General scenario of relaxation in periodically driven quantum systems

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Periodically driven systems

Quantum systems subjected to periodic driving field (e.g. laser)

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

$$H(t) = H(t+T) \qquad \omega = \frac{2\pi}{T}$$

Even if the instantaneous Hamiltonian is simple, periodically driven systems can show rich phenomena.

Periodically driven systems

Quantum systems subjected to periodic driving field (e.g. laser)

Rich phenomena

- dynamical localization — Dunlap and Kenkre, PRB (1986)
- coherent destruction of tunneling — Grossmann, et al. PRL (1991)
- dynamical phase transitions — Prosen and Ilievski, PRL (2011)
- nontrivial topological phase — Lindner, et. al. Nature Phys. (2011)

Floquet engineering:
To realize interesting properties of matter by applying periodic driving

Thermalization in isolated quantum systems

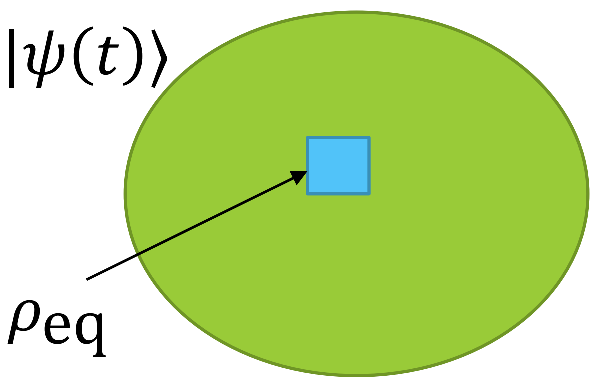
$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

Unitary time evolution

There is no relaxation to equilibrium in the strong sense

$$|\psi(t)\rangle\langle\psi(t)| \neq \rho_{\mathrm{eq}} \equiv \frac{e^{-\beta H}}{Z}$$

However, the system can show thermalization *in the weak sense.*



$$\langle\psi(t)|O|\psi(t)\rangle \approx \mathrm{Tr} O \rho_{\mathrm{eq}}$$

O: any local observable

β varies according to the initial state

Thermalization in isolated periodically driven quantum systems

Floquet theory: $|\psi(t)\rangle = \sum_\alpha C_\alpha e^{-\frac{i}{\hbar}\varepsilon_\alpha t}|u_\alpha(t)\rangle \quad |u_\alpha(t)\rangle = |u_\alpha(t+T)\rangle$

Stroboscopic observation: $|\psi(mT)\rangle = \sum_\alpha C_\alpha e^{-\frac{i}{\hbar}\varepsilon_\alpha mT}|u_\alpha(0)\rangle$

Floquet Hamiltonian

$$\mathcal{T} e^{-\frac{i}{\hbar}\int_0^T H(t)dt} \equiv e^{-iH_F T}$$

$\varepsilon_\alpha \in \left[-\frac{\omega}{2}, \frac{\omega}{2}\right)$ Floquet energy

$$H_F|u_\alpha(0)\rangle = \varepsilon_\alpha|u_\alpha(0)\rangle$$

$|u_\alpha(t)\rangle$ Floquet state

long-time asymptotic state $m \gg 1$

$\langle\psi(mT)|O|\psi(mT)\rangle \approx \mathrm{Tr}O\rho_\infty$ $\quad O$: any local observable

There is no conservation of energy: $\rho_\infty = \frac{\hat{1}}{D}$

D: dimension of the Hilbert space

Eigenstate thermalization hypothesis

Thermalization in the weak sense is possible when
each energy eigenstate looks thermal!

