Plan of Lecture

- Part 0:
  General introduction

- Part 1:
  Fluctuation theorems

- **Part 2:**
  Classical/quantum measurement and information

- Part 3:
  Information thermodynamics

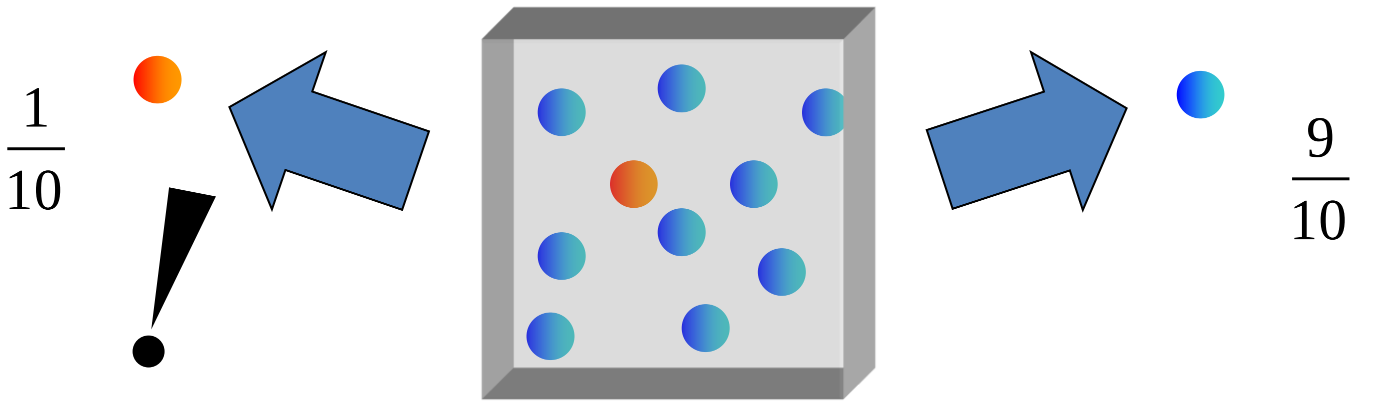
Part 2: Classical/quantum measurement and information

- Shannon information
- Mutual information
- Quantum measurement
- Quantum mutual information
- QC-mutual information

Part 2:
Classical/quantum measurement and information

- **Shannon information**
- Mutual information
- Quantum measurement
- Quantum mutual information
- QC-mutual information

Shannon Information (1)



Information content with event k: $\ln \frac{1}{p_k}$

Average

Shannon information: $H = \sum_k p_k \ln \frac{1}{p_k}$

Shannon Information (2)

