Entanglement formation under random interactions

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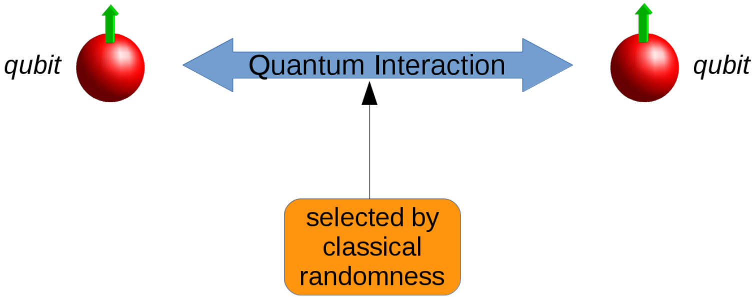
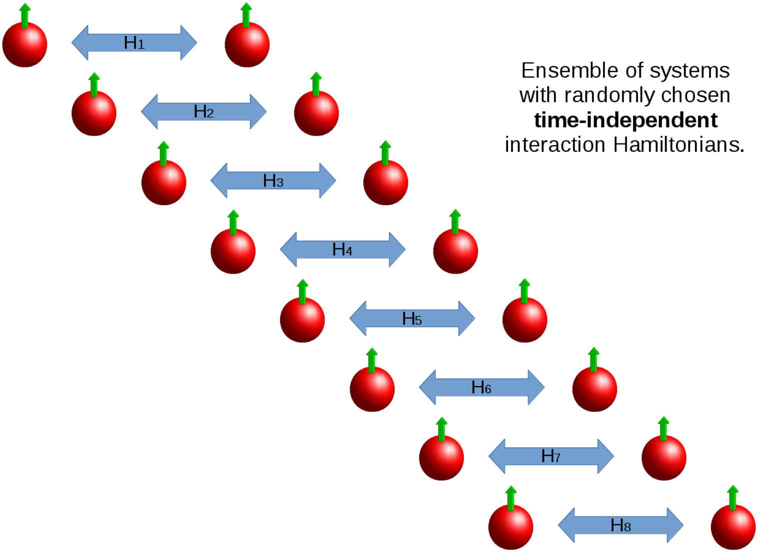
Outline:

Random interactions

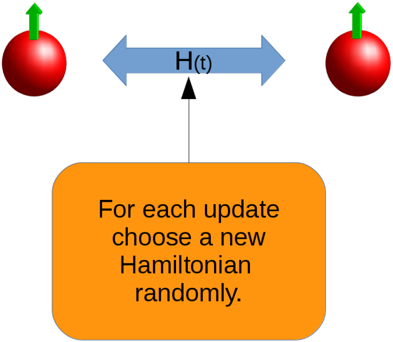


Case (a):



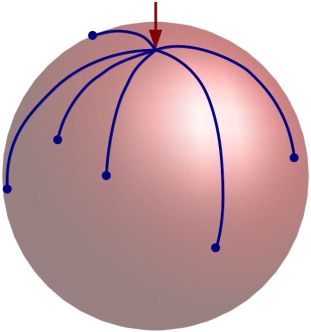
Ensemble of systems with randomly chosen **time-independent** interaction Hamiltonians.

Case (b):

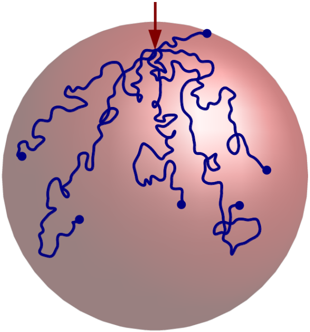


System with a **time-dependent** random Hamiltonian.

The two cases:



(a) quenched



(b) temporal

We draw the SU(4) manifold as if it was a Bloch sphere.