"Eigenstate Thermalization Hypothesis" (ETH) $\langle \phi_\alpha | O | \phi_\alpha \rangle \approx \mathrm{Tr} O \rho_{\mathrm{eq}}$

Each energy eigenstate is indistinguishable from the (micro)canonical ensemble

extension to periodically driven systems

D'Alessio and Rigol, PRX (2014)

Lazarides, Das, and Moessner, PRE (2014)

Ponte, Chandran, Papic, and Abanin, Ann. Phys. (2015)

"Floquet ETH" $\langle u_\alpha(t) | O | u_\alpha(t) \rangle \approx \mathrm{Tr}\, O \frac{\hat{1}}{D}$

Each Floquet eigenstate is indistinguishable from the state of infinite temperature (i.e. completely random state)

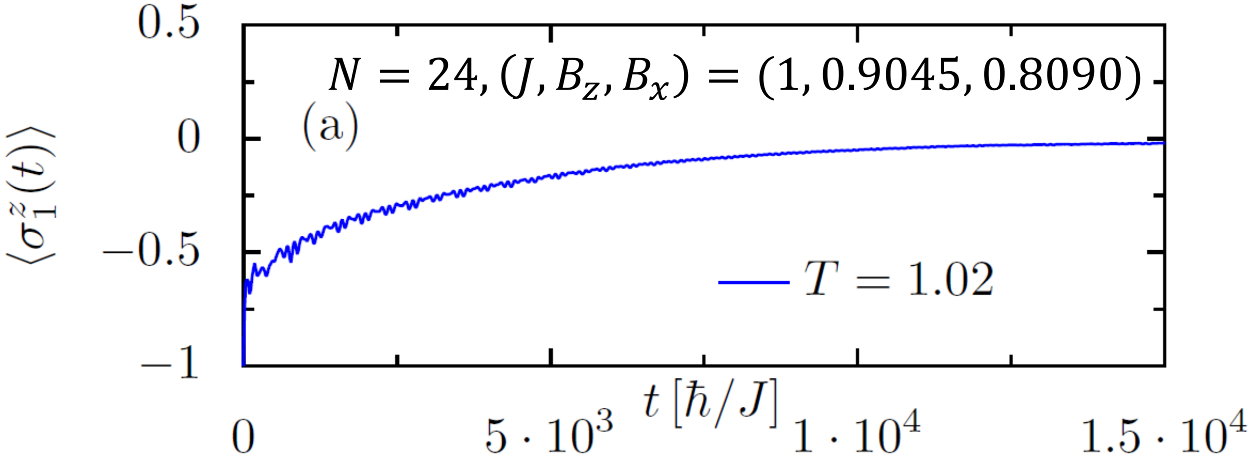
Heating up to infinite temperature

Kim, Ikeda, Huse, PRE (2014)

$$H(t) = \begin{cases} H_z \text{ for } 0 \le t \le \frac{T}{2} \\ H_x \text{ for } \frac{T}{2} \le t \le T \end{cases}$$

