THE POSITIVE JACOBIAN CONSTRAINT IN ELASTICITY THEORY AND ORIENTATION-PRESERVING YOUNG MEASURES

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ABSTRACT. In elasticity theory, one naturally requires that the Jacobian determinant of the deformation is positive or even a-priori prescribed (for example in the case of incompressibility). However, such strongly non-linear and non-convex constraints are difficult to deal with in mathematical models. In this minicourse, I will present various recent results on how this constraint can be manipulated in subcritical Sobolev spaces, where the integrability exponent is less than the dimension. In particular, I will give a characterization theorem for Young measures under this side constraint, which are widely used in the Calculus of Variations to model limits of nonlinear functions of weakly converging "generating" sequences. This is in the spirit of the celebrated Kinderlehrer–Pedregal Theorem and based on convex integration and "geometry" in matrix space. Finally, applications to the minimization of integral functionals, the theory of semiconvex hulls, incompressible extensions, and approximation of weakly orientation-preserving maps by strictly orientation-preserving ones in Sobolev spaces are given. This course is based on joint work with K. Koumatos (L'Aquila) and E. Wiedemann (Bonn).

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Consider a standard minimization problem for an integral functional of the form

$$\mathscr{F}[u] := \int_\Omega f(x, \nabla u(x)) \,\mathrm{d}x \to \min \qquad \text{over all } u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d).$$

Assume for the moment that $f\colon \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is continuous, say, and satisfies

$$|f(x,A)| \le C(1+|A|^p),$$

where here and in all of the following, $1 < p < \infty$ and $C > 0$ is a generic constant (which may change from line to line). The gradient of the vector-valued function u is

$$\nabla u = \begin{pmatrix} \partial_1 u^1 & \partial_2 u^1 & \cdots & \partial_d u^1 \\ \partial_1 u^2 & \partial_2 u^2 & \cdots & \partial_d u^2 \\ \vdots & \vdots & & \vdots \\ \partial_1 u^d & \partial_2 u^d & \cdots & \partial_d u^d \end{pmatrix}.$$

We interpret u in a mechanical context as a **deformation**. Additionally, for example in elasticity theory, one often also has a side constraint of the form

$$\det \nabla u > 0 \quad \text{a.e. in } \Omega \tag{1.1}$$

on all admissible functions u. In mechanics, one can consider the Jacobian (determinant) $\det \nabla u$ as the local "stretching factor" of a deformation, as is also expressed in the multi-dimensional transformation formula for integrals. Then, the above constraint means that no volume of material is compressed into a point or even turned "inside out", which would make the Jacobian zero or negative, respectively. In elasticity theory this constraint is of utmost importance since it entails that the deformation does not have a reflection component (which should be excluded from physical reasoning) and also is a local version of the fact that matter should not inter-penetrate (of course, this really is a global constraint, but positivity of the Jacobian is at least necessary).

We can also re-write (1.1) in the form of a pointwise **differential inclusion**:

$$\nabla u \in S := \bigl\{ A \in \mathbb{R}^{d \times d} \ \mathbf{:}\ \det A > 0 \bigr\} \quad \text{a.e. in } \Omega.$$

The reason for the inherent difficulty of this constraint is the following: While it is possible to use standard methods to deal with pointwise constraints $\nabla u_j \in K$ a.e. with K *convex*, these techniques fail for the above *non-convex* constraint. Indeed, this non-convexity is easy to see:

Example 1. For

$$A := \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} -1 & 1/2 \\ 1/2 & -1 \end{pmatrix}, \qquad \frac{1}{2}A + \frac{1}{2}B = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},$$

we have $\det A = \det B = 3/4$, but $\det(A/2 + B/2) = -1/4$. Hence, $A, B \in S$, but $A/2 + B/2 \notin S$.

1.1. Approximate and exact solutions.

Assume that $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ satisfies the constraint (1.1) "approximately", that is, there exists a sequence

$$(u_j) \subset \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d) \qquad \text{(that is, } (u_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \text{ with } u_j|_{\partial\Omega} = u|_{\partial\Omega})$$

with

$$\int_{\{\det \nabla u_j < 0\}} \underbrace{|\det \nabla u_j(x)|^{p/d}}_{\in \mathrm{L}^1} \,\mathrm{d}x \ \to \ 0.$$

Notice that if u satisfies (1.1) exactly, then we may choose $u_j = u$. Conversely, one can ask:

Question Q1. Given a sequence (u_j) of approximate solutions as above, can we find new sequence $(v_j) \subset \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ that satisfied (1.1) *exactly*, that is,

$$\det \nabla v_j > 0 \quad \text{a.e.} \qquad \text{and} \qquad \|u_j - v_j\|_{\mathrm{W}^{1,p}} \to 0 \qquad ?$$

Let us first consider how this question is sensitive to the value of the integrability exponent p: Say we are in the special situation $u_j = u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ for all j and

$$\det \nabla u = 0 \quad \text{a.e.}$$

This clearly satisfies the constraint $\int_{\{\det \nabla u < 0\}} \bigl|\det \nabla u(x)\bigr|^{p/d} \,\mathrm{d}x = 0$. So Question Q1 here reads: Can one find $v_j \to u$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ such that $v_j|_{\partial\Omega} = u|_{\partial\Omega}$ and

$$\det \nabla v_j > 0 \quad \text{a.e. ?}$$

We can convince ourselves quickly that this is unsolvable if $p \geq d$: In this case, $\det \nabla v_j \in \mathrm{L}^1(\Omega)$ and by Stokes' Theorem (this can also be computed in a more elementary way, see Theorem 2.3 in [Mül99] for another proof):

$$\begin{aligned}\int_\Omega \det \nabla v_j \,\mathrm{d}x &= \int_\Omega dv_j^1 \wedge \cdots \wedge dv_j^d = \int_{\partial\Omega} v_j^1 \wedge dv_j^2 \wedge \cdots \wedge dv_j^d \\ &= \int_{\partial\Omega} u^1 \wedge du^2 \wedge \cdots \wedge du^d = \cdots = \int_\Omega \det \nabla u \,\mathrm{d}x = 0.\end{aligned}$$

Thus, the value of $\int_\Omega \det \nabla v_j \,\mathrm{d}x$ only depends on the (boundary) values of $\det \nabla v_j$ and they are fixed, hence the last equality to zero. Hence, $\det \nabla v_j > 0$ a.e. is impossible.