SU(4) representation

- Lie algebra of 15 generators $\lambda_1, \ldots, \lambda_{15}$.
- 15 Euler-like angles $\vec{\alpha} = \{\alpha_1, \ldots, \alpha_{15}\}$
- Group elements of SU(4):

$$
\begin{aligned}
U_{\vec{\alpha}} &= e^{i\lambda_3\alpha_1}\, e^{i\lambda_2\alpha_2}\, e^{i\lambda_3\alpha_3}\, e^{i\lambda_5\alpha_4}\, e^{i\lambda_3\alpha_5}\, e^{i\lambda_{10}\alpha_6}\, e^{i\lambda_3\alpha_7}\, e^{i\lambda_2\alpha_8} \\
&\quad \times e^{i\lambda_3\alpha_9}\, e^{i\lambda_5\alpha_{10}}\, e^{i\lambda_3\alpha_{11}}\, e^{i\lambda_2\alpha_{12}}\, e^{i\lambda_3\alpha_{13}}\, e^{i\lambda_8\alpha_{14}}\, e^{i\lambda_{15}\alpha_{15}}
\end{aligned}
$$

- Well-defined integration ranges for the α_i without overlaps.

T. Tilma, M. Byrd, and E. C. G. Sudarshan, J. Phys. A 2002.

Integration over SU(4)

Average over a probability density on the unit sphere in $\mathbb{R}^3$:

$$\left\langle f \right\rangle_p = \frac{1}{A} \int f\, p \,\mathrm{d}A = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(\theta,\phi) p(\theta,\phi) \underbrace{\sin\theta}_{\mu(\theta,\phi)} \,\mathrm{d}\theta \,\mathrm{d}\phi$$

Average over a probability density on the SU(4) group manifold:

$$\begin{aligned} \left\langle f \right\rangle_p &= \frac{1}{V_{SU(4)}} \int f\, p \,\mathrm{d}V_{SU(4)} \\ &= \frac{1}{V_{SU(4)}} \int \cdots \int f(\vec{\alpha})\, p(\vec{\alpha})\ \mu(\vec{\alpha}) \,\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{15} \end{aligned}$$

$$\mu(\vec{\alpha}) = \sin(2\alpha_2) \sin(\alpha_4) \sin^5(\alpha_6) \sin(2\alpha_8) \sin^3(\alpha_{10}) \sin(2\alpha_{12}) \cos^3(\alpha_4) \cos(\alpha_6) \cos(\alpha_{10})\,.$$

$V_{SU(4)} = \frac{\sqrt{2}\,\pi^9}{3}$

M. S. Marinov, J. Phys. A 14 (1981).

SU(4) representation

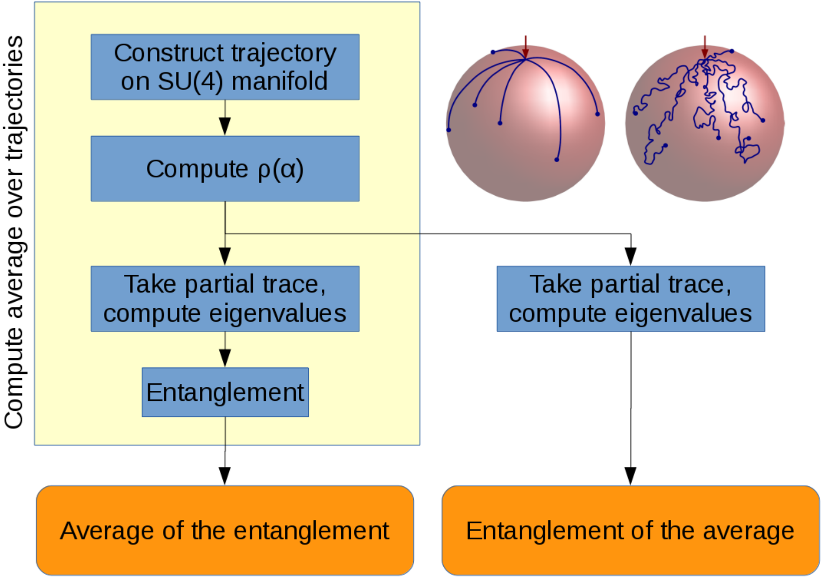
Initial state: $\rho(0) = |11\rangle\langle 11|$