$$H_z = \sum_{i=1}^{N} [-J\sigma_i^z \sigma_{i+1}^z + B_z \sigma_i^z]$$

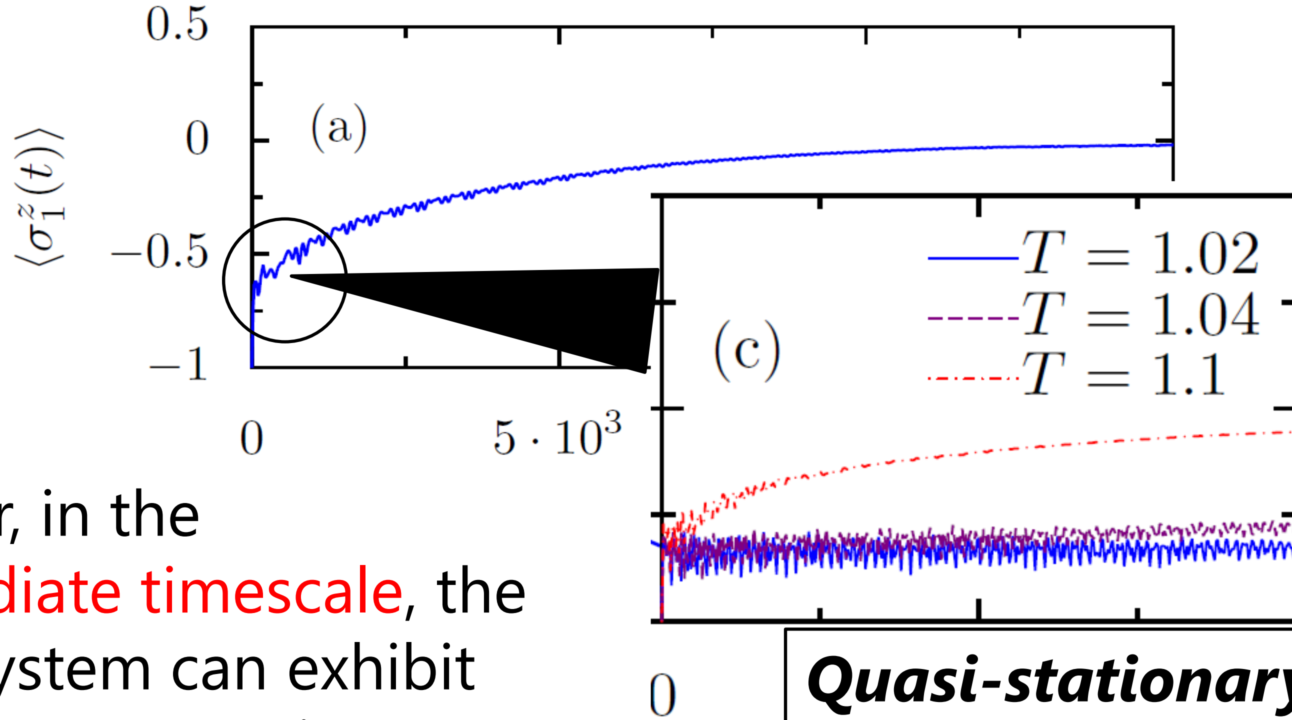
$$H_x = B_x \sum_{i=1}^{N} \sigma_i^x$$



Mori, Kuwahara, Saito, arXiv:1509.03968v2

Transient dynamics

The state of infinite temperature is **featureless**
No interesting phenomenon…



However, in the intermediate timescale, the driven system can exhibit interesting properties (→ Floquet engineering)

Quasi-stationary state
Timescale of intermediate region becomes longer as the frequency increases.

Motivation of the study

- What we observe in experiment is transient dynamics. (NOT the long-time limit!)
- It is important to understand how the intermediate state is described.
- Only from the numerical simulations, We cannot get a decisive conclusion on the long-time behavior of a *macroscopic* driven system.

We want to understand the behavior of driven quantum systems in the intermediate timescale in a mathematically rigorous way!

Setup

System on an arbitrary d-dimensional lattice

$i = 1,2,\dots,N$: index of site

Up to k-site interactions
("k-local Hamiltonian")

$$H(t) = \sum_{X:|X|\le k} h_X(t) \qquad X = \{i_1, i_2, \dots, i_{|X|}\}$$

Bounded single-site energy
("g-extensive")

$$\sum_{X\ni i} \|h_X(t)\| \le g \text{ for } \forall\, i$$

Example: the most general spin-1/2 Hamiltonian with $k = 2$

$$H(t) = \sum_{i=1}^{N} \vec{B}_i(t)\cdot\vec{\sigma}_i + \sum_{i<j}^{N} \sum_{\alpha,\gamma=x,y,z} J_{ij}^{\alpha,\gamma}(t)\sigma_i^\alpha \sigma_j^\gamma$$

Remark: We don't assume the range of interactions

$J_{ij}^{\alpha,\gamma} = \frac{\delta_{\alpha,\gamma}}{N}$: all-to-all Heisenberg interactions with a bounded energy per spin