$$0 \le H \le \ln N$$

N: the number of k's

$p_i = 1$ for a single i

$p_k = 0$ for $k \neq i$

$p_k = 1/N$ for all k

$$H = -\sum_k p_k \ln p_k$$

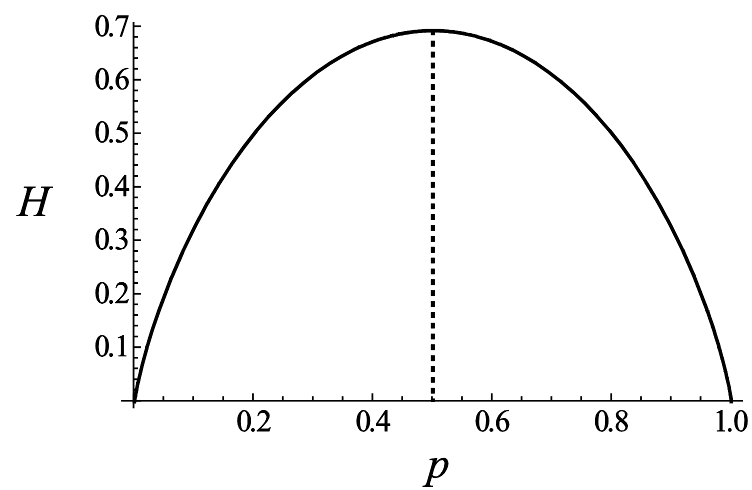
Ex. Binary system

"0": probability p

"1": probability 1-p

$$H = -p \ln p - (1-p)\ln(1-p)$$

$$0 \le H \le \ln 2$$

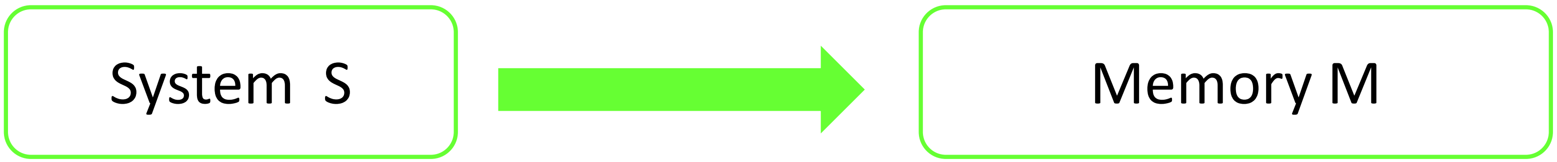


Characterizes the **randomness** of the system

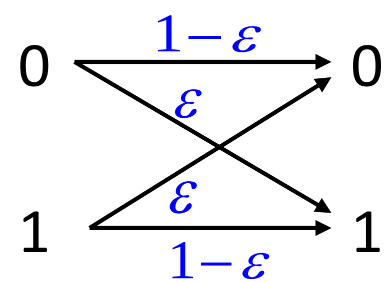
Part 2: Classical/quantum measurement and information

- Shannon information
- **Mutual information**
- Quantum measurement
- Quantum mutual information
- QC-mutual information

Mutual Information (1)



Measurement or communication
(with stochastic error, in general)



Ex. Binary symmetric channel

$p(s)$: distribution of the measured state of S

$p(m)$: distribution of the outcome in M

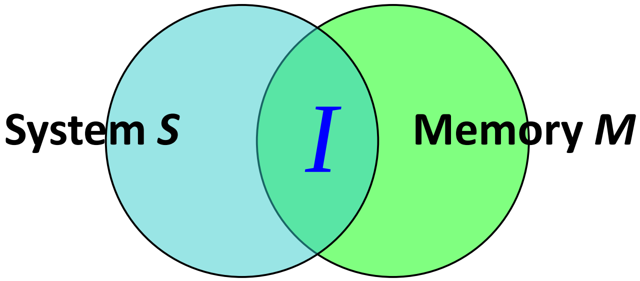
$p(m|s)$: conditional probability characterizing the error

$p(s,m) = p(m|s)p(s)$: joint distribution of S and M

Mutual Information (2)

$$I(S:M) \equiv H(S) + H(M) - H(SM)$$

$$= \sum_{sm} p(s,m) \ln \frac{p(s,m)}{p(s)\,p(m)}$$



$p(s)$: distribution of the measured state of S

$p(m)$: distribution of the outcome in M

$p(m\,|\,s)$: conditional probability characterizing the error

$p(s,m) = p(m\,|\,s)\,p(s)$: joint distribution of S and M

$$H(S) = -\sum_{s} p(s) \ln p(s) \quad H(M) = -\sum_{m} p(m) \ln p(m) \quad H(SM) = -\sum_{sm} p(s,m) \ln p(s,m)$$

Mutual Information (3)

