Smoothings of complex normal surface singularities

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Suppose $f(x,y,z)$ is a complex polynomial which has a zero and an isolated critical point at the origin. Then the two-dimensional complex hypersurface $V = \{f = 0\}$ has a singular point at 0. Topologically, it is the cone over its neighborhood boundary Σ, the compact 3-manifold which is the intersection of V with a small sphere in $\mathbb{C}^3$. Σ may be understood explicitly by resolution of singularities, and its intrinsic topology gives much (but not enough!) information about the geometry of the singular point. "Smoothing" of the singularity takes place by considering $\{f = \delta\}$, and the change in topology as δ goes to 0 is described by the "Milnor fibre" M (the intersection of $\{f = \delta\}$ with a small ball); it is a compact 4-manifold with boundary Σ. Milnor's classical work proves that M is simply-connected, and has the homotopy type of a bouquet of μ 2-spheres, where the Milnor number μ is the colength of the Jacobian ideal of f.

Our goal is to consider the more general and richer situation of a "smoothing of a (germ of a) normal surface singularity $(V, 0)$". The local topology of V is again determined by its neighborhood boundary Σ, and a smoothing (if one exists) gives a 4-manifold M with boundary Σ. Already for cyclic quotient singularities, in which case Σ is a lens space, the study of smoothings is quite involved, and particularly interesting phenomena may appear. For example, in some cases M is a "rational homology disk:" its rational homology is trivial. This situation is already interesting from the symplectic point of view, and played the key role in the work of Fintushel-Stern on "rational blow-down" in symplectic geometry. Recent work has yielded a complete list of all surface singularities possessing rational homology disk smoothings.

The first lecture will discuss the basic definitions of normal surface singularities, introducing special classes (ICIS, rational, Gorenstein). Deformations and smoothings will be treated in the second lecture, while the third will discuss the recent work on rational homology disk smoothings. The last talk will deal with a number of conjectures about Milnor and Tjurina numbers.