Quantum state: $\rho(\vec{\alpha}) = U_{\vec{\alpha}}\, \rho(0)\, U_{\vec{\alpha}}^{\dagger}$

$$
\begin{aligned}
\rho_{11}(\alpha) &= \cos^2(\alpha_2)\cos^2(\alpha_4)\sin^2(\alpha_6) \\
\rho_{12}(\alpha) &= -\frac{1}{2}e^{2i\alpha_1}\cos^2(\alpha_4)\sin(2\alpha_2)\sin^2(\alpha_6) \\
\rho_{13}(\alpha) &= -\frac{1}{2}e^{i(\alpha_1+\alpha_3)}\cos(\alpha_2)\sin(2\alpha_4)\sin^2(\alpha_6) \\
\rho_{14}(\alpha) &= e^{i(\alpha_1+\alpha_3+\alpha_5)}\cos(\alpha_2)\cos(\alpha_4)\cos(\alpha_6)\sin(\alpha_6) \\
\rho_{22}(\alpha) &= \cos^2(\alpha_4)\sin^2(\alpha_2)\sin^2(\alpha_6) \\
\rho_{23}(\alpha) &= e^{-i(\alpha_1-\alpha_3)}\cos(\alpha_4)\sin(\alpha_2)\sin(\alpha_4)\sin^2(\alpha_6) \\
\rho_{24}(\alpha) &= -e^{-i(\alpha_1-\alpha_3-\alpha_5)}\cos(\alpha_4)\cos(\alpha_6)\sin(\alpha_2)\sin(\alpha_6) \\
\rho_{33}(\alpha) &= \sin^2(\alpha_4)\sin^2(\alpha_6) \\
\rho_{34}(\alpha) &= -e^{i\alpha_5}\cos(\alpha_6)\sin(\alpha_4)\sin(\alpha_6) \\
\rho_{44}(\alpha) &= \cos^2(\alpha_6)
\end{aligned}
$$

$\rho(\vec{\alpha})$ depends only on six angles $\alpha_1, \ldots, \alpha_6$ out of 15.

Strategy:


Compute average over trajectories
Construct trajectory on SU(4) manifold
Compute ρ(α)
Take partial trace, compute eigenvalues
Entanglement
Take partial trace, compute eigenvalues
Average of the entanglement
Entanglement of the average

Quenched random interactions

Choose a fixed random Hamiltonian

$$H = \sum_{i=1}^{4} E_i |\phi_i\rangle\langle\phi_i| \quad \Rightarrow \quad U(t) = e^{-iHt} = \sum_{i=1}^{4} e^{-iE_i t} |\phi_i\rangle\langle\phi_i| \,,$$

from a **Gaussian Unitary Ensemble** (GUE).

Compute:

- *Entanglement of the average:* $H\Big[Tr_1\Big[\langle\rho(t)\rangle_{\text{GUE}}\Big]\Big]$
- *Average of the entanglement:* $\Big\langle H\big[Tr_1[\rho(t)]\big]\Big\rangle_{\text{GUE}}$

In the GUE, the eigenvalue and eigenvector statistics are independent:

$$\Big\langle \ldots \Big\rangle_{\text{GUE}} = \Big\langle \Big\langle \ldots \Big\rangle_{\vec{\alpha}} \Big\rangle_{E} = \Big\langle \Big\langle \ldots \Big\rangle_{E} \Big\rangle_{\vec{\alpha}}$$

(i) Energy eigenvalue statistics:

$$\Big\langle f \Big\rangle_{E} \; = \; \int \mathrm{d}E_1 \cdots \int \mathrm{d}E_4 \, e^{-A\sum_i E_i^2} \prod_{n>m} (E_n - E_m)^2 \;\; f(E_1, \ldots, E_4)$$