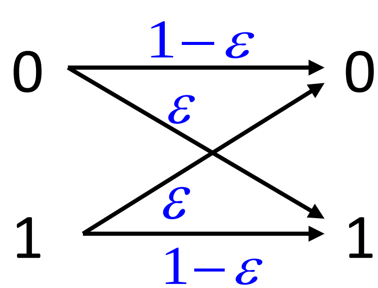
$$I(S:M) \equiv H(S) + H(M) - H(SM)$$

Correlation between S and M

$$0 \leq I \leq H(M)$$

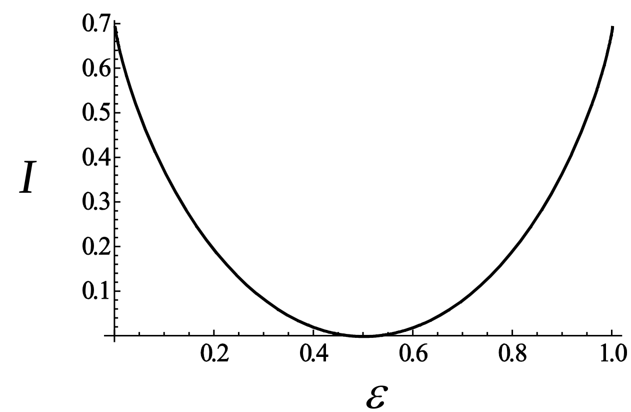
No information

No error

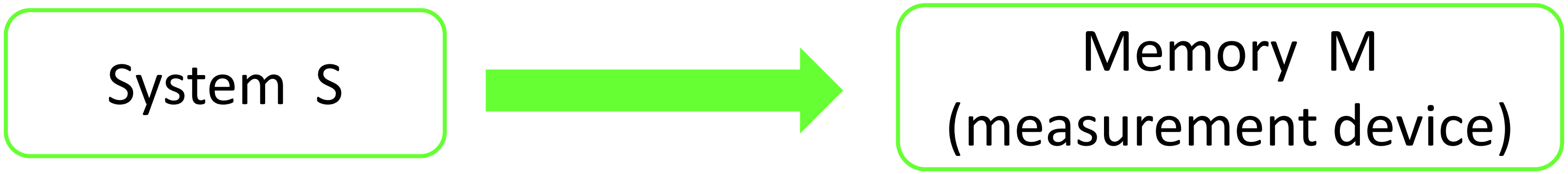


Ex. Binary symmetric channel

$$I = \ln 2 + \varepsilon \ln \varepsilon + (1-\varepsilon)\ln(1-\varepsilon)$$

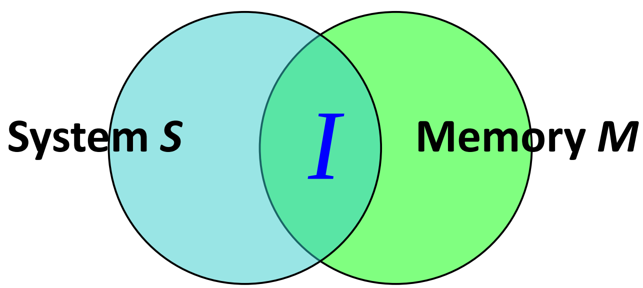


Mutual Information: Summary



Measurement with stochastic errors

$$I(S:M) \equiv H(S) + H(M) - H(SM)$$



$$0 \leq I \leq H(M)$$

No information

Error-free

Shannon information:
Randomness of the system

Mutual information:
Correlation between the system and the memory

Part 2:
Classical/quantum measurement and information

- Shannon information
- Mutual information
- **Quantum measurement**
- Quantum mutual information
- QC-mutual information

Quantum State

Density operator: $\rho = \sum_i q_i \left| \varphi_i \right\rangle\left\langle \varphi_i \right|$

Statistical mixture of state vector $\left| \varphi_i \right\rangle$ with probability q_i

---

Non-negativity: $\rho \geq 0$

Unity: $\mathrm{tr}[\rho] = \sum_i q_i = 1$

Quantum Measurement

Projection measurement (error-free)

Observable $A = \sum_k \alpha_k P_k$

Measurement outcome k

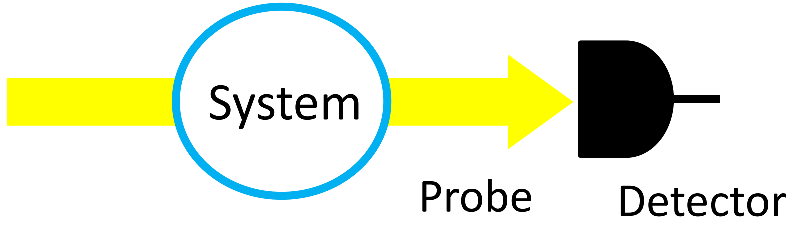
Projection operators $\{P_k\}$

$$\sum_k P_k = I$$

Probability $p_k = \mathrm{tr}[\rho P_k]$

Born's rule

Indirect measurement (with error)