Undriven part and driving part

$$H(t) = H_0 + V(t)$$

<u>undriven part</u>

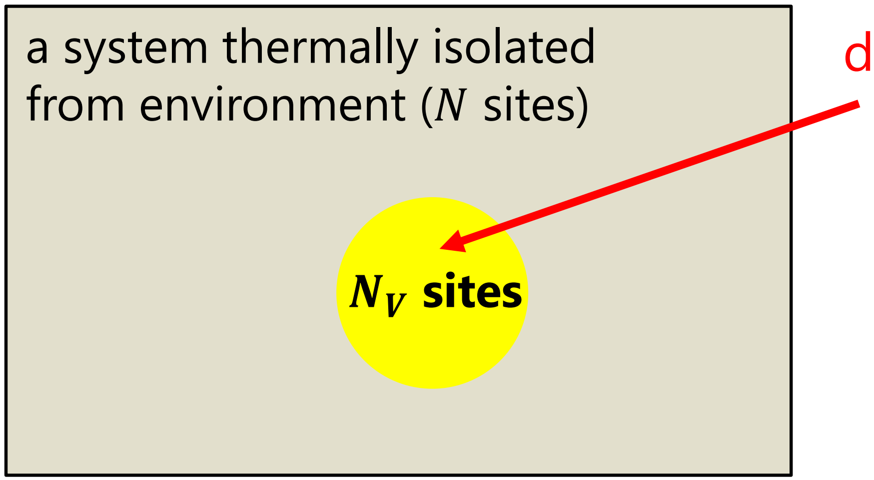
$$H_0 \equiv \frac{1}{T}\int_0^T H(t)dt$$

<u>driving part</u>

$V(t)$: acting on N_V sites

$N_V \ll N$
local driving

$N_V = \mathcal{O}(N)$
global driving



Method: truncation of Magnus expansion

We focus on the **high-frequency** regime

Magnus expansion: formal expansion of the Floquet Hamiltonian in the power series of the period T

$$H_F = \sum_{m=0}^{\infty} T^m \Omega_m$$

$$\Omega_0 = \frac{1}{T}\int_0^T H(t)dt = H_0$$

$$\Omega_1 = \frac{1}{2iT^2}\int_0^T dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]$$

Truncation of Magnus expansion

$$H_F^{(n)} = \sum_{m=0}^{n} T^m \Omega_m$$

BUT it is known that the Magnus expansion is generally **divergent**. The use of the Magnus expansion is not justified.

When does it work?

General form of Magnus expansion

$$H_F = \sum_{m=0}^{\infty} T^m \Omega_m$$

$$\Omega_n = \sum_{\sigma} \frac{(-1)^{n-\theta[\sigma]}\theta[\sigma]!\,(n-\theta[\sigma])!}{i^n (n+1)^2 n!\, T^{n+1}} \int_0^T dt_{n+1} \dots \int_0^{t_2} dt_1$$

$$\times \left[H\left(t_{\sigma(n+1)}\right), \left[H\left(t_{\sigma(n)}\right), \dots, \left[H\left(t_{\sigma(2)}\right), H\left(t_{\sigma(1)}\right)\right] \dots\right]\right]$$

Inequalities for local operators

$A: k_A-$local and g_A-extensive, $B: k_B-$local

$$\|[A,B]\| \leq 2 g_A k_B \|B\|$$

Compare with the familiar inequality: $\|[A,B]\| \leq 2\|A\| \cdot \|B\|$

In many-body problem, $\|A\| \propto N$, but $g_A k_B = \mathcal{O}(1)$!

$$\|[A_n,[A_{n-1},\dots,[A_1,B]\dots]]\| \leq 2\left(\prod_{i=1}^{n} g_{A_i} K_i\right)\|B\|$$