In conclusion, for $p \geq d$ the Jacobian constraint is (strongly) **rigid** in the sense that one cannot improve approximate solutions to exact solutions of (1.1).

However, we will see below that this questions can be positively solved if $p < d$.

1.2. **Young measures.** We next briefly introduce Young measures, a now widely-used and very useful tool in the Calculus of Variations, see [Ped97] for an introduction and more information.

Consider the following situation:

(i) $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is continuous,
(ii) $|f(x,A)| \leq C(1+|A|^p)$ for a $1 < p < \infty$ and a constant $C > 0$,
(iii) $(v_j) \subset \mathrm{L}^p(\Omega;\mathbb{R}^N)$ is norm-bounded (hence weakly compact).

For many applications it is important to answer:

$$\textbf{What is} \quad \operatorname*{w\text{-}lim}_{j\to\infty} f\big(x, v_j(x)\big) \quad \textbf{?}$$

One can see quickly that even if (v_j) converges weakly, the answer is not always $f(\bullet, \text{w-lim}\, v_j)$, so non-linear operations and weak limits do not in general commute (easy examples show this: try $f(x,A) = |A|^2$). On the other hand, if v_j converges strongly to v in L^p, one can show that $f(x,v_j(x))$ converges strongly to $f(x,v(x))$ (use Pratt's extension of Lebesgue's dominated convergence theorem or Vitali's Convergence Theorem or argue "by hand").

A slightly more detailed analysis reveals that there are two reasons for this phenomenon:

- **Oscillations**, e.g. $v_j(x) = \sin(jx)$.
- **Concentrations**, e.g. $v_j(x) = j^{1/p}\mathbb{1}_{(0,1/j)}$.

There are no other obstructions! Indeed, Vitali's Convergence Theorem says that non-oscillating (pointwise or in measure convergent) and non-concentrating (equiintegrable) sequences converge in norm, whereby (in principle) nonlinear expressions of our sequence converge.

If we exclude concentrations and assume that (v_j) is p-equiintegrable, L. C. Young proved in the 30s and 40s the following theorem:

Theorem 2 (Fundamental Theorem of Young measure theory). *Let (v_j) be as above, and additionally assumed to be p-equiintegrable. Then, up to a subsequence, there exists a family $(\nu_x)_{x\in\Omega}$ of probability measures ν_x on $\mathbb{R}^N$ (with some measurability properties making the following expressions well-defined) such that*

$$\int_\Omega f(x,v_j(x))\,\mathrm{d}x \quad \to \quad \int_\Omega \int_{\mathbb{R}^N} f(x,A)\,\mathrm{d}\nu_x(A)\,\mathrm{d}x$$

for all (!) *continuous $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ with p-growth ($|f(x,A)| \leq C(1+|A|^p)$).*

In this case we say that the (sub)sequence (v_j) **generates** the Young measure $\nu = (\nu_x)$. If ν_x is the same measure for almost every $x \in \Omega$, then $\nu = \nu_x$ is called **homogeneous**.

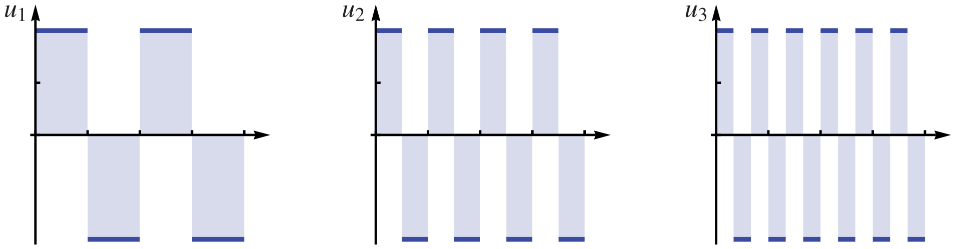

FIGURE 1. An oscillating sequence.

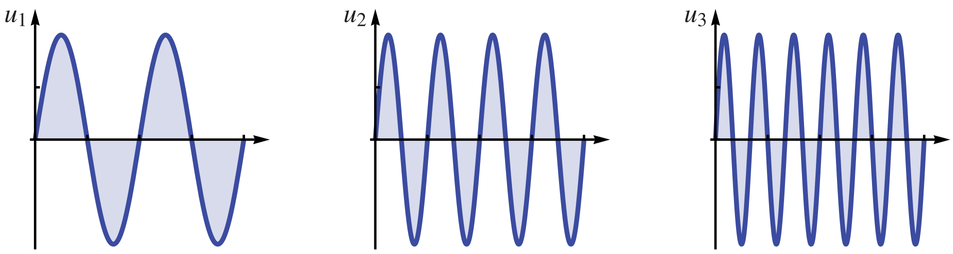

FIGURE 2. Another oscillating sequence.

Example 3. In $\Omega := (0,1)$ define $u := \mathbb{1}_{(0,1/2)} - \mathbb{1}_{(1/2,1)}$ and extend this function periodically to all of $\mathbb{R}$. Then, the functions $u_j(x) := u(jx)$ for $j \in \mathbb{N}$ (see Figure 1) generate the homogeneous Young measure ν with

$$\nu_x = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1} \qquad \text{for a.e. } x \in (0,1).$$

Indeed, for $\varphi \otimes h \in \mathrm{C}_0((0,1)\times\mathbb{R})$ we have that φ is uniformly continuous, say $|\varphi(x) - \varphi(y)| \le \omega(|x-y|)$ with a modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ (that is, ω is continuous, increasing, and $\omega(0) = 0$). Then,

$$\begin{aligned}\lim_{j\to\infty} \int_0^1 \varphi(x)h(u_j(x))\,\mathrm{d}x &= \lim_{j\to\infty} \sum_{k=0}^{j-1} \left(\int_{k/j}^{(k+1)/j} \varphi(k/j)h(u_j(x))\,\mathrm{d}x + \frac{\omega(1/j)\|h\|_\infty}{j} \right) \\ &= \lim_{j\to\infty} \sum_{k=0}^{j-1} \frac{1}{j}\varphi(k/j) \int_0^1 h(u(y))\,\mathrm{d}y \\ &= \int_0^1 \varphi(x)\,\mathrm{d}x \cdot \left(\frac{1}{2}h(-1) + \frac{1}{2}h(+1) \right)\end{aligned}$$

since the Riemann sums converge to the integral of φ. The last line implies the assertion.