Measurement outcome k

POVM $\{E_k\}$ (positive operators)

$$\sum_k E_k = I$$

Probability $p_k = \mathrm{tr}[\rho E_k]$

Projection Measurement

Projection measurement on density operator $\rho = \sum_i q_i |\varphi_i\rangle\langle\varphi_i|$

Observable $A = \sum_k \alpha_k P_k$

P_k : projection operator

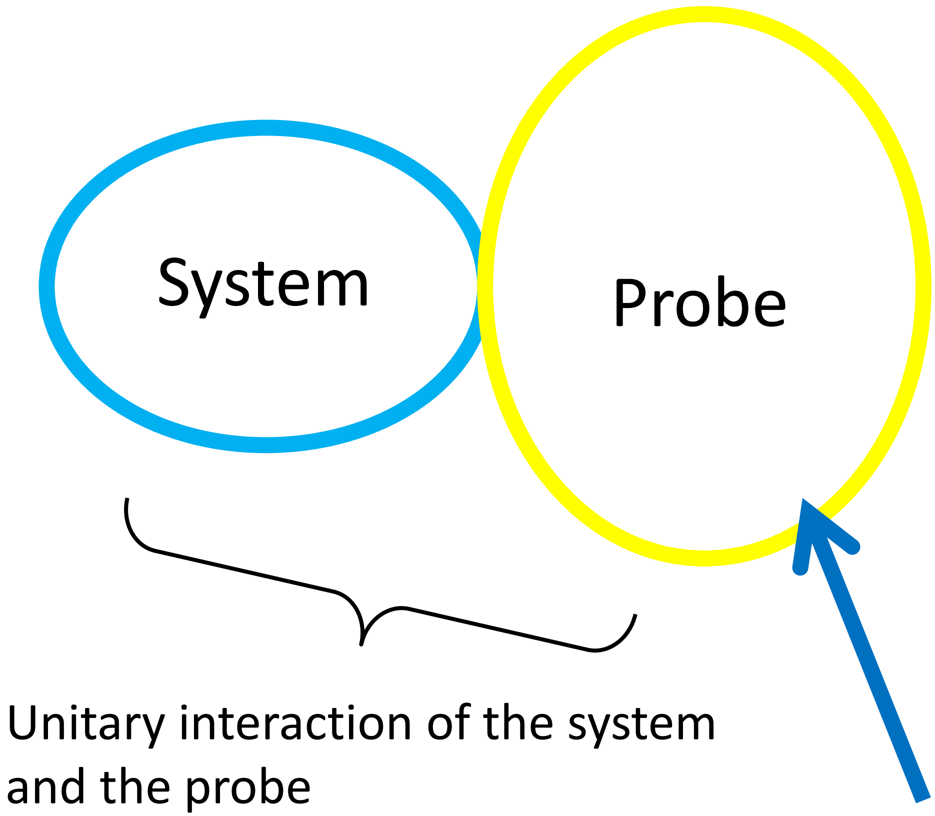
Probability of outcome k

Born's rule $$p_k = \sum_i q_i \langle\varphi_i|P_k|\varphi_i\rangle = \mathrm{tr}[P_k \rho]$$

Post-measurement state with outcome k

Projection postulate $$\frac{1}{p_k} P_k \rho P_k$$

Schematic of Indirect Measurement



$$\rho \otimes \rho_{\mathrm{P}}$$

$$U \rho \otimes \rho_{\mathrm{P}} U^{\dagger}$$

Assume: $\rho_{\mathrm{P}} = |\psi\rangle\langle\psi|$

Projection measurement on the probe

$P_k = |\psi_k\rangle\langle\psi_k|$ $\{|\psi_k\rangle\}$: orthonormal basis

Probability of Indirect Measurement

Probability of outcome k :

$$p_k = \mathrm{tr}_{\mathrm{SP}}[P_k U\rho \otimes |\psi\rangle\langle\psi| U^\dagger P_k]$$