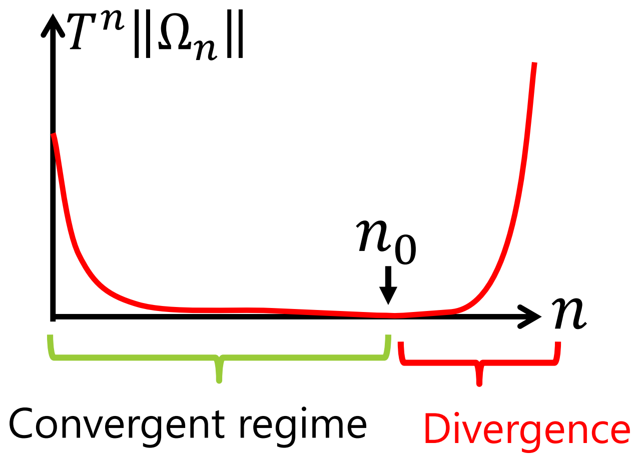
$$K_i = k_B + \sum_{j=1}^{i-1} k_{A_i}$$

Application of the inequality to the Magnus expansion

$$H_F^{(n)} = \sum_{m=1}^{n} \Omega_m T^m \quad \Omega_n = \sum_{\sigma} \frac{(-1)^{n-\theta[\sigma]}\theta[\sigma]!\,(n-\theta[\sigma])!}{i^n (n+1)^2 n!\,T^{n+1}} \int_0^T dt_{n+1} \ldots \int_0^{t_2} dt\_1$$

$$\times \boxed{\left[H\left(t_{\sigma(n+1)}\right), \left[H\left(t_{\sigma(n)}\right), \ldots, \left[H\left(t_{\sigma(2)}\right), H\left(t_{\sigma(1)}\right)\right] \ldots\right]\right]}$$

$$\|\Omega_n\| \leq 2gN_V \frac{(2gk)^n n!}{(n+1)^2}$$

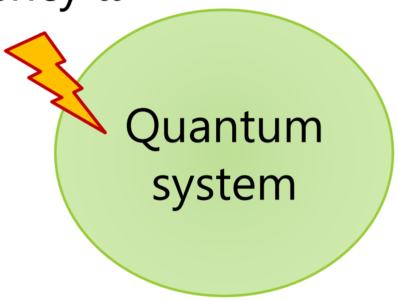


$$n_0 \sim \frac{1}{2gkT} \sim \frac{\hbar\omega}{gk}$$

Intuitive picture behind analysis

Energy absorption is taken into account in the Magnus expansion when $(n+1)k \geq N^*$, namely, $n \gtrsim \hbar\omega/gk \sim n_0$.

Periodic driving frequency ω



- Energy absorption and emission are quantized into $\hbar\omega$ (*energy quantum*)
- Single-spin energy is bounded by g
- In order to absorb the single energy quantum, $N^* \sim \frac{\hbar\omega}{g} \gg 1$ spins must flip cooperatively.

General form of the Magnus coefficient $H_F^{(n)} = \sum_{m=0}^{n} T^m \Omega_m$

$$\Omega_n = \sum_{\sigma} \frac{(-1)^{n-\theta[\sigma]}\theta[\sigma]!\,(n-\theta[\sigma])!}{i^n (n+1)^2 n!\, T^{n+1}} \int_0^T dt_{n+1} \dots \int_0^{t_2} dt_1 \times \left[H(t_{\sigma(n+1)}), \left[H(t_{\sigma(n)}), \dots, \left[H(t_{\sigma(2)}), H(t_{\sigma(1)})\right] \dots\right]\right.$$

$(n+1)$k-spin simaltaneous flips

Intuitive picture behind analysis

Energy absorption is taken into account in the Magnus expansion when $(n+1)k \geq N^*$, namely, $n \gtrsim \hbar\omega/gk \sim n_0$.

The divergence of the Magnus expansion has a clear physical meaning: it stems from the heating effect!

We can eliminate the effect of heating most efficiently by truncating the Magnus expansion at n_0-th order.

In order to analyze the timescale of heating, this n_0 plays an important role.