(ii) Eigenvector statistics: uniform under SU(4) (Haar measure)

$$\Big\langle f \Big\rangle_{\alpha} \;\; = \;\; \frac{1}{V_{SU(4)}} \int \cdots \int \; f(\vec{\alpha}) \underbrace{p(\vec{\alpha})}_{=1} \; \mu(\vec{\alpha}) \, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{15}$$

(a) Entanglement of the average

Compute the average density matrix

$$\left\langle \rho(t) \right\rangle_{\text{GUE}} = \sum_{j,k=1}^{4} \underbrace{\left\langle e^{-i\left(E_j - E_k\right)t} \right\rangle_E}_{R_{jk}} \underbrace{\left\langle |\phi_j\rangle\langle\phi_j|\, \rho(0)\, |\phi_k\rangle\langle\phi_k| \right\rangle_\alpha}_{\mathbf{T}_{jk}}$$

⬇

$$R_{jk} = f(\tau) + \Big(1 - f(\tau)\Big)\delta_{jk} = \begin{cases} 1 & \text{for } j = k \\ f(\tau) & \text{for } j \neq k \end{cases}$$

$$f(\tau) = \frac{1}{72}\, e^{-\tau^2} \left(-2\tau^{10} + 25\tau^8 - 128\tau^6 + 276\tau^4 - 288\tau^2 + 72\right)$$

where $\tau := t/\sqrt{2A}$ is the scaled time.

Entanglement of the average

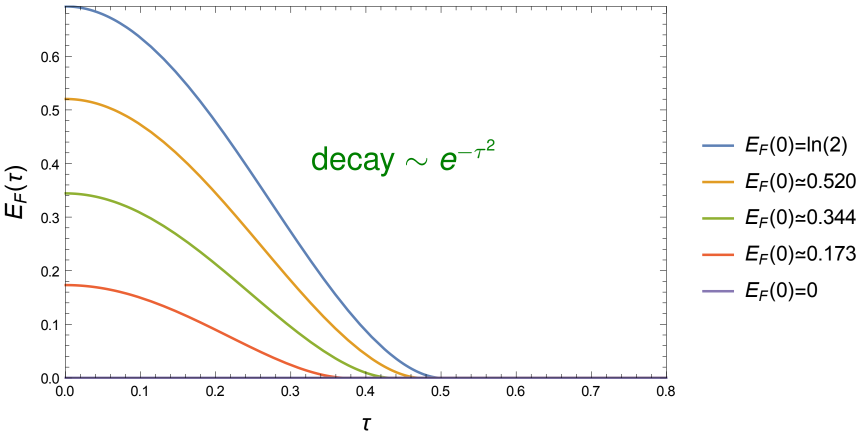
Compute the average density matrix

$$\left\langle \rho(t) \right\rangle_{\text{GUE}} = \sum_{j,k=1}^{4} \underbrace{\left\langle e^{-i(E_j - E_k)t} \right\rangle_E}_{R_{jk}} \underbrace{\left\langle |\phi_j\rangle\langle\phi_j|\, \rho(0)\, |\phi_k\rangle\langle\phi_k| \right\rangle_\alpha}_{\mathbf{T}_{jk}}$$

$$\left\langle \rho(t) \right\rangle_{\text{GUE}} = f(\tau) \underbrace{\sum_{j,k=1}^{4} \mathbf{T}_{jk}}_{=\rho(0)} + \Big(1 - f(\tau)\Big) \underbrace{\sum_{j=1}^{4} \mathbf{T}_{jj}}_{=\frac{1}{5}(\mathbf{1}+\rho(0))}$$

Entanglement of the average

$$\left\langle \rho(t) \right\rangle_{\text{GUE}} = \frac{1 - f(\tau)}{5}\mathbf{1} + \frac{1 + 4f(\tau)}{5}\rho(0)$$