$$= \mathrm{tr}_{\mathrm{S}}[\langle\psi_k| U\rho \otimes |\psi\rangle\langle\psi| U^\dagger |\psi_k\rangle]$$

$$= \mathrm{tr}_{\mathrm{S}}[\langle\psi_k| U |\psi\rangle \rho \langle\psi| U^\dagger |\psi_k\rangle]$$

$$= \mathrm{tr}_{\mathrm{S}}[M_k \rho M_k^\dagger]$$

$$= \mathrm{tr}_{\mathrm{S}}[E_k \rho]$$

POVM:

$$E_k = M_k^\dagger M_k$$

Kraus operator:

$$M_k \equiv \langle\psi_k| U |\psi\rangle$$

$$\sum_k p_k = 1 \Longleftrightarrow \sum_k E_k = I_d$$

Back-action of Indirect Measurement

Post-measurement state with outcome k :

$$\rho_k = \frac{1}{p_k} \mathrm{tr}_\mathrm{P}[P_k U \rho \otimes |\psi\rangle\langle\psi| U^\dagger P_k]$$

$$= \frac{1}{p_k} \langle\psi_k| U \rho \otimes |\psi\rangle\langle\psi| U^\dagger |\psi_k\rangle$$

$$= \frac{1}{p_k} M_k \rho M_k^\dagger$$

$$M_k \equiv \langle\psi_k|U|\psi\rangle$$

Example 1: Projection Measurement

Two-level atom

Kraus operators: $M_0 = |0\rangle\langle 0|$ $M_1 = |1\rangle\langle 1|$ (projection)

POVM: $E_0 = |0\rangle\langle 0|$ $E_1 = |1\rangle\langle 1|$ (projection)

$$\rho = q_0|0\rangle\langle 0| + q_1|1\rangle\langle 1| + \alpha|0\rangle\langle 1| + \alpha^*|1\rangle\langle 0|$$

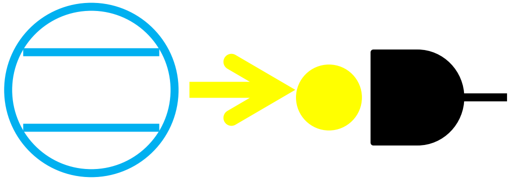
$\Rightarrow$

$p_0 = q_0$ $p_1 = q_1$ (error-free)

$\rho_0 = |0\rangle\langle 0|$ $\rho_1 = |1\rangle\langle 1|$ (projection)

Example 2: Measurement by Photon Emission

Two-level atom



Kraus operators: $M_0 = |0\rangle\langle 0|$ $M_1 = |0\rangle\langle 1|$ (not projection)

POVM: $E_0 = |0\rangle\langle 0|$ $E_1 = |1\rangle\langle 1|$ (projection)

$$\rho = q_0|0\rangle\langle 0| + q_1|1\rangle\langle 1| + \alpha|0\rangle\langle 1| + \alpha^*|1\rangle\langle 0|$$

$p_0 = q_0$ $p_1 = q_1$ (error-free)

$\rho_0 = |0\rangle\langle 0|$ $\rho_1 = |0\rangle\langle 0|$ (not projection)

Example 3: Measurement with Error

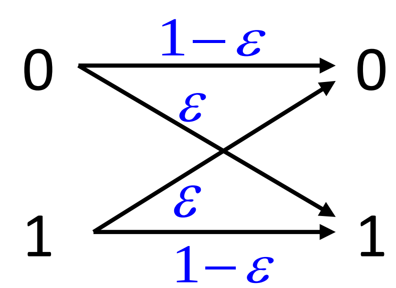
Two-level atom

Kraus operators: $M_0 = \sqrt{1-\varepsilon}|0\rangle\langle 0| + \sqrt{\varepsilon}|1\rangle\langle 1|$ $\quad M_1 = \sqrt{\varepsilon}|0\rangle\langle 0| + \sqrt{1-\varepsilon}|1\rangle\langle 1|$