Mori, Kuwahara, Saito, arXiv:1509.03968

Rigorous Theorem (Hereafter we put $\hbar = 1$)

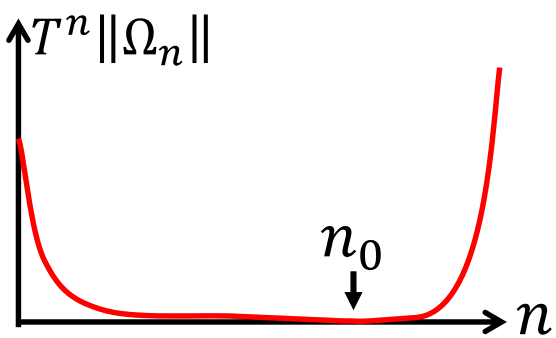
The truncated Floquet Hamiltonian at n_0-th order is an almost-conserved quantity in exponentially long timescale.

$$\frac{1}{N}\left\| H_F^{(n_0)}(t) - H_F^{(n_0)}(0) \right\| \le \left(\frac{N_V}{N}\right) 16 g^2 k 2^{-n_0} t \sim e^{-\mathcal{O}(\omega/g)} t$$

Heisenberg picture

$$n_0 \approx \frac{1}{8gkT} = \mathcal{O}(\omega/g)$$

The n-th order truncated Floquet Hamiltonian is very close to the n_0-th order one as long as $n < n_0$:



$$\frac{1}{N}\left\| H_F^{(n)} - H_F^{(n_0)} \right\| = \mathcal{O}(T^{n+1})$$

$$\Rightarrow \quad \frac{1}{N}\left\| H_F^{(n)}(t) - H_F^{(n)}(0) \right\| \le \left(\frac{N_V}{N}\right) \left[16 g^2 k 2^{-n_0} t + \mathcal{O}(T^{n+1})\right]$$

Exponentially slow heating

$n = 0 \qquad H_F^{(0)} = H_0 \qquad\qquad H_0 \equiv \frac{1}{T}\int_0^T H(t)dt$

$$\frac{1}{N}\|H_0(t) - H_0(0)\| \leq \left(\frac{N_V}{N}\right)[16g^2k2^{-n_0}t + \mathcal{O}(T)]$$

$$\Rightarrow \quad \langle\psi(t)|H_0|\psi(t)\rangle \approx \langle\psi(0)|H_0|\psi(0)\rangle$$

up to an exponentially long timescale $\tau_h = e^{\mathcal{O}(\omega/g)}$

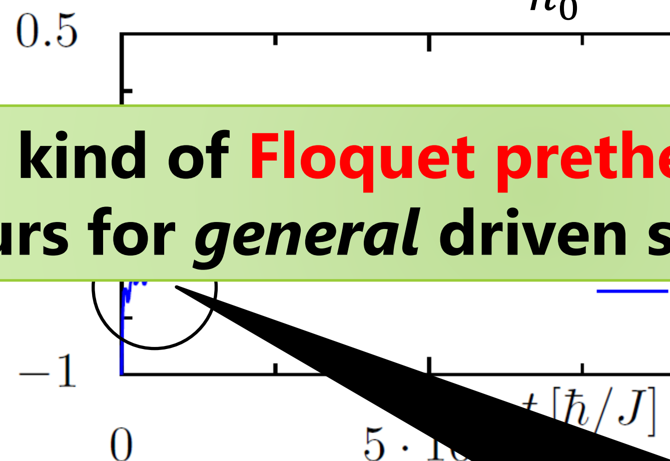
Heating can occur, but is extremely slow in high-frequency regime