Example 4. Take $\Omega := (0,1)$ again and let $u_j(x) = \sin(2\pi jx)$ for $j \in \mathbb{N}$ (see Figure 2). The sequence (u_j) generates the homogeneous Young measure ν with

$$\nu_x = \frac{1}{\pi\sqrt{1-y^2}} \mathscr{L}^1_y \llcorner (-1,1) \qquad \text{for a.e. } x \in (0,1),$$

as should be plausible from the oscillating sequence (there is less mass close to the horizontal axis than farther away).

Example 5. Take a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, let $A, B \in \mathbb{R}^{2\times 2}$ with $B - A = a \otimes n$ (this is equivalent to $\operatorname{rank}(A-B) \leq 1$), where $a, n \in \mathbb{R}^2$, and for $\theta \in (0,1)$ define

$$u(x) := Ax + \left(\int_0^{x\cdot n} \chi(t)\,\mathrm{d}t \right) a, \qquad x \in \mathbb{R}^2,$$

where $\chi := \mathbb{1}_{\bigcup_{z\in\mathbb{Z}}[z,z+1-\theta)}$. If we let $u_j(x) := u(jx)/j$, $x \in \Omega$, then (∇u_j) generates the homogeneous Young measure ν with

$$\nu_x = \theta\delta_A + (1-\theta)\delta_B \qquad \text{for a.e. } x \in \Omega.$$

This example can also be extended to include multiple scales, cf. [Mül99].

For several purposes (e.g. relaxation theorems), one is interested in the following:

Can one characterize the class of Young measures that can be generated by a sequence of ***gradients*** *of* $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$*-functions?*

This is indeed possible and was first achieved by D. Kinderlehrer and P. Pedregal [KP91,KP94]:

Theorem 6 (Kinderlehrer–Pedregal). *A Young measure $\nu = (\nu_x)$ is a* ***gradient p-Young measure****, that is there exists a sequence $(u_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ such that (∇u_j) generates $\nu = (\nu_x)$, if and only if the following conditions hold:*

(KP1) $\displaystyle\int_\Omega \int |A|^p \,\mathrm{d}\nu_x(A)\,\mathrm{d}x < \infty.$

(KP2) *The barycenter $\int A\,\mathrm{d}\nu_x(A)$ is a gradient, i.e. there exists $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ with*

$$\int A\,\mathrm{d}\nu_x(A) = \nabla u(x) \qquad \textit{for a.e. } x \in \Omega.$$

(KP3) *For every quasiconvex function $h\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ with $|h(A)| \leq C(1+|A|^p)$, the* ***Jensen-type inequality***

$$h(\nabla u(x)) \leq \int h(A)\,\mathrm{d}\nu_x(A) \qquad \textit{holds for a.e. } x \in \Omega.$$

Here, a locally bounded Borel-measurable function $h\colon \mathbb{R}^{d\times d} \to \mathbb{R}$ is called **quasiconvex** if

$$h(M) \leq \fint_{B(0,1)} h(M + \nabla\psi(z))\,\mathrm{d}z$$

for all $M \in \mathbb{R}^{d\times d}$ and all $\psi \in \mathrm{C}_c^\infty(B(0,1);\mathbb{R}^d)$ (compactly supported). It can be shown easily using the classical Jensen-inequality that all convex functions are also quasiconvex.

For our purposes, it is *not* necessary to understand (KP1)–(KP3) in detail, it is only important to realize that such conditions exist.

The Kinderlehrer–Pedregal Theorem has a variety of applications, for example for relaxation theorems: if no minimizer for original problem exists, it sometimes makes sense to extend it to a minimization problem on Young measures. It is also very useful for theoretical investigations into the properties of quasiconvexity, which is fundamental in the modern theory of minimization problems over vector-valued functions in the Calculus of Variations.

A proof of this theorem can be found in [Ped97]. We only remark that the necessity of (KP1)–(KP3) is essentially equivalent to a lower semicontinuity theorem in $\mathrm{W}^{1,p}$, for the sufficiency one needs the Hahn–Banach Theorem and several sophisticated techniques.

Returning to our original side constraint, it is natural to ask the following:

Question Q2. Can one characterize the class of Young measures that can be generated by a sequence (∇u_j) of gradients of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^m)$-functions satisfying the side constraint

$$\det \nabla u_j > 0 \quad \text{a.e.} \qquad ? \tag{1.2}$$

This is essentially a weaker version of Question Q1.

We will show that this question can be solved and prove the following theorem from [KRW13]:

Theorem 7. *Let $p < d$. A Young measure $\nu = (\nu_x)$ is an **orientation-preserving gradient** p-**Young measure**, that is, there exists a sequence $(u_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ with $\nabla u_j \overset{\mathrm{Y}}{\to} \nu = (\nu_x)$ and satisfying* (1.2)*, if and only if the following conditions hold:*

(KP1)–(KP3) *from the Kinderlehrer–Pedregal Theorem.*

(SUPP) $\operatorname{supp} \nu_x \subset \{ A \in \mathbb{R}^{d\times d} \;:\; \det A \geq 0 \}$.