Average of the entanglement

- The von-Neumann entanglement is too difficult to compute.
- Use the so-called **linear entropy** instead:

$$\Big\langle L(t) \Big\rangle_{\text{GUE}} = 1 - \Big\langle \text{Tr}[\rho_1^2(t)] \Big\rangle_{\text{GUE}}$$

$$\Big\langle L(t) \Big\rangle_{\text{GUE}} = 1 - \sum_{\mu,\beta,\gamma,\delta=1}^{2} \Big\langle \langle \mu\beta|\rho(t)|\gamma\beta\rangle \langle \gamma\delta|\rho(t)|\mu\delta\rangle \Big\rangle_{\text{GUE}}$$

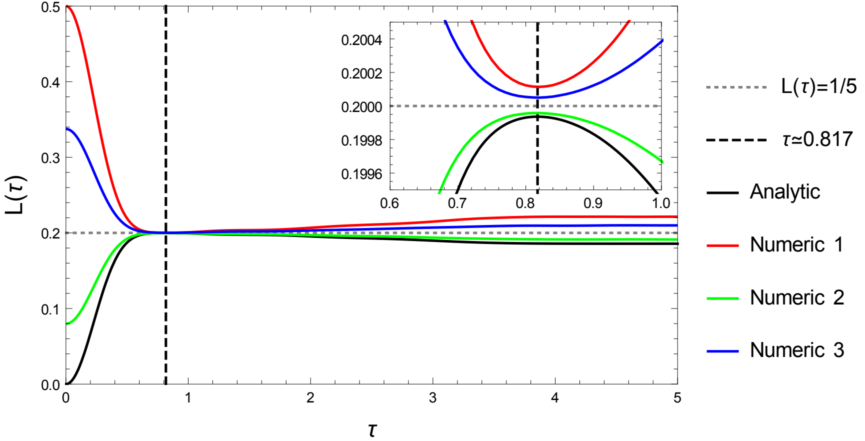
Average of the entanglement

Technically similar:

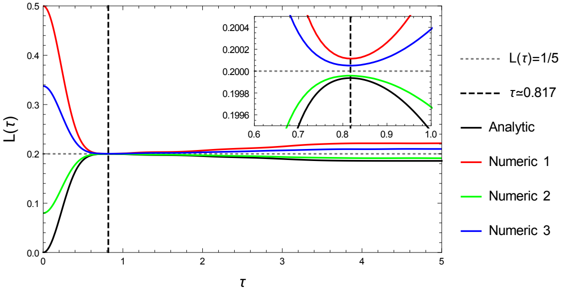
$$\left\langle L(t) \right\rangle_{\mathrm{GUE}} = 1 - \sum_{\mu,\beta,\gamma,\delta=1}^{2} \sum_{j,k,l,m=1}^{4} \underbrace{\left\langle e^{-i(E_j - E_k + E_l - E_m)\tau\sqrt{2A}} \right\rangle_E}_{R_{ijkl}(\tau)} \times \underbrace{\left\langle c_j^{\mu\beta *} \, c_j^{11} \, c_k^{11 *} \, c_k^{\gamma\beta} \, c_l^{\gamma\beta *} \, c_l^{11} \, c_m^{11 *} \, c_m^{\delta\mu} \right\rangle_\alpha}_{\mathbf{T}_{jklm}^{\mu\beta\gamma\delta}} .$$