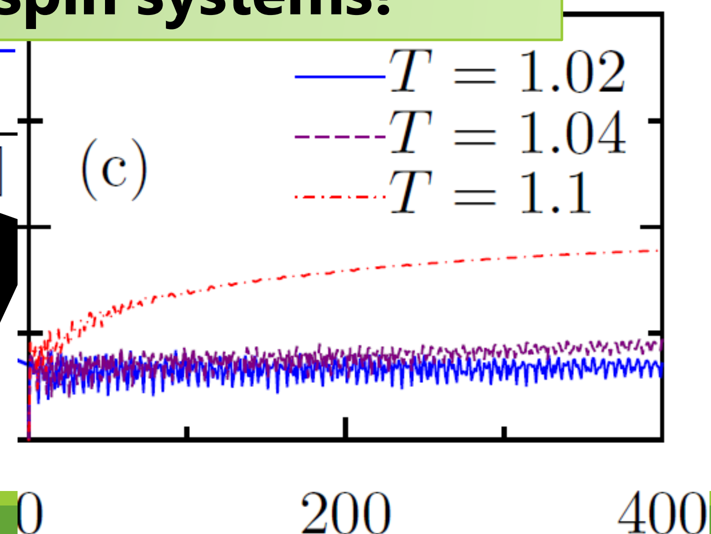
Quasi-stationary states

Because the energy is almost conserved up to τ_h, the system first reaches a ***quasi-stationary state*** with a **finite** temperature!

$$\rho_{QSS} = \frac{e^{-\beta H_F^{(n_0)}}}{Z_{n_0}} \approx \frac{e^{-\beta H_0}}{Z_0}$$

This kind of Floquet prethermalization occurs for *general* driven spin systems!

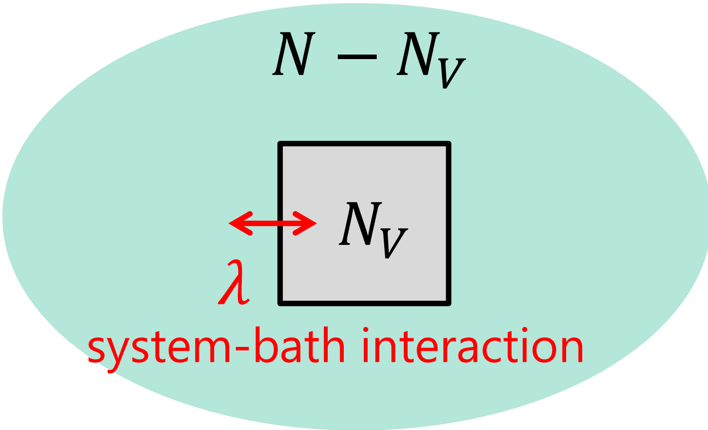




What is numerically and experimentally observed is this quasi-stationary state!

Shirai, Mori, Miyashita, Phys. Rev. E (2015)

Remark: open quantum systems



$N \to \infty$ with N_V fixed

open quantum system

system of interest

$\rho_S(t)$ reduced density matrix

thermal bath $\rho_B = \dfrac{e^{-\beta H_B}}{\mathrm{Tr}(e^{-\beta H_B})}$

$$\rho_S(t) \to \frac{e^{-\beta H_F^{(n)}}}{Z} \qquad t < e^{\mathcal{O}(\omega/g)}$$

$$\rho_{SS} = \lim_{t\to\infty} \lim_{N\to\infty} \rho_S(t)$$

In the van Hove limit $\lambda \to 0, t \to \infty$ with $\lambda^2 t = \tau$ held fixed

→ Floquet Lindblad master equation

$$\rho_{SS} \neq \frac{e^{-\beta H_F^{(n)}}}{Z}$$

heating rate dissipation

$$\exp[-\mathcal{O}(\omega)] \gg \lambda^2$$

(in the van Hove limit)

Summary

- Periodically driven quantum systems from a general point of view.
- Heating is extremely slow in high-frequency regime.
- Quasi-stationary states appear, which are characterized by the truncated Floquet Hamiltonian.
  → the use of the truncation of Magnus expansion is justified!

Mori, Kuwahara, Saito, arXiv:1509.03968v2

Kuwahara, Mori, Saito, arXiv:1508.05797v2