1.3. **More general constraints, prescribed Jacobians.** Finally, one can also consider the following, more general constraints:

$$\det \nabla u \geq r > 0 \quad \text{a.e.}$$

or

$$\det \nabla u = r > 0 \quad \text{a.e.}$$

or even

$$J_1(x) \leq \det \nabla u(x) \leq J_2(x) \quad \text{for a.e. } x \in \Omega.$$

where

(i) $J_1 \colon \Omega \to [-\infty,+\infty)$, $J_2 \colon \Omega \to (-\infty,+\infty]$ measurable,
(ii) $J_1(x) \leq J_2(x)$ for a.e. $x \in \Omega$, and
(iii) $\displaystyle\int_\Omega |J_1^+(x)|^{p/d} \,\mathrm{d}x < \infty \quad \text{and} \quad \int_\Omega |J_2^-(x)|^{p/d} \,\mathrm{d}x < \infty.$

Here, J_1^+ and J_2^- are the positive part of J_1 and the negative part of J_2, respectively. These conditions are rather natural (the only non-trivial one is (iii) and this one is simply an integrability condition, only relevant in very special cases). Notice that the lower and upper bound can also be non-active for certain x, for example if $J_1(x) = -\infty$, there is no lower bound at x.

A question that is related to the well-known Dacorogna–Moser Theory [DM90] then is:

Question Q3. Given $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega;\mathbb{R}^d)$ (the trace space to $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$, that is the space of all boundary values of $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$-functions), does there exist

$$u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \quad \text{with} \quad u|_{\partial\Omega} = g \quad \text{and} \quad \det \nabla u > 0 \text{ a.e.} \qquad ?$$

The same question can also be asked with the even stricter requirement $\det \nabla u = 1$ instead of mere positivity of the Jacobian. This expresses **incompressibility** and is relevant in the theory of fluids.

Notice that here we have no "compatibility" assumption on g. Again, one can show with a similar argument as before that this question is unsolvable if $p \geq d$. However, as a corollary to the methods described in these notes we will also get solvability for this question as long as $p < d$.

The remainder of these notes is concerned with proving some of the statements above.

2. GEOMETRY OF THE DETERMINANT CONSTRAINT

As a first step toward investigating the above questions, in this section we examine the "geometry" of the set $\{A \in \mathbb{R}^{d\times d} \,:\, \det A = 0\}$, which has a central place in our arguments. First, we make the simple observation that any square matrix $M_0 \in \mathbb{R}^{d\times d}$ with $\det M_0 < 0$ can be written as the barycenter of a probability measure μ on $\mathbb{R}^{d\times d}$ with

$$\operatorname{supp}\mu \subset \big\{A \in \mathbb{R}^{d\times d} \,:\, \det A = 0\big\}.$$

Indeed, if (and we will see in the proof of Proposition 8 below that we can always reduce to this case)

$$M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \qquad \text{with} \qquad 0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_d,$$

then trivially,

$$\begin{aligned} M_0 = \;& \frac{1}{2}\begin{pmatrix} 0 & & & & \\ & 2\sigma_2 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2\sigma_1 & & & & \\ & 0 & & & \\ & & \sigma_3 & & \\ & & & \ddots & \\ & & & & \sigma_d \end{pmatrix} \\ =:\;& \frac{1}{2}M_1 + \frac{1}{2}M_2. \end{aligned}$$

It is clear that $\det M_1 = \det M_2 = 0$, and so,

$$\mu := \frac{1}{2}\delta_{M_1} + \frac{1}{2}\delta_{M_2}$$

fulfills the above assertion.

A more intricate question is whether this can also be achieved if μ is restricted to be a gradient Young measure or even a p-laminate. A p-laminate is a special type of *gradient* homogeneous Young measure (constant in x) that originates from the Dirac mass δ_{M_0} by recursively splitting matrices along rank-one lines (this terminology will be made clearer below), for a precise definition see [KRW13]. Here we only need that if we are given a Dirac mass and we split it along a rank-one line, and possibly iterate this by splitting further matrices along rank-one lines, then the resulting (finite) probability measure is a gradient p-Young measure, see [Ped97].

We will see a bit later that it is indeed always possible to write M_0 as the barycenter of a p-laminate for $p < d$, albeit one with *infinite* order, and certain good estimates hold.

Proposition 8. *Let $M_0 \in \mathbb{R}^{d\times d}$ with $\det M_0 < 0$. Then, there exists a homogeneous Young measure ν (a probability measure on $\mathbb{R}^{d\times d}$) that is a p-laminate of infinite order for every $1 \le p < d$ and such that the following assertions hold:*

(i) $[\nu] = \displaystyle\int \mathrm{id}\, \mathrm{d}\nu = M_0$,

(ii) $\operatorname{supp}\nu \subset \big\{A \in \mathbb{R}^{d\times d} \,:\, \det A = 0\big\}$,

(iii) $\displaystyle\int |A|^p \, \mathrm{d}\nu(A) \le C_p |M_0|^p$,

(iv) $\displaystyle\int |A - M_0|^p \, \mathrm{d}\nu(A) \le C_p |\det M_0|^{p/d}$,

where $C_p = C(d,p) > 0$ is a constant.

Remark 9. (1) Note that ν does not depend on p.
(2) We remark that in (iii) and (iv) the constant blows up, $C_p \to \infty$, as $p \uparrow d$.

Proof. The idea of the proof is to employ recursive lamination constructions to furnish a sequence of homogeneous Young measures $\nu_0 = \delta_{M_0}, \nu_1, \nu_2, \ldots$, which push more and more of the total mass into the set of zero-determinant matrices, and then use weak*-precompactness of the sequence (ν_j) to pass to an infinite-order p-laminate ν, which satisfies all the properties in the proposition.

Step 1. We first transform M_0 to diagonal form. Let $M_0 = \tilde{P}\tilde{D}_0\tilde{Q}^T$ be the *real* singular value decomposition, that is, $\tilde{D}_0 = \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ with $0 < \sigma_1 \le \sigma_2 \le \cdots \le \sigma_d$, and $\tilde{P}, \tilde{Q}$ orthogonal matrices. As $0 > \det M_0 = \det \tilde{P} \cdot \det \tilde{D}_0 \cdot \det \tilde{Q}$, either $\tilde{P}$ or $\tilde{Q}$ has negative determinant, say $\det \tilde{P} < 0$ (the other case is similar). With

$$D_0 := \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} \qquad P := \tilde{P} \cdot \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad Q := \tilde{Q},$$

we have $M_0 = PD_0Q^T$, where now $P, Q \in \mathrm{SO}(d)$ and $\det D_0 < 0$. Now, if D_0 can be written as a laminate, i.e. a hierarchical decomposition along rank-one lines, then the same holds true for M_0 since $P(a \otimes b)Q^T = (Pa) \otimes (Qb)$ for any $a, b \in \mathbb{R}^d$.

