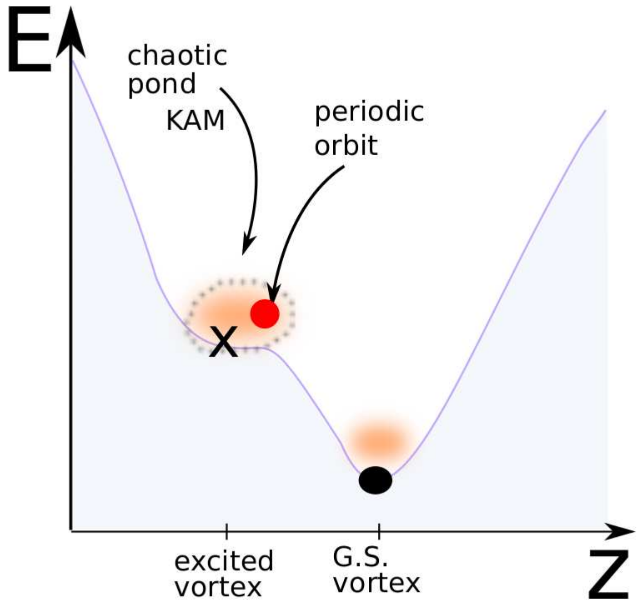


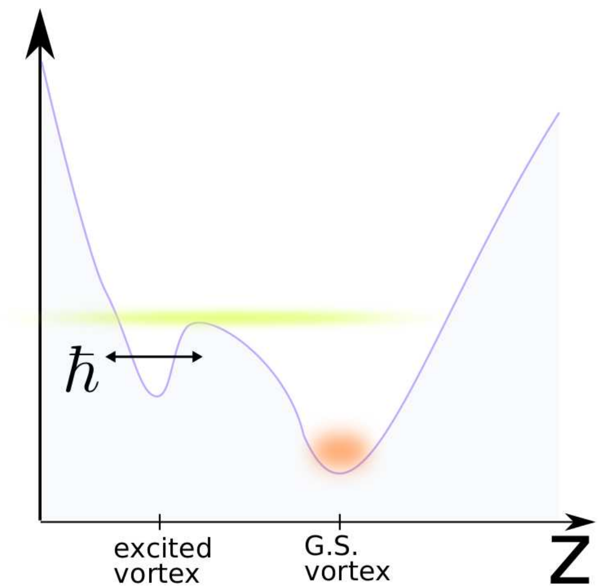




Beyond the traditional view

- Dynamical instability of a vortex state does imply that superfluidity is diminished. Kolmogorov-Arnold-Moser (KAM) structures $\rightsquigarrow$ Chaotic and irregular vortex states.
- Dynamical stability of a vortex state does not imply in general strict stability. For $M > 3$ the KAM tori do not block transport (Arnold diffusion).
- One should take into account quantum fluctuations (uncertainty width of a coherent state). Stability is required within a Plank cell around the fixed-point. Regime-diagram is $\hbar$ dependent.





Regime Diagram for $M = 3$

A stable vortex state carries current:

$$I_m \;=\; \frac{N}{M} K \,\sin\left(\frac{1}{M}(2\pi m - \Phi)\right)$$

Here: $M{=}3$; $m{=}1$; $I_m \sim \frac{N}{M} K$

Energetic stability (solid line):

$$u > \frac{3 - 12\sin^2\left(\frac{\Phi}{3} - \frac{\pi}{6}\right)}{4\sin\left(\frac{\Phi}{3} - \frac{\pi}{6}\right)}$$

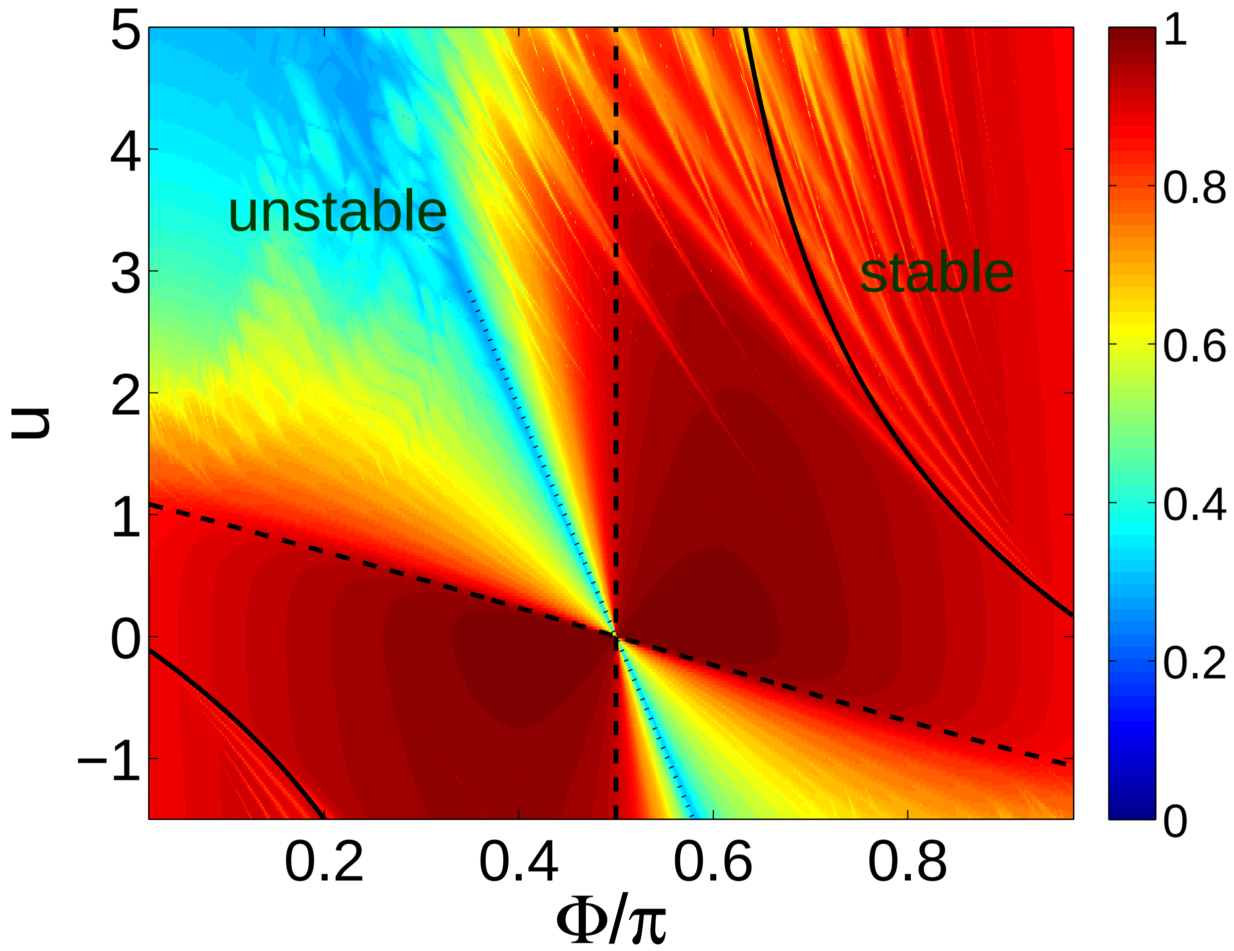
Dynamical instability (dashed line):