POVM: $E_0 = (1-\varepsilon)|0\rangle\langle 0| + \varepsilon|1\rangle\langle 1|$ $\quad E_1 = \varepsilon|0\rangle\langle 0| + (1-\varepsilon)|1\rangle\langle 1|$

ε : error rate

$$\rho = q_0|0\rangle\langle 0| + q_1|1\rangle\langle 1| + \alpha|0\rangle\langle 1| + \alpha^*|1\rangle\langle 0|$$



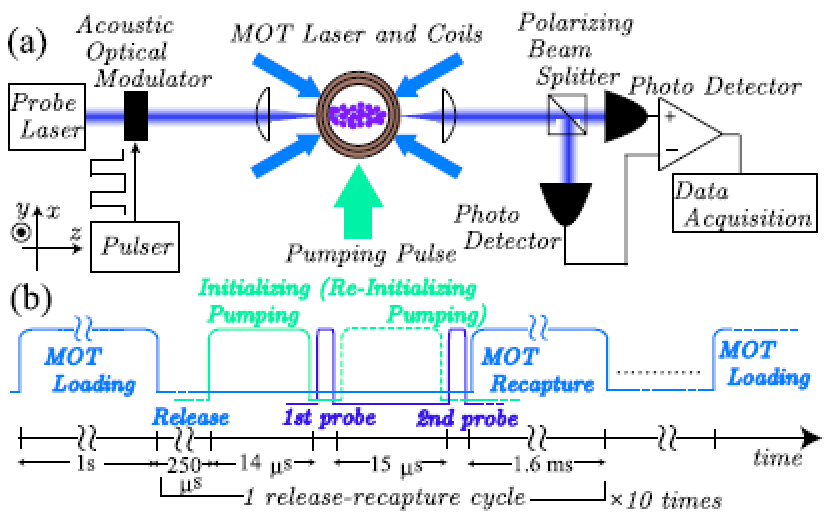
$$p_0 = (1-\varepsilon)q_0 + \varepsilon q_1$$

$$p_1 = \varepsilon q_0 + (1-\varepsilon)q_1$$

Real Example: QND Measurement of Atomic Spin

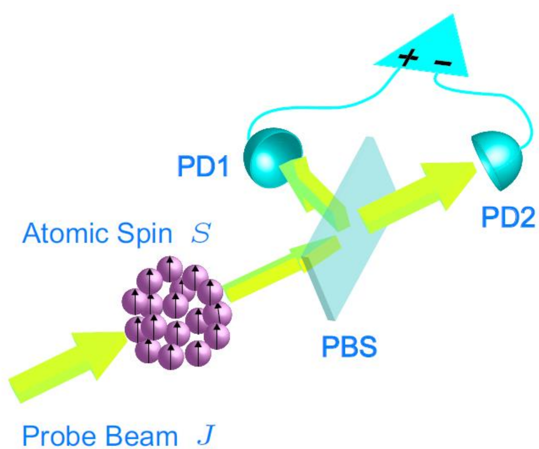
Takano *et al.*, Phys. Rev. Lett. **102**, 033601 (2009)

QND = Quantum Non-Demolition



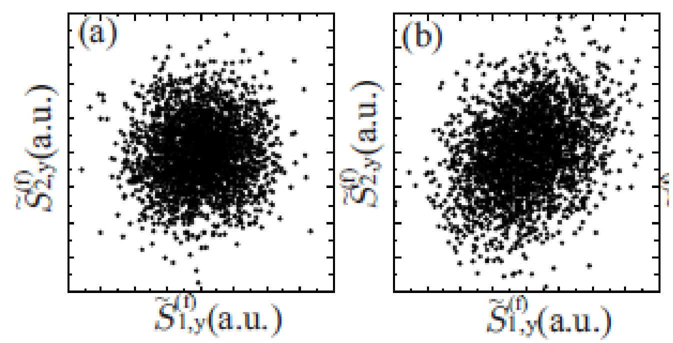
System = Atomic Nuclear Spins

Probe = Polarized photons



Correlation of two-time measurements

Without atoms **With atoms**



No correlation

Correlation by wave function collapse

Quantum Measurement: Summary

Projection measurement (error-free)

Observable $A = \sum_k \alpha_k P_k$

Measurement outcome k

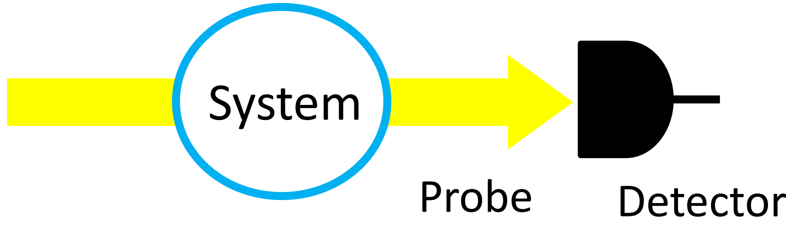
Projection operators $\{P_k\}$

$$\sum_k P_k = I$$

Probability $p_k = \mathrm{tr}[\rho P_k]$

Born's rule

Indirect measurement (with error)



Measurement outcome k

POVM $\{E_k\}$ (positive operators)

$$\sum_k E_k = I$$

Probability $p_k = \mathrm{tr}[\rho E_k]$

Part 2:
Classical/quantum measurement and information

- Shannon information
- Mutual information
- Quantum measurement
- **Quantum mutual information**
- QC-mutual information

Quantum Entropy

Von Neumann entropy: $$S(\rho) = -\mathrm{tr}\left[\rho \ln \rho\right]$$

ρ : density operator

$$\rho = \sum_i q_i \left|\varphi_i\right\rangle\left\langle\varphi_i\right| \quad \Rightarrow \quad S(\rho) = -\sum_i q_i \ln q_i$$

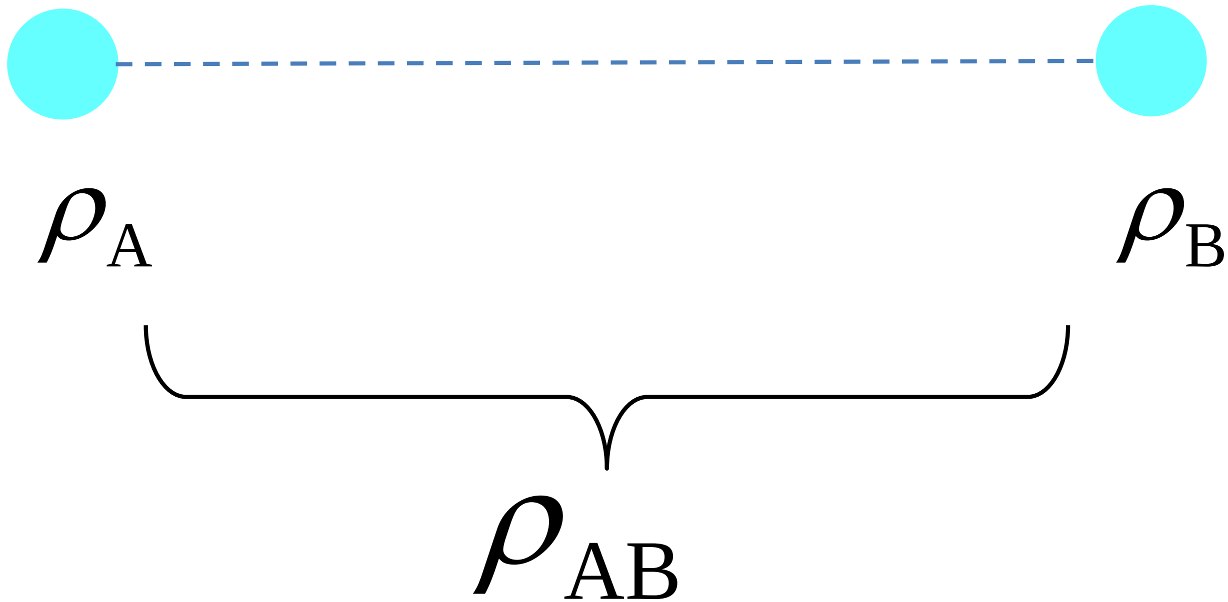
with an orthonormal basis

Characterizes the randomness of the classical mixture in the density operator

Variety of Quantum Information

- Quantum mutual information
- Quantum discord
- Holevo's chi
- QC-mutual information (Groenewold information)

Bipartite Quantum System



$$\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}[\rho_{\mathrm{AB}}] \qquad \rho_{\mathrm{B}} = \mathrm{tr}_{\mathrm{A}}[\rho_{\mathrm{AB}}]$$

Quantum Mutual Information (1)

$$I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) = S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) - S(\rho_{\mathrm{AB}})$$

$$I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) \geq 0 \quad \Leftrightarrow \quad S(\rho_{\mathrm{A}}) + S(\rho_{\mathrm{B}}) \geq S(\rho_{\mathrm{AB}})$$

(Subadditivity of von Neumann entropy)

Purely classical correlation ⇒ Classical mutual information

$$\rho_{\mathrm{AB}} = \sum_{xy} q_{xy} |\varphi_x\rangle\langle\varphi_x| \otimes |\phi_y\rangle\langle\phi_y|$$ with orthonormal bases

$$\Rightarrow \quad I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) = \sum_{xy} q_{xy} \ln \frac{q_{xy}}{q_x q_y}$$

Quantum Mutual Information (2)

Example: 2 qubits

$$\rho_{\mathrm{AB}} = |00\rangle\langle 00| \quad \text{(no correlation)} \quad \Rightarrow \quad I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) = 0$$

$$\rho_{\mathrm{AB}} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) \quad \Rightarrow \quad I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) = \ln 2$$

(classical correlation)

$$\rho_{\mathrm{AB}} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11| + |11\rangle\langle 00| + |00\rangle\langle 11|\right)$$

(entanglement)

$$\Rightarrow \quad I_{\mathrm{A:B}}(\rho_{\mathrm{AB}}) = 2\ln 2$$

Part 2: Classical/quantum measurement and information

- Shannon information
- Mutual information
- Quantum measurement
- Quantum mutual information
- **QC-mutual information**

QC-mutual Information (1)

Information flow from **Q**uantum system to **C**lassical outcome by quantum measurement

$$I_{\rm QC} \equiv S(\rho) - \sum_y p_y S(\rho_y)$$

ρ : measured density operator

$p(y) = \mathrm{tr}[\rho M_y^\dagger M_y]$: probability of obtaining outcome k

$\rho_y = \frac{1}{p(y)} M_y \rho M_y^\dagger$: post-measurement state with outcome k

Assumed a single Kraus operator for each outcome

H. J. Groenewold, Int. J. Theor. Phys. **4**, 327 (1971).
M. Ozawa, J. Math. Phys. **27**, 759 (1986).
TS and M. Ueda, PRL **100**, 080403 (2008).

QC-mutual Information (2)

$$0 \le I_{\mathrm{QC}} \le H$$

$$H = -\sum_y p(y) \ln p(y)$$

No information

Error-free & classical

Any POVM element is identity operator

Any POVM element is projection and commutable with measured state

---

Classical measurement, I_{QC} reduces to the classical mutual information

---

If the measured state is a pure state: $I_{\mathrm{QC}} = 0$

QC-mutual Information (3)

The QC-mutual information gives an upper bound of the accessible classical information x encoded in ρ

F. Buscemi, M. Hayashi, and M.Horodecki, PRL **100**, 210504 (2008).

$$\rho = \sum_x q(x)\rho_x$$: an arbitrary decomposition

ρ_x are not necessarily orthogonal

$$p(x,y) = \mathrm{tr}[M_y^\dagger M_y \rho_x] q(x) \qquad p(y) = \mathrm{tr}[M_y^\dagger M_y \rho] = \sum_x p(x,y)$$

$$I = \sum_{xy} p(x,y) \ln \frac{p(x,y)}{q(x)p(y)}$$

Theorem: $I \le I_{\mathrm{QC}}$

A variant (a "dual") of the Holevo bound