We remark in this context that the procedure to reduce to a diagonal matrix does not change the (Frobenius) matrix norm, since the latter only depends on the singular values, which trivially are not changed by the singular value decomposition. Also, as $P, Q \in \mathrm{SO}(d)$, the determinant is also not changed in this process.

Step 2. Owing to Step 1, in the following we can assume that M_0 is already diagonal, the first diagonal entry is negative and all others are positive. We will write the first 2×2 block of M_0 as an infinite hierarchy of convex combinations along rank-one lines such that all resulting matrices have zero determinant. Write

$$M_0 = \begin{pmatrix} -\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix},$$

for which $\sigma_i > 0$ as in Step 1.

Set $r := 2^{\frac{p}{d}-1}$ and observe that since $p < d$, we have $2^{(1-d)/d} \le r < 1$. We also set $\gamma := \sqrt{\sigma_1\sigma_2}$. Then, we can decompose M_0 twice along rank-one lines as follows:

$$\begin{aligned}
M_0 &= \frac{1}{2}\big[M_0 + \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2)\big] + \frac{1}{2}\big[M_0 - \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2)\big] \\
&= \frac{1}{4}\big[M_0 + \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2) + \gamma(\mathrm{e}_2 \otimes \mathrm{e}_1)\big] + \frac{1}{4}\big[M_0 + \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2) - \gamma(\mathrm{e}_2 \otimes \mathrm{e}_1)\big] \\
&\quad + \frac{1}{4}\big[M_0 - \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2) + \gamma(\mathrm{e}_2 \otimes \mathrm{e}_1)\big] + \frac{1}{4}\big[M_0 - \gamma(\mathrm{e}_1 \otimes \mathrm{e}_2) - \gamma(\mathrm{e}_2 \otimes \mathrm{e}_1)\big] \\
&=: \frac{1}{4}M_{1,B1} + \frac{1}{4}M_{1,G1} + \frac{1}{4}M_{1,G2} + \frac{1}{4}M_{1,B2}.
\end{aligned}$$

We can compute

$$
\begin{aligned}
\det M_{1,G1} &= \det M_{1,G2} = (-\sigma_1\sigma_2 + \sigma_1\sigma_2)\prod_{i=3}^{d}\sigma_i = 0.\\
\det M_{1,B1} &= \det M_{1,B2} = (-\sigma_1\sigma_2 - \sigma_1\sigma_2)\prod_{i=3}^{d}\sigma_i = -2\sigma_1\sigma_2\prod_{i=3}^{d}\sigma_i < 0\\
|\det M_{1,B1}| &= |\det M_{1,B2}| = 2|\det M_0| = (2r)^{d/p}\,|\det M_0|. \qquad (2.1)
\end{aligned}
$$

Thus, the "good" matrices $M_{1,G1}, M_{1,G2}$ already satisfy our constraint of having zero determinant, the "bad" matrices $M_{1,B1}, M_{1,B2}$ will be further decomposed later on. Moreover, note that

$$
|M_{1,J} - M_0| = 2^{1/2}(\sigma_1\sigma_2)^{1/2} \leq 2^{1/2}\,|\det M_0|^{1/d}, \qquad J \in \{G1, G2, B1, B2\}, \qquad (2.2)
$$

since $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_d$ and hence $(\sigma_1\sigma_2)^{d/2} \leq |\det M_0|$.

Step 3. Define

$$
\nu_0 := \delta_{M_0}, \qquad \nu_1 := \frac{1}{4}\delta_{M_{1,G1}} + \frac{1}{4}\delta_{M_{1,G2}} + \frac{1}{4}\delta_{M_{1,B1}} + \frac{1}{4}\delta_{M_{1,B2}},
$$

and, as detailed above, we observe that ν_1 is derived from ν_0 by two additional lamination steps. Moreover, $[\nu_1] = \int A\,\mathrm{d}\nu_1(A) = [\nu_0] = M_0$.

Now recursively apply the procedure from the preceding steps to decompose the "bad" matrices $M_{1,B1}$ and $M_{1,B2}$ in turn taking the role of M_0. This yields matrices $M_{2,G1}, \ldots, M_{2,G4}$, $M_{2,B1}, \ldots, M_{2,B4}$ such that

$$
\begin{aligned}
M_{1,B1} &= \frac{1}{4}M_{2,G1} + \frac{1}{4}M_{2,G2} + \frac{1}{4}M_{2,B1} + \frac{1}{4}M_{2,B2},\\
M_{1,B2} &= \frac{1}{4}M_{2,G3} + \frac{1}{4}M_{2,G4} + \frac{1}{4}M_{2,B3} + \frac{1}{4}M_{2,B4}.
\end{aligned}
$$

We define ν_2 accordingly as

$$
\begin{aligned}
\nu_2 := \frac{1}{4}\delta_{M_{1,G1}} + \frac{1}{4}\delta_{M_{1,G2}} &+ \frac{1}{4^2}\left[\delta_{M_{2,G1}} + \delta_{M_{2,G2}} + \delta_{M_{2,B1}} + \delta_{M_{2,B2}}\right]\\
&+ \frac{1}{4^2}\left[\delta_{M_{2,G3}} + \delta_{M_{2,G4}} + \delta_{M_{2,B3}} + \delta_{M_{2,B4}}\right].
\end{aligned}
$$

Then, still $[\nu_2] = M_0$ and ν_2 is a finite-order laminate.

Now iterate this scheme of first bringing the matrix to diagonal form via Step 1 and then laminating via Step 2, in every step defining a new finite-order laminate ν_j, $j \in \mathbb{N}$, with $[\nu_j] = M_0$. In this context recall that the reduction to a diagonal form does not change the matrix norm or determinant.