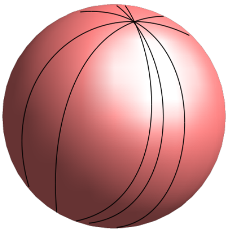
$$\begin{aligned} \left\langle L(\tau) \right\rangle_{\mathrm{GUE}} = & -\frac{1}{840} e^{-\tau^2} \left(-2\tau^{10} + 25\tau^8 - 128\tau^6 + 276\tau^4 - 288\tau^2 + 72 \right) \\ & -\frac{1}{630} e^{-2\tau^2} \left(32\tau^8 - 128\tau^6 + 168\tau^4 - 72\tau^2 + 9 \right) \\ & -\frac{1}{420} e^{-3\tau^2} \left(-54\tau^{10} + 387\tau^8 - 832\tau^6 + 828\tau^4 - 288\tau^2 + 24 \right) \\ & -\frac{1}{315} e^{-4\tau^2} \left(-256\tau^{10} + 800\tau^8 - 1024\tau^6 + 552\tau^4 - 144\tau^2 + 9 \right) + \frac{13}{70} . \end{aligned}$$

Average of the entanglement



Average of the entanglement



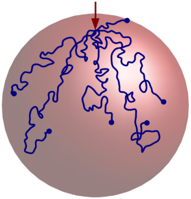


Why does the mean entanglement memorize the initial state?

- Unitary transformations $U(t) = e^{iHt}$ are recurrent.
- Trajectories keep on intersecting at the “north pole”.
- $\Rightarrow$ Non-uniform density on the SU(4) manifold.

Time-dependent random interactions

The generalized Euler angles $\vec{\alpha} = \alpha_1, \ldots, \alpha_{15}$ change randomly in time.



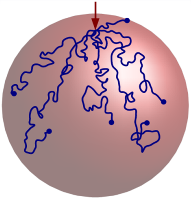
Random walk $\Rightarrow$ Probability distribution $p(\vec{\alpha}, t)$

Diffusion equation on the SU(4) manifold:

$$\frac{\partial p(\vec{\alpha}, t)}{\partial t} - D\Delta_\alpha p(\vec{\alpha}, t) = 0$$

Time-dependent random interactions

Limit $t \to \infty$: Stationary state with a homogeneous distribution (Haar measure).



Known result:

Let $|\psi\rangle$ be a random state in $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^m$.

$$\Rightarrow \left\langle E_{|\psi\rangle} \right\rangle_{GUE} = \left(\left(\sum_{k=n+1}^{mn} \frac{1}{k} \right) - \frac{m-1}{2n} \right)$$

$$\Rightarrow \left\langle E_{|\psi\rangle} \right\rangle_{GUE} = \frac{1}{3} \quad \text{for two qubits.}$$

Page, PRL 71,1291 (1993); Foong and Kanno, PRL 72,1148 (1994); Sen, PRL 77,1 (1996).

Time-dependent random interactions

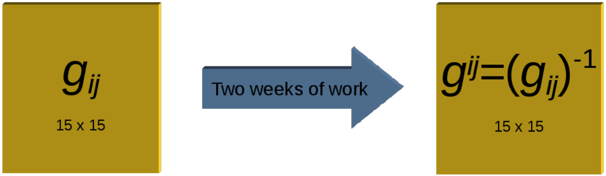
Diffusion equation on curved space: $\frac{\partial p(\vec{\alpha},t)}{\partial t} - D\Delta_\alpha p(\vec{\alpha}, t) = 0$

Laplace-Beltrami operator: $\Delta_\alpha f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \alpha_i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial \alpha_j} f \right)$

SU(4) metric tensor:

$$\begin{aligned} ds^2 &= \sum_{i,j=1}^{15} g_{ij} d\alpha_i d\alpha_j \\ &= \mathrm{Tr}\left[\mathrm{d}U \mathrm{d}U^\dagger\right] = \mathrm{Tr}\left[U(\mathrm{d}\vec{\alpha}) U^\dagger(\mathrm{d}\vec{\alpha})\right] \end{aligned}$$

Time-dependent random interactions



see Mathematica notebook (paper supplement)

Time-dependent random interactions

Consider a function f on the SU(4) manifold:

$$\left\langle f(t) \right\rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\partial^n}{\partial \tilde{t}^n} \left\langle f(\tilde{t}) \right\rangle \Big|_{t=0}$$