$$u > \frac{9}{4}\sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right) \qquad \& \qquad \Phi < \frac{\pi}{2}$$

Swap transition (dotted line):

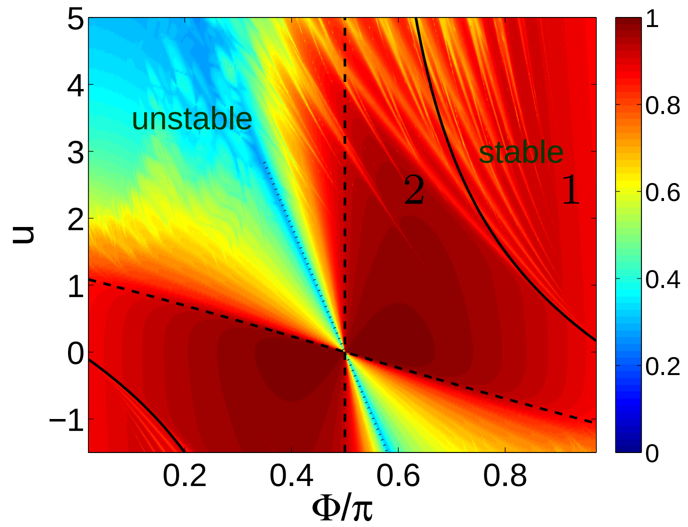
$$u \;=\; 18\sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right)$$

I of maximal current state:



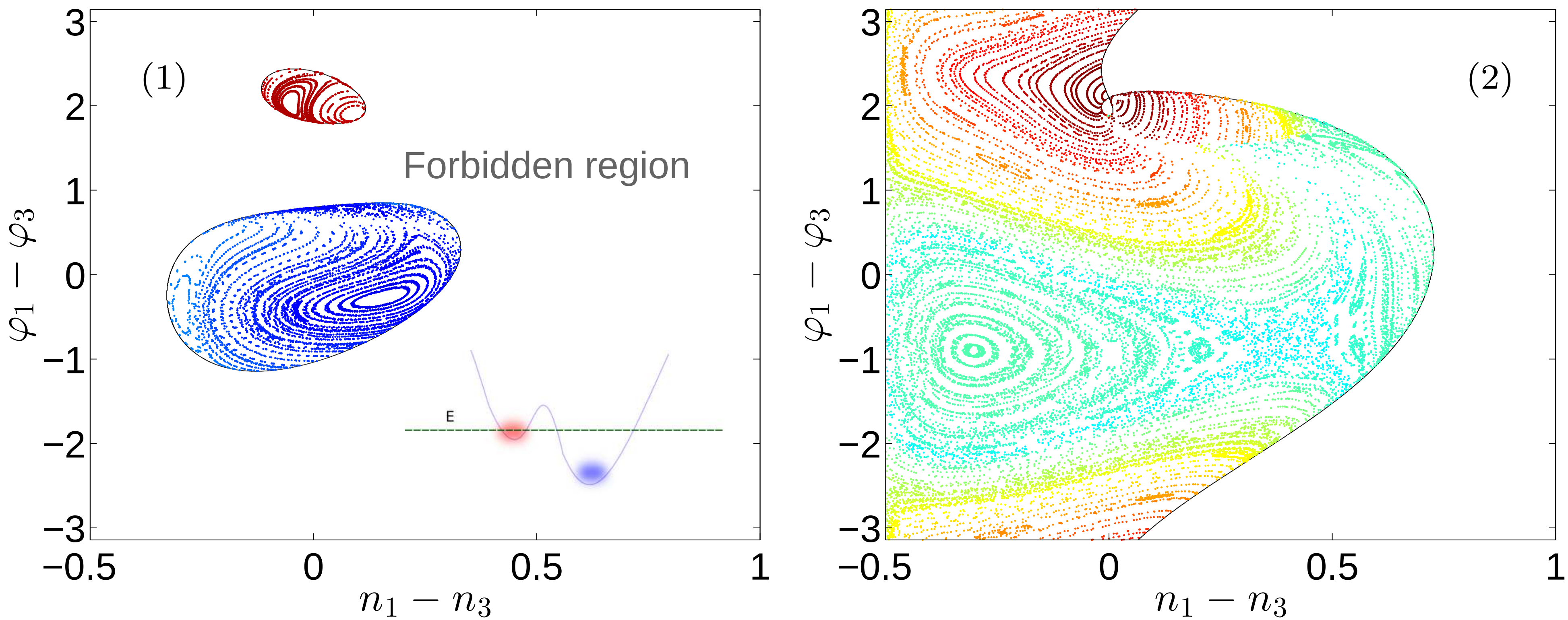
Energetic vs Dynamical stability

Poincare section $n_2 = n_3$ at the vortex energy.
(1) Energetic stability; **(2)** Dynamical stability.
red trajectories = large positive current
blue trajectories = large negative current



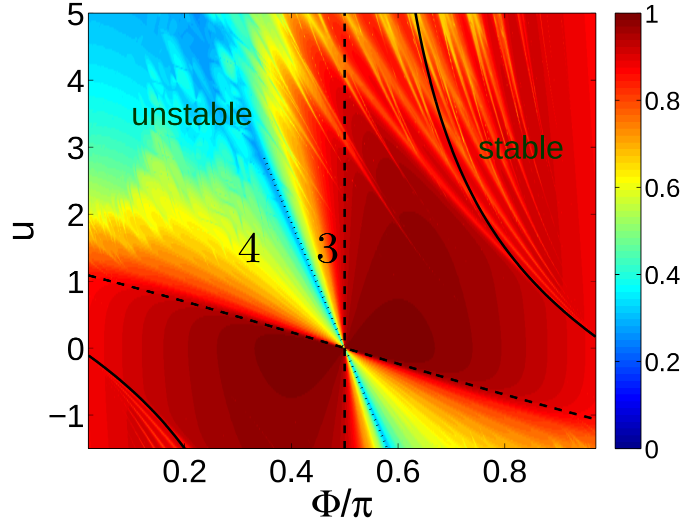
The Vortex fixed-points are located along the symmetry axis:

$$n_1 = n_2 = \cdots = N/M, \qquad \varphi_i - \varphi_{i-1} = \left(\frac{2\pi}{M}\right) m$$



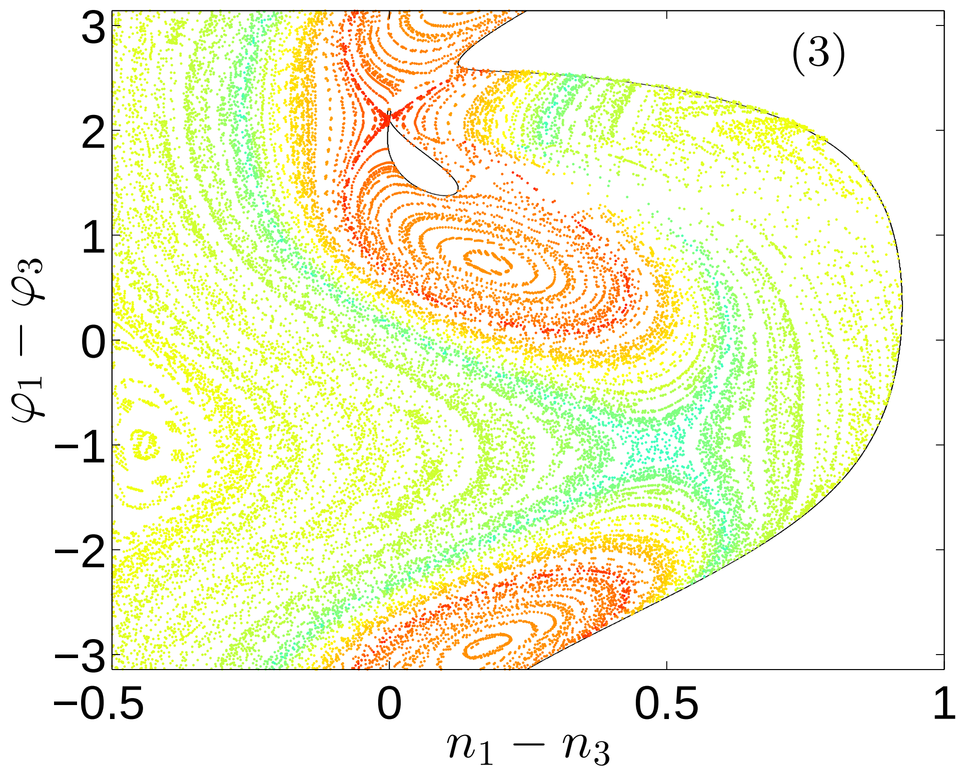
Swap transition

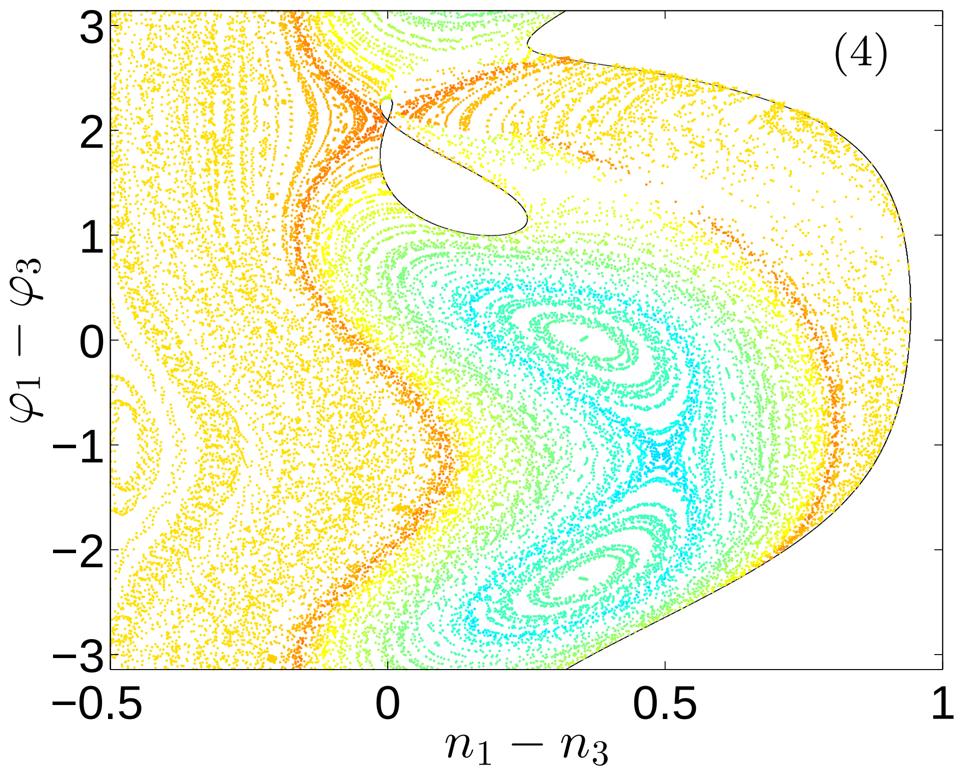
In **(3)** and **(4)** dynamical stability is lost $\rightsquigarrow$ chaotic motion.
But the chaotic trajectory is confined within a chaotic pond;
uni-directional chaotic motion; superfluidity persists!
At the separatrix swap-transition superfluidity diminishes.



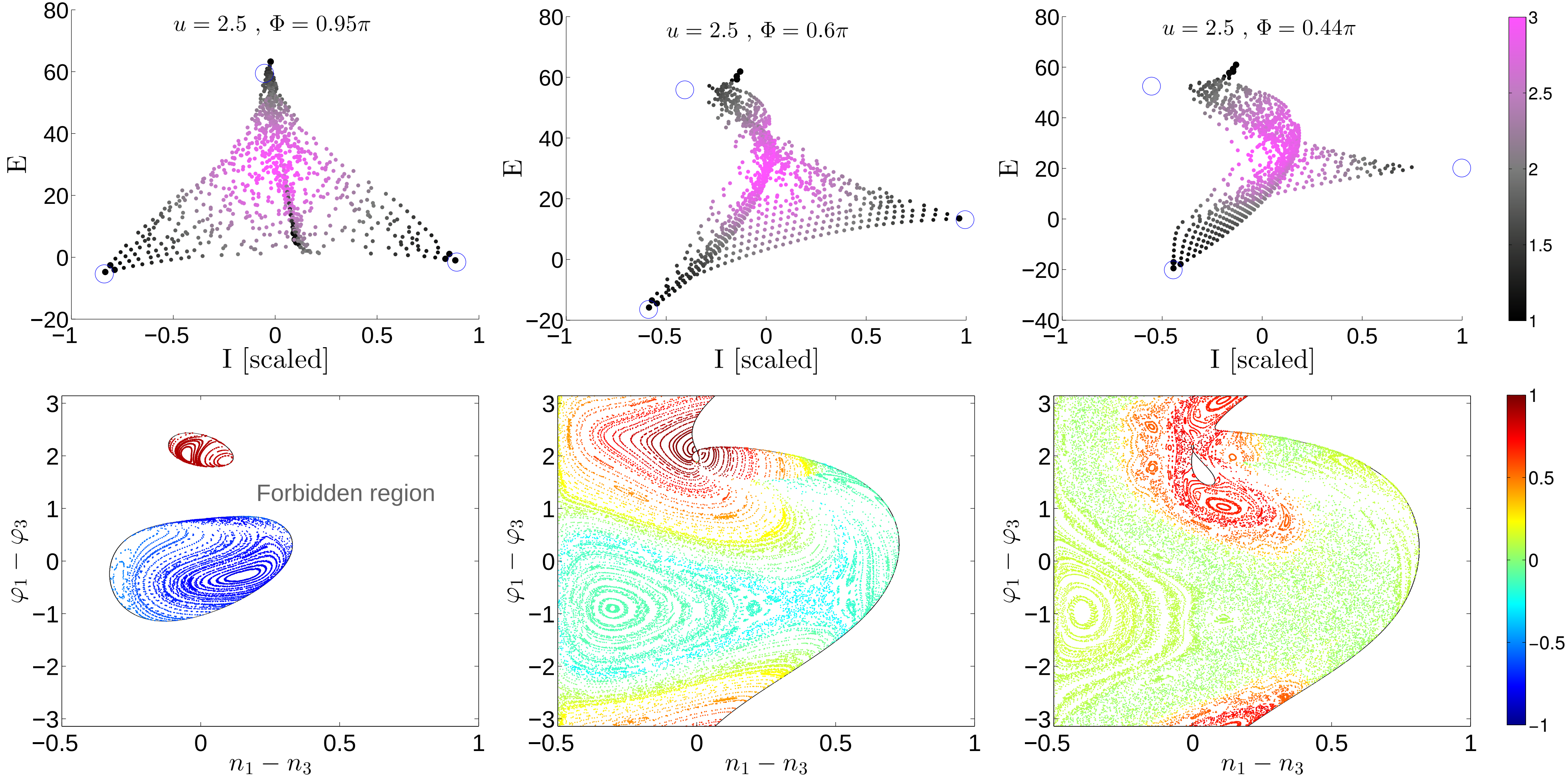
Swap transition (dotted line):

$$u \;=\; 18\sin\left(\frac{\pi}{6}-\frac{\Phi}{3}\right) \qquad \text{(non-linear resonance)}$$

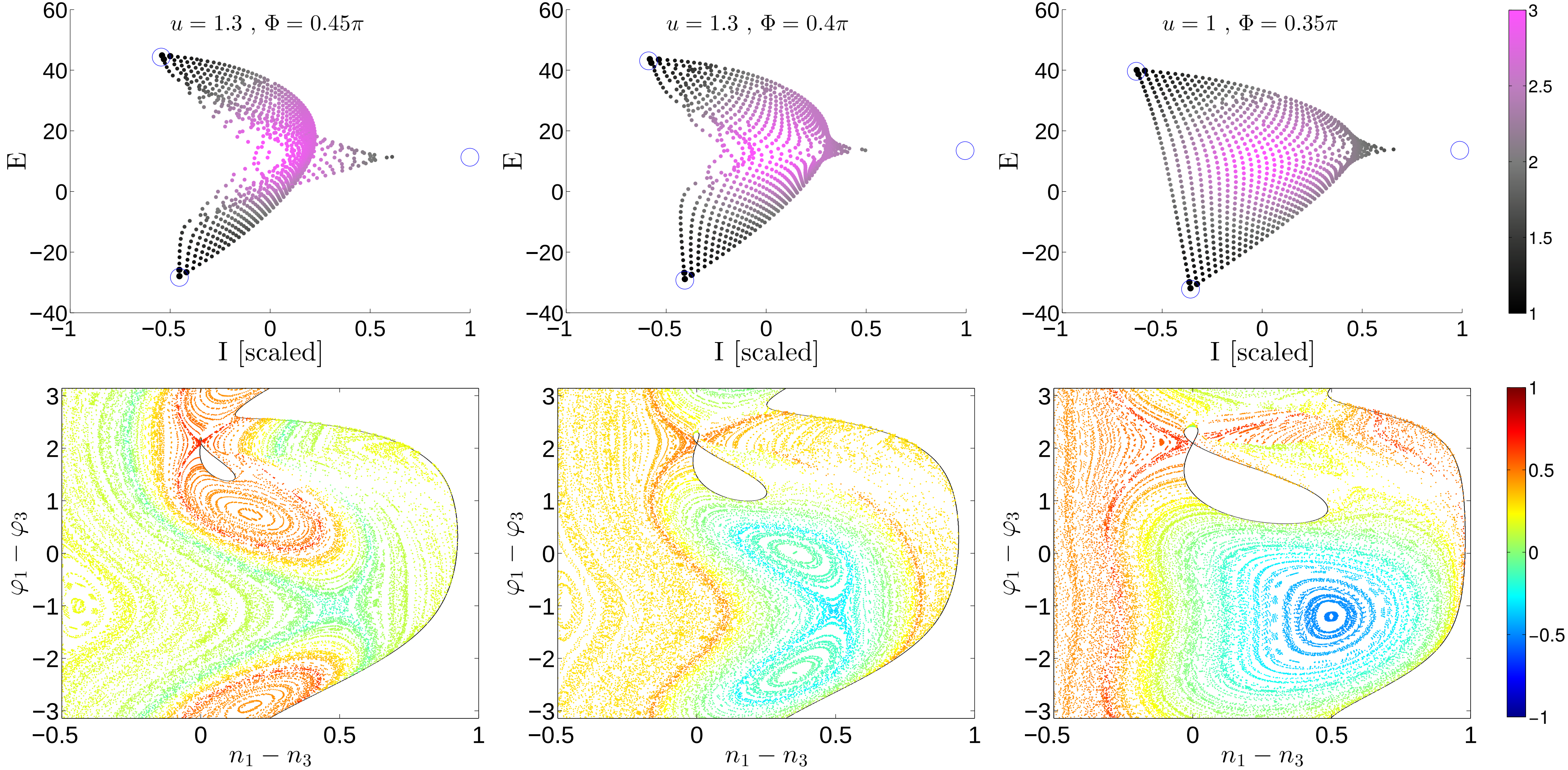




Phase space tomography (I)

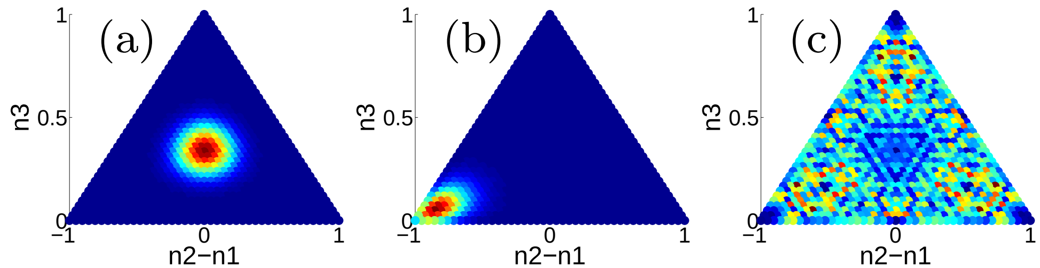


Phase space tomography (II)



Representative Wavefunctions ($M = 3$)

We use standard Fock basis representation.
Images of $|\psi(\boldsymbol{n})|^2 = |\langle \boldsymbol{n}|E_\alpha\rangle|^2$

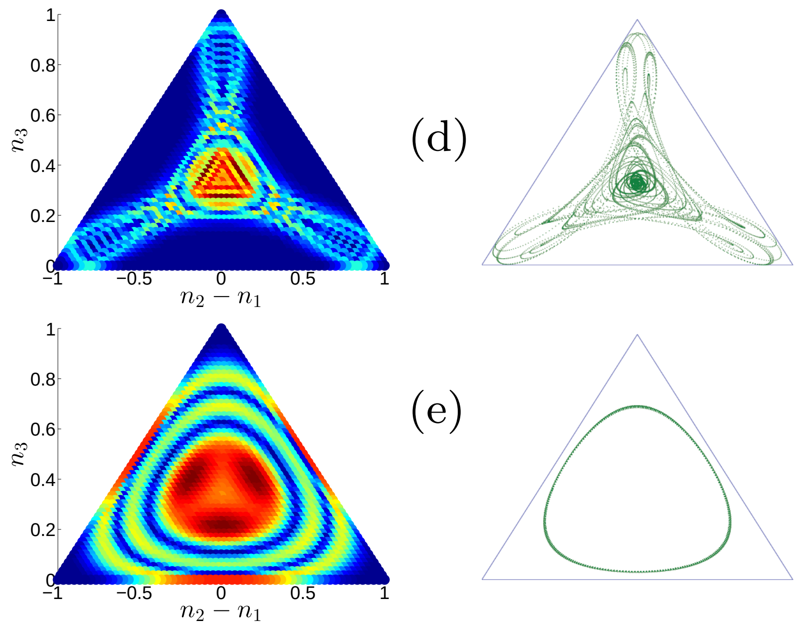


(a) Regular coherent vortex state.
(b) Self-trapped state ("bright soliton").
(c) Typical state in the chaotic sea.

Launching trajectories at the vicinity of the vortex fixed-point we encounter 3 possibilities.

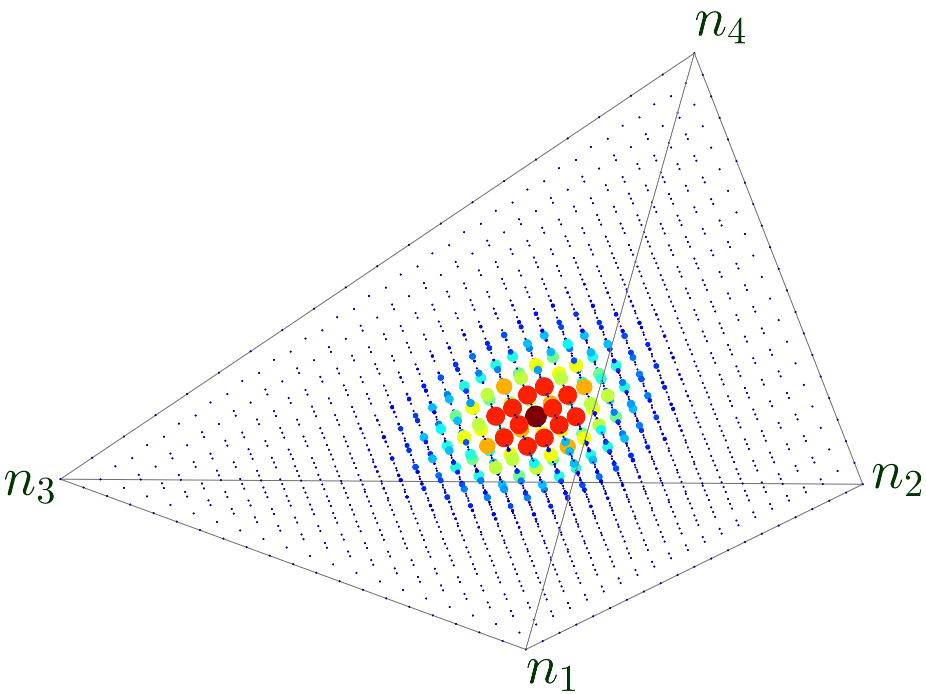
A trajectories might be:

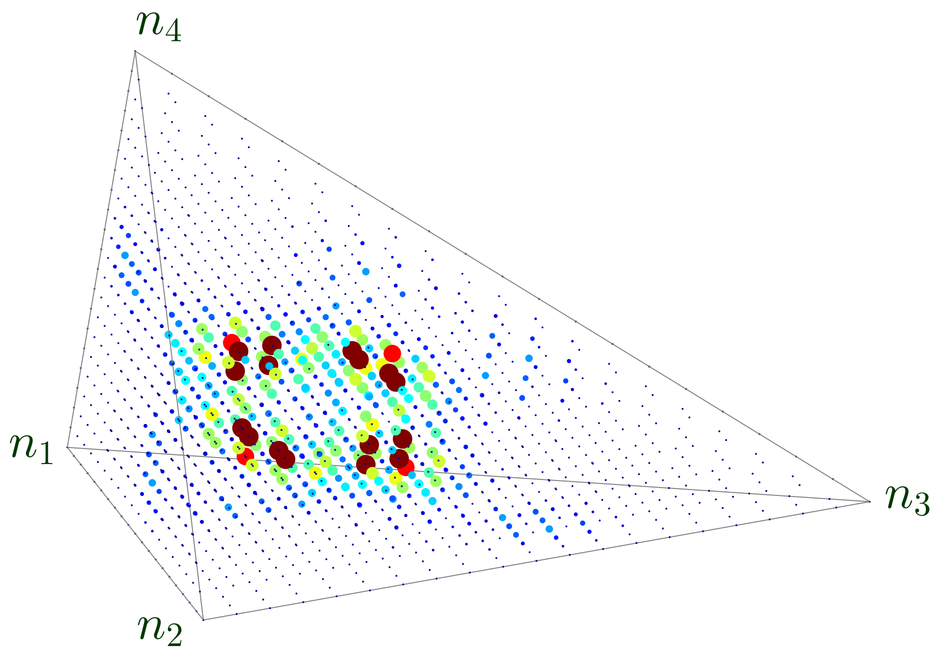
- locked at the vortex fixed point
  (regular vortex state (a))
- chaotic but unidirectional
  (chaotic vortex state (d))
- quasi-periodic in phase-space
  (breathing vortex state (e))



Panels of (d) and (e):
Left: quantum eigenstates.
Right: underlying classical dynamics.

What about $M = 4$?





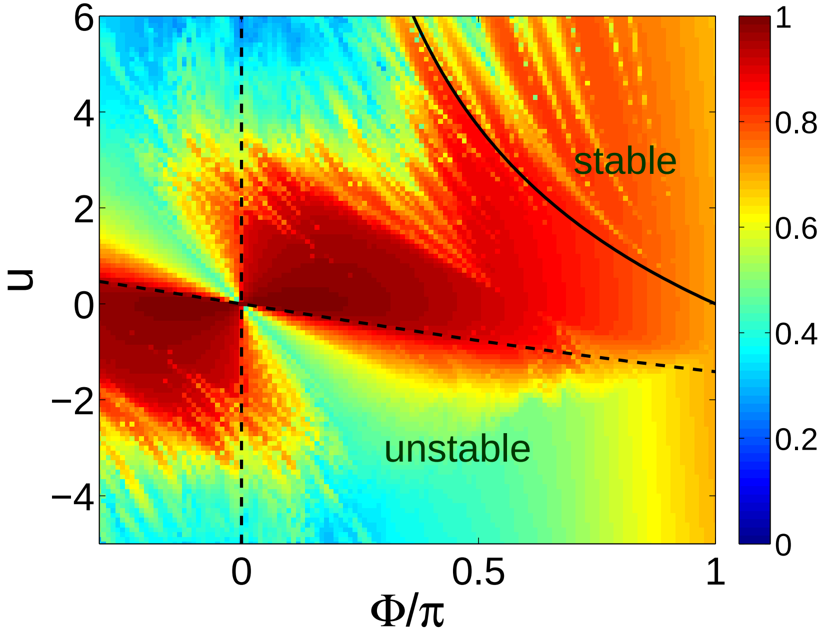
Regular vortex state

Irregular vortex state

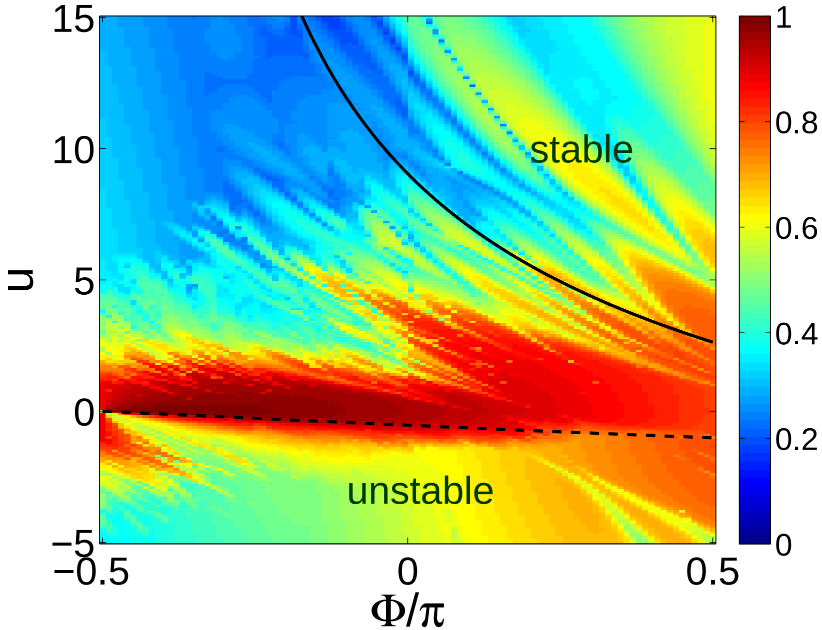
But there is a dramatic difference compared to $M = 3$

"Large" rings ($M > 3$)

$M = 4\ ,\ \ N = 16$



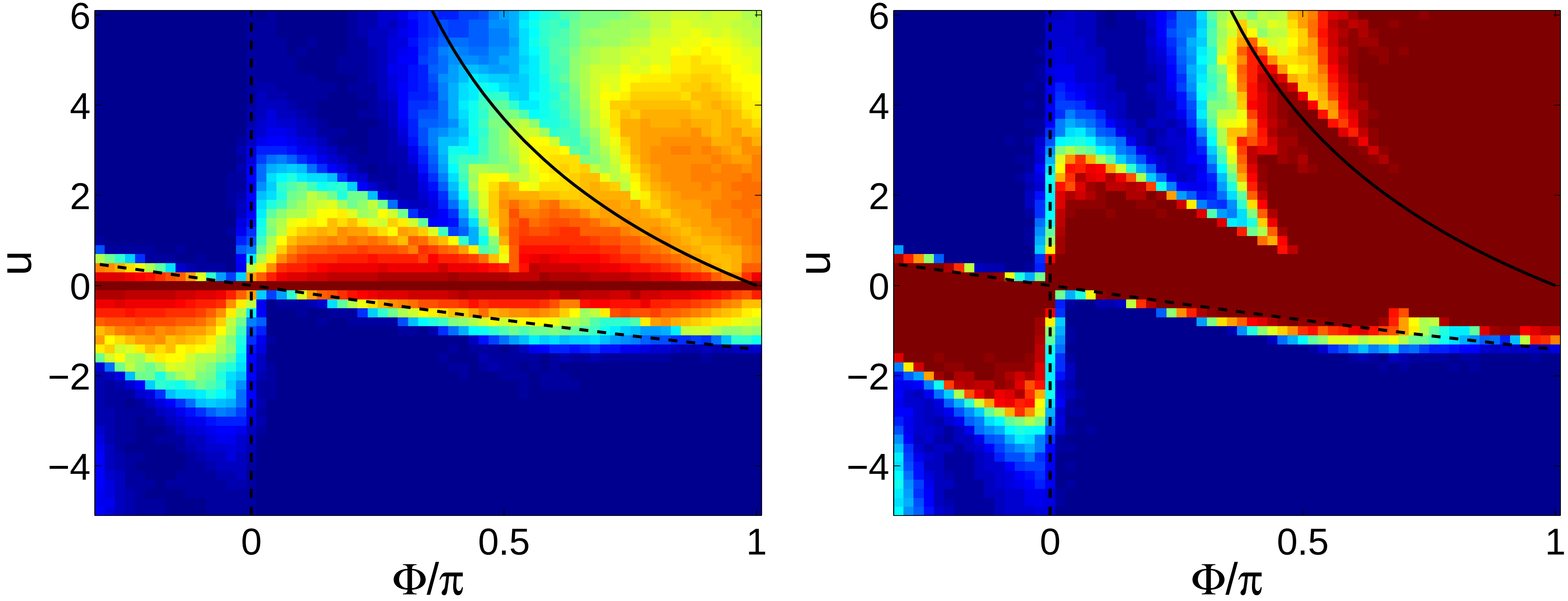
$M = 5\ ,\ \ N = 11$



- Energy surface is $2d - 1$ dimensional (reminder: $d = M - 1$)
- KAM tori are d dimensional
- Arnold diffusion: the KAM tori in phase space are not effective in blocking the transport on the energy shell if $d > 2$.
- As u becomes larger this non-linear leakage effect is enhanced, stability of the motion is deteriorated, and the current is diminished.
- Due to the finite uncertainty width of the vortex state superfluidity can diminish even in the energetically stable region.

Semiclassical reproduction of the regime diagram $M = 4$

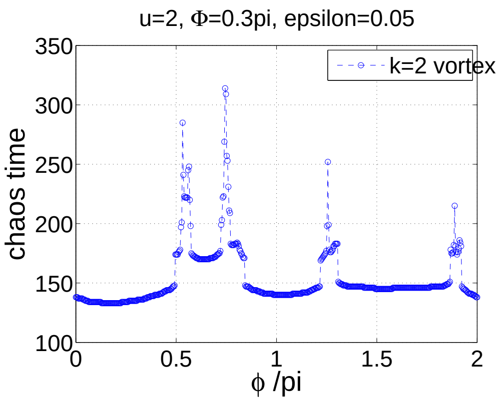
We launch a Gaussian cloud of trajectories that has an uncertainty width that corresponds to N.
Then we calculate the cloud-averaged current $I(t)$.
Sufficient criterion for quasi-stability is $I(t_H) \gtrsim (1/2)I(0)$ where $t_H \propto N^d$ (Heisenberg time)
In practice: the fraction of trajectories that escape is used as a measure for the stability.

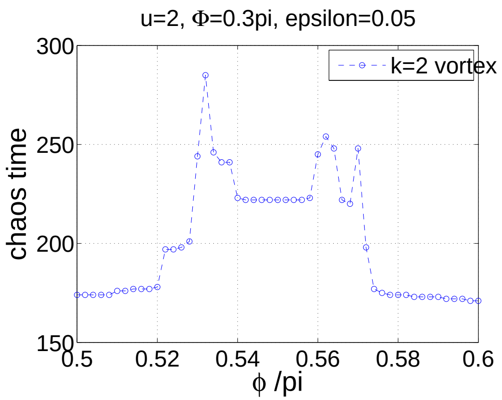


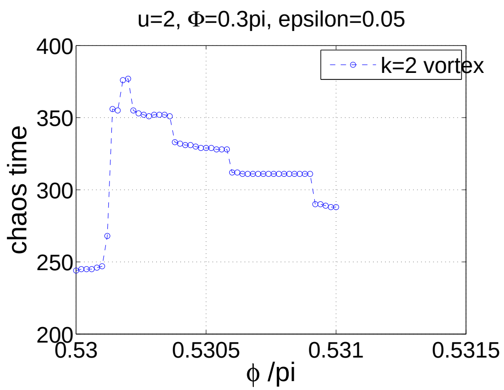
Results are displayed for clouds that have uncertainty width $\Delta\varphi \sim \pi/2$ (left) and $\Delta\varphi \sim \pi/4$ (right).

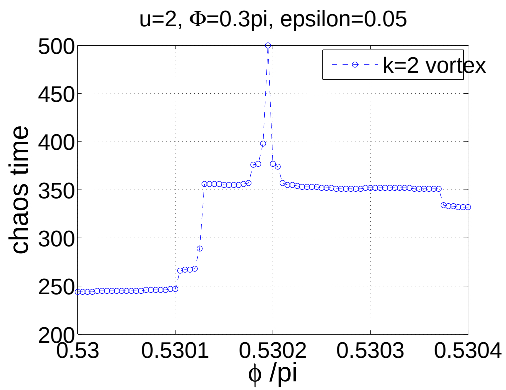
Indication for an underlying multi-fractal structure

The escape time of classical trajectories: section along the cloud.



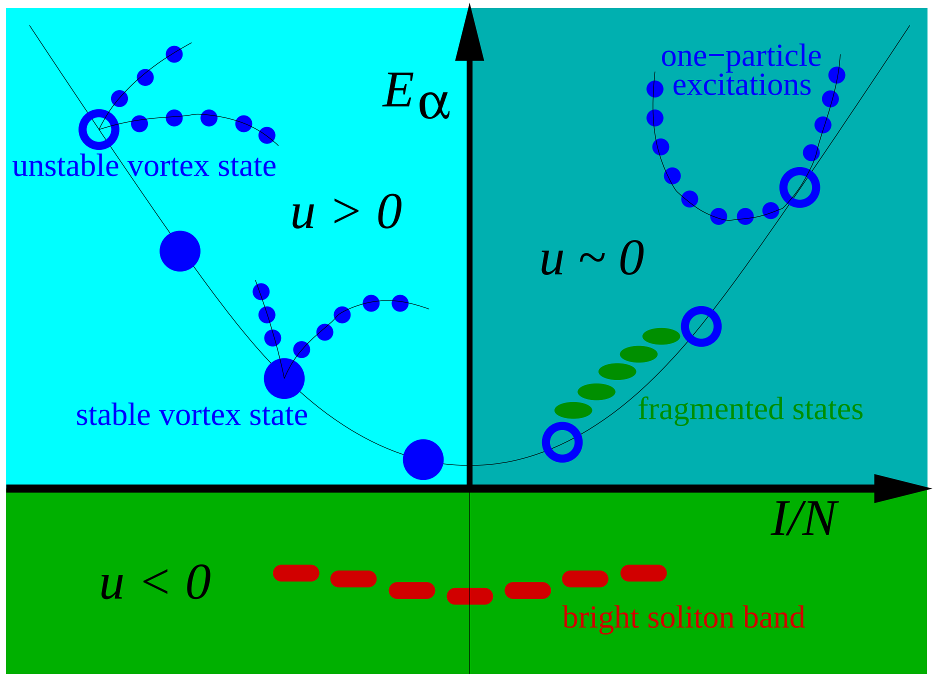






Concluding Remarks regarding superfluidity

- The essence of superfluidity is the possibility to witness metastable vortex states ("dissipationless current")
- The standard energetic stability analysis implies that vortex states whose rotation velocity is less than a critical velocity are metastable ("Landau criterion")
- We challenge the application of the traditional BdG analysis to low-dimensional superfluid circuits.
- We have highlighted a novel type of superfluidity that is supported by irregular or chaotic or breathing vortex states.



- We emphasize that the role of chaos should be recognized in the analysis of superfluidity.