In more detail, we get in the first two iterations (adding appropriate indices to the matrices P,Q,D):

$$\begin{aligned}
M_0 &= P_0 D_0 Q_0^T \\
&= P_0\left(\frac{1}{4}M_{1,G1} + \frac{1}{4}M_{1,G2} + \frac{1}{4}M_{1,B1} + \frac{1}{4}M_{1,B2}\right)Q_0^T \\
&= P_0\left(\frac{1}{4}M_{1,G1} + \frac{1}{4}M_{1,G2} + \frac{1}{4}P_{1,B1}D_{1,B1}Q_{1,B1}^T + \frac{1}{4}P_{1,B2}D_{1,B2}Q_{1,B2}^T\right)Q_0^T \\
&= \frac{1}{4}\underbrace{P_0 M_{1,G1} Q_0^T}_{\det=0} + \frac{1}{4}\underbrace{P_0 M_{1,G2} Q_0^T}_{\det=0} + \frac{1}{4}P_0 P_{1,B1} D_{1,B1} Q_{1,B1}^T Q_0^T + \frac{1}{4}P_0 P_{1,B2} D_{1,B2} Q_{1,B2}^T Q_0^T \\
&= \frac{1}{4}P_0 M_{1,G1} Q_0^T + \frac{1}{4}P_0 M_{1,G2} Q_0^T \\
&\qquad + \frac{1}{4}P_0 P_{1,B1}\left(\frac{1}{4}M_{2,G1} + \frac{1}{4}M_{2,G2} + \frac{1}{4}M_{2,B1} + \frac{1}{4}M_{2,B2}\right)Q_{1,B1}^T Q_0^T + \underbrace{\cdots}_{1,B2\text{-part}}
\end{aligned}$$

In every step of bringing matrices to diagonal form, the mean value M_0 of the Young measures ν_j associated to these splittings is preserved. Further, note that we only split along rank-one lines, hence

$$P_0 M_{1,G1/G2/B1/B2} Q_0^T = M_0 \pm \gamma(P_0 \mathrm{e}_1) \otimes (Q_0 \mathrm{e}_2) \pm \gamma(P_0 \mathrm{e}_2) \otimes (Q_0 \mathrm{e}_1),$$

and we preserve the property for the ν_j's to be finite-order laminates.

Without proof we recall the fact that splitting Dirac masses along rank-one lines as above preserves the property of the ν_j to be gradient Young measures (essentially this holds because these rank-one connections can be generated by sequences of gradients, a formal proof can proceed via the Kinderlehrer–Pedregal Theorem 6). Thus, all of our ν_j are gradient Young measures.

Step 4. Let us consider the distance integral in (iv):

$$\begin{aligned}
\int |A - M_0|^p \,\mathrm{d}\nu_j(A) &= \sum_{i=1}^{j}\sum_{k=1}^{2^i} \frac{1}{4^i}|M_{i,Gk} - M_0|^p + \sum_{k=1}^{2^j} \frac{1}{4^j}|M_{j,Bk} - M_0|^p \\
&\le \sum_{i=1}^{j}\sum_{k=1}^{2^i} \frac{1}{4^i}\left(\sum_{\ell=1}^{i} |X_\ell - X_{\ell-1}|\right)^p + \sum_{k=1}^{2^j} \frac{1}{4^j}\left(\sum_{\ell=1}^{j} |Y_\ell - Y_{\ell-1}|\right)^p,
\end{aligned}$$

where in the innermost summations we defined $X_i := M_{i,Gk}$, $X_0 := M_0$, and $X_{\ell-1}$ is the $M_{\ell-1,Bk}$ with $k \in \{1,\dots,2^{\ell-1}\}$ such that X_ℓ originated from $X_{\ell-1}$ through the lamination construction from the previous proof step (with the understanding $M_{0,B1} := M_0$); similarly, $Y_j := M_{j,Bk}$, $Y_0 := M_0$, and $Y_{\ell-1}$ defined analogously to $X_{\ell-1}$. Then, $\sum_{\ell=1}^{i} X_\ell - X_{\ell-1} = M_{i,Gk} - M_0$ and $\sum_{\ell=1}^{j} Y_\ell - Y_{\ell-1} = M_{j,Bk} - M_0$, and so the second line in the estimate follows from the first by virtue of the triangle inequality. Now, to bound $|X_\ell - X_{\ell-1}|$ we use (2.2) and then (2.1) recursively. Thus,

$$\begin{aligned}
\sum_{\ell=1}^{i} |X_\ell - X_{\ell-1}| &\le \sum_{\ell=1}^{i} 2^{1/2}\,|\det X_{\ell-1}|^{1/d} \le \sum_{\ell=1}^{i} 2^{1/2}\cdot(2r)^{(\ell-1)/p}\,|\det M_0|^{1/d} \\
&\le \frac{2^{1/2}\,|\det M_0|^{1/d}}{(2r)^{1/p} - 1}\cdot(2r)^{i/p}
\end{aligned}$$

and a similar estimate holds for the second inner summation involving the Y_ℓ's. Hence, we can plug this into the previous estimate to get

$$\begin{aligned}\int |A-M_0|^p \, \mathrm{d}\nu_j(A) &\le \left[\frac{2^{1/2}}{(2r)^{1/p}-1}\right]^p \cdot |\det M_0|^{p/d} \cdot \left[\sum_{i=1}^{j} \frac{2^i(2r)^i}{4^i} + \frac{2^j(2r)^j}{4^j}\right] \\ &\le \left[\frac{2^{1/2}}{(2r)^{1/p}-1}\right]^p \cdot |\det M_0|^{p/d} \cdot \left[\frac{1}{1-r} + r^j\right] \\ &\le C_p |\det M_0|^{p/d}. \end{aligned} \tag{2.3}$$

Moreover, by (2.3) and the fact that the ν_j's are probability measures,

$$\begin{aligned}\int |A|^p \, \mathrm{d}\nu_j(A) &\le 2^p \left[\int |A-M_0|^p \, \mathrm{d}\nu_j(A) + |M_0|^p\right] \\ &\le 2^p C_p |\det M_0|^{p/d} + 2^p |M_0|^p \\ &\le C_p |M_0|^p, \end{aligned} \tag{2.4}$$

which is uniformly bounded. In particular, the ν_j are (sequentially) weakly*-precompact as measures, hence there exists a subsequence and a cluster point ν, which is a p-laminate, $1 < p < d$, and satisfies $[\nu] = M_0$. Moreover, this ν is also a (homogeneous) *gradient* Young measures since the ν_j's have this property and the property of being a *gradient* Young measures is preserved under the operation of taking a weak* limit (this can be proved by an easy diagonal argument).

Passing to the limit in (2.3) and (2.4) yields (iii) and (iv).

Finally, it can be seen easily that the mass of ν_j that is carried by "bad" matrices, i.e. those with negative determinant, is

$$|\nu_j|\big(\big\{A \in \mathbb{R}^{d\times d} \;:\; \det A < 0\big\}\big) = \frac{2^j}{4^j} \to 0 \qquad \text{as } j \to \infty.$$

Thus, also (ii) follows, concluding the proof. □

Remark 10. By a similar, slightly more intricate, strategy one can also show that there exist (finite-order) laminates ν_j, with $\int |A|^p \, \mathrm{d}\nu_j(A)$ uniformly bounded, and ν_j can be split as

$$\nu_j = \nu_j^+ + \nu_j^- \qquad \text{with} \qquad \operatorname{supp} \nu_j^\pm \subset \big\{A \in \mathbb{R}^{d\times d} \;:\; \det A \gtrless 0\big\},$$

where $\int |A|^p \, \mathrm{d}\nu_j^-(A) \to 0$ as $j \to \infty$. In particular, $\nu_j \xrightarrow{*} \nu$ (in the weak* Young measure or measure convergence) where ν is as in Proposition 8 but $\operatorname{supp} \nu \subset \big\{A \in \mathbb{R}^{d\times d} \;:\; \det A > 0\big\}$.

3. A CONVEX INTEGRATION PRINCIPLE

Employing our investigation into the geometry of the zero-determinant constraint in matrix space from the previous section and the fact that p-laminates are gradient Young measures (which, as mentioned before, follows from the Kinderlehrer–Pedregal Theorem), in this section we prove the following proposition, which directly entails a weaker variant of Theorem 7 with the generating sequence consisting of gradients with nonnegative determinant only; the full strength of the main theorem is proved in [KRW13] (the extension is just a bit more technical).

We first prove a key proposition which makes the proof of our main result almost trivial. This can be considered a variant of convex integration, whose origins lie in the Nash–Kuiper C^1-Embedding Theorem in Differential Geometry, but which has recently found many interesting applications, cf. [Gro86, EM02, MŠ03, Kir03, DLS12] for more information.

The rough idea of convex integration is to go from an approximate solution to an exact solution to a differential inclusion (as in Question Q1) by adding "fast oscillations" to push the values into the set, but at the same time not changing the approximate solution by "too much".

Proposition 11. *Let* $1 < p < d$ *and* $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$*. Then there exists* $v \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ *such that*

(i) $\det \nabla v(x) \geq 0$ *for a.e.* $x \in \Omega$,

(ii) $v \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$,

(iii) $\|\nabla v - \nabla u\|_p^p \leq C_p \int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d} \,\mathrm{d}x$,

where $C_p = C(d,p) > 0$ *is a constant independent of u.*

Note that this immediately gives a solution to the (weakly orientaion-preserving version of) Question Q1.

Remark 12. The preceding proposition also holds in more general situations, see [KRW15].

Proof of Proposition 11. We can assume that (otherwise there is nothing to prove)

$$\int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d} \,\mathrm{d}x > 0.$$

We construct a sequence of functions $(v^l)_{l\in\mathbb{N}}$, bounded in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$, such that

$$v^l \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d), \tag{3.1}$$

$$\int_{\{\det \nabla v^l<0\}} |\det \nabla v^l(x)|^{p/d} \,\mathrm{d}x \leq 2^{-lp} \int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d} \,\mathrm{d}x, \tag{3.2}$$

$$\int_\Omega |\nabla v^{l+1}(x) - \nabla v^l(x)|^p \,\mathrm{d}x \leq 2^{-(l-1)p} C \int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d} \,\mathrm{d}x, \tag{3.3}$$

where $C > 0$ is a (l-independent) constant.

Let us construct the sequence inductively. Set $v^0 = u$ so that (3.1) and (3.2) are satisfied. If $v^l \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ has been constructed to satisfy (3.1) and (3.2), we find v^{l+1} in the following way: By Proposition 8 for a.e. $x \in \Omega$ with $\det \nabla v^l(x) < 0$, there exists a homogeneous gradient p-Young measure ν^l_x with $[\nu^l_x] = \nabla v^l(x)$, with support in the set $\{ A \in \mathbb{R}^{d\times d} \;:\; \det A \geq 0 \}$. By (iii) from Proposition 8 we may moreover assume

$$\int |A|^p \,\mathrm{d}\nu^l_x(A) \leq C |\nabla v^l(x)|^p,$$

the constant C being independent of x and v^l. For $x \in \Omega$ such that $\det \nabla v^l(x) \geq 0$, simply set $\nu^l_x = \delta_{\nabla v^l(x)}$.

Define $\nu^l = (\nu^l_x)_{x\in\Omega}$ and note that $\int_\Omega \int |A|^p \,\mathrm{d}\nu^l_x(A) \,\mathrm{d}x < \infty$, $[\nu^l] = \nabla v^l$, and ν^l_x is a gradient p-Young measure for almost every $x \in \Omega$. Then, by the Kinderlehrer–Pedregal Theorem, ν^l is itself a gradient p-Young measure with $\operatorname{supp} \nu^l_x \subset \{ A \in \mathbb{R}^{d\times d} \;:\; \det A \geq 0 \}$ a.e. and, by standard Young measure arguments (see for example [Ped97]), there exists a sequence $(v^{l,m})_m \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ with $(\nabla v^{l,m})_m$ p-equiintegrable and generating ν^l. Furthermore, $v^{l,m} - v^l \in \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^d)$ and hence $v^{l,m} \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ for all $m \in \mathbb{N}$.

We define $g : \Omega \times \mathbb{R}^{d\times d} \to \mathbb{R}$ by

$$g(x,A) = \mathbb{1}_{\{\det A<0\}} |\det A|^{p/d} = \begin{cases} |\det A|^{p/d} & \text{if } \det A < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Using g as a test function and the fact that ν_x^l is supported in $\left\{A \in \mathbb{R}^{d\times d} \,:\, \det A \geq 0\right\}$, by Young measure representation, we may choose m large enough, say $m = M$, and define $\nabla v^{l+1} := \nabla v^{l,M}$ such that

$$\int_{\{\det \nabla v^{l+1}<0\}} |\det \nabla v^{l+1}(x)|^{p/d}\,\mathrm{d}x \leq 2^{-(l+1)p} \int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d}\,\mathrm{d}x,$$

i.e. (3.1) as well as (3.2) hold for v^{l+1}. Also, by taking M even larger if necessary, we can ensure that also

$$\int_\Omega |\nabla v^{l+1}(x) - \nabla v^l(x)|^p\,\mathrm{d}x \leq 2^p \int_\Omega \int |A - \nabla v^l(x)|^p\,\mathrm{d}\nu_x^l(A)\,\mathrm{d}x \tag{3.5}$$

Indeed, this follows again from Young measure representation for the integrand $|A - \nabla v^l(x)|^p$.

Next, for any $l \in \mathbb{N}$, by property (iv) in Proposition 8 and (3.2) we infer that

$$\begin{aligned}\int_\Omega \int |A - \nabla v^l(x)|^p\,\mathrm{d}\nu_x^l(A)\,\mathrm{d}x &\leq C \int_{\{\det \nabla v^l<0\}} |\det \nabla v^l(x)|^{p/d}\,\mathrm{d}x \\ &\leq 2^{-lp} C \int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d}\,\mathrm{d}x\end{aligned}$$

for a constant $C > 0$ independent of x. Combining with (3.5) we get the estimate

$$\int_\Omega |\nabla v^{l+1}(x) - \nabla v^l(x)|^p\,\mathrm{d}x \leq 2^{-(l-1)p} C \int \mathbb{1}_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d}\,\mathrm{d}x,$$

which is (3.3), completing the definition of our sequence.

The result now follows easily: by (3.3) in conjunction with the Poincaré–Friedrichs inequality, $(v^l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ and therefore has a strong $\mathrm{W}^{1,p}$-limit v. In particular, it holds that $v \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ and (ii) follows. Using the triangle inequality and (3.3), we deduce that

$$\begin{aligned}\|\nabla v - \nabla u\|_{\mathrm{L}^p} &\leq \sum_{l=0}^{\infty} \|\nabla v^{l+1} - \nabla v^l\|_{\mathrm{L}^p} \\ &\leq C^{1/p} \left(\int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d}\,\mathrm{d}x \right)^{1/p} \sum_{l=0}^{\infty} 2^{-(l-1)} \\ &= 4C^{1/p} \left(\int_{\{\det \nabla u<0\}} |\det \nabla u(x)|^{p/d}\,\mathrm{d}x \right)^{1/p},\end{aligned}$$

proving (iii). Lastly, $(\nabla v^l)_l$ is p-equiintegrable (being Cauchy in L^p), and since it holds that $|\det \nabla v^l(x)|^{p/d} \leq C|\nabla v^l|^p$, also $|\det \nabla v^l(x)|^{p/d}\}_{l\in\mathbb{N}}$ is equiintegrable and converges, up to a subsequence, to $|\det, \nabla v|^{p/d}$. Therefore, by Vitali's Convergence Theorem,

$$\int_{\{\det \nabla v<0\}} |\det \nabla v(x)|^{p/d}\,\mathrm{d}x = 0,$$

which implies $\det \nabla v(x) \geq 0$ for a.e. $x \in \Omega$, i.e. (i), and the proof is complete. □

We are now in a position to prove Theorem 7 with the weaker constraint that the deformation is only *weakly* orientation-preserving, that is, for $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ it holds that

$$\det \nabla u \geq 0 \quad \text{a.e. in } \Omega.$$

Proof of Theorem 7 (Question Q2) for the weak orientation-preserving constraints.
(i) $\Rightarrow$ (ii): Conditions (KP1)–(KP3) follow from the usual Kinderlehrer–Pedregal Theorem 6.

Regarding (SUPP), let $h \in \mathrm{L}^\infty(\Omega \times \mathbb{R}^{d\times d})$ be Carathéodory and with the property $\operatorname{supp} h(x,\cdot) \subset\subset \{A \in \mathbb{R}^{d\times d} \ \colon\ \det A < 0\}$ for almost every x. Then, by the assumptions on ∇u_j,

$$\int_\Omega \int h(x,A)\,\mathrm{d}\nu_x(A)\,\mathrm{d}x = \lim_{j\to\infty} \int_\Omega h(x,\nabla u_j(x))\,\mathrm{d}x = 0.$$

Varying h, we infer that $\operatorname{supp}\nu_x \subset \{A \in \mathbb{R}^{d\times d} \ \colon\ \det A \geq 0\}$ for a.e. $x \in \Omega$.

(ii) $\Rightarrow$ (i): For $1 < p < \infty$ let ν be a gradient p-Young measure with

$$\operatorname{supp}\nu_x \subset \{A \in \mathbb{R}^{d\times d} \ \colon\ \det A \geq 0\} \qquad \text{for a.e. } x \in \Omega.$$

Standard results yield that there exists a sequence $(u_j) \subset \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ with $\nabla u_j \overset{\mathrm{Y}}{\to} \nu$ and (∇u_j) p-equiintegrable and satisfies $u_j \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ where $\nabla u(x) = [\nu_x]$. By Young measure representation applied to the test function g in (3.4) and the assumption on the support of ν, we may assume (after passing to a subsequence if necessary) that

$$\int_{\{\det \nabla u_j < 0\}} |\det \nabla u_j(x)|^{p/d}\,\mathrm{d}x < \frac{1}{j^p}. \tag{3.6}$$

Applying Proposition 11 to each u_j, we obtain a new sequence $(v_j) \subset \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$ such that $\det \nabla v_j(x) \geq 0$ a.e., and, by (3.6) together with part (iii) of Proposition 11,

$$\|\nabla u_j - \nabla v_j\|_{\mathrm{L}^p} < \frac{C^{1/p}}{j}.$$

Hence (∇