Then the diffusion equation implies that

$$\frac{\partial^n}{\partial t^n} \langle f(t) \rangle \Big|_{t=0} = D^n \Delta_\alpha^n f(\vec{\alpha}) \Big|_{\alpha=\alpha_0}$$

$\Rightarrow$ All we need to know is $\Delta_\alpha^n f(\alpha)$

Time-dependent random interactions

First surprise:

$$\rho(\vec{\alpha}) \;=\; U_{\vec{\alpha}}\,\rho(0)\,U_{\vec{\alpha}}^{\dagger}$$

is some kind of eigenvector of the Laplace-Beltrami operator:

$$\triangle_{\alpha}\rho(\vec{\alpha}) \;=\; 2\cdot\mathbf{1} - 8\,\rho(\vec{\alpha})$$

⬇

$\Rightarrow$ All derivatives $\Delta_{\alpha}^{n}\rho(\vec{\alpha})$ are known!

⬇

$$\Big\langle \rho(t) \Big\rangle = \frac{1}{4}\cdot\mathbf{1} + \left(\rho(\vec{\alpha}_0) - \frac{1}{4}\cdot\mathbf{1}\right) e^{-8Dt}$$

Entanglement of the average vanishes.

Time-dependent random interactions

Second surprise:

The linear entropy $L(\vec{\alpha}) \; = \; 1 - \mathrm{Tr}[\rho_1(\vec{\alpha})^2]$

is also an eigenvector of the Laplace-Beltrami operator:

$$\triangle_\alpha L(\vec{\alpha}) \; = \; 4 \cdot \mathbf{1} - 20\, L(\vec{\alpha})$$

⇓

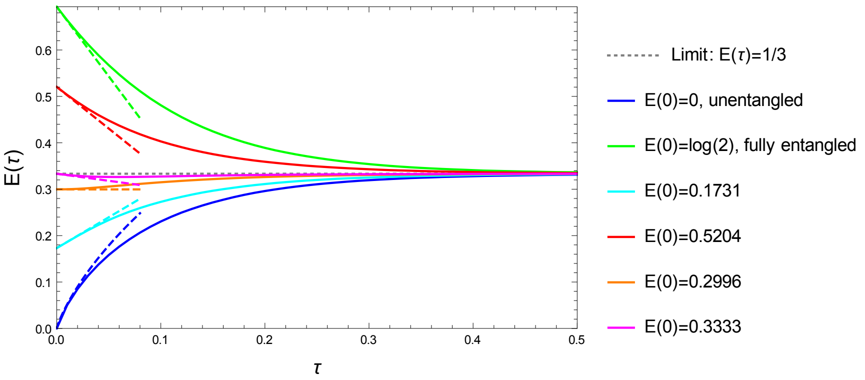
$$\left\langle L(t) \right\rangle = \frac{1}{5} + \left(L(\vec{\alpha}_0) - \frac{1}{5} \right) e^{-20Dt}.$$

Non-entangled initial state: $\left\langle L(t) \right\rangle = \frac{1}{5} - \frac{1}{5} e^{-20Dt}$.

Full entangled initial state: $\left\langle L(t) \right\rangle = \frac{1}{5} + \frac{3}{10} e^{-20Dt}$.

Time-dependent random interactions

Von-Neumann entanglement entropy is too difficult and can only be calculated to first order in τ:



Conclusions

- Random interactions may lead to a buildup of entanglement.
- Two cases:
  (a) Quenched (time-independent) randomness
  (b) Fluctuating (time-dependent) randomness
- Two types of averages
  - Entanglement of the average (more physical)
  - Average of the entanglement (less physical)
- By-product: Metric and Laplacian of SU(4)
- Result: (a) $\sim e^{-\tau^2}$, (b) $\sim e^{-\tau}$

arxiv/1508.01652, to appear in J. Phys. A

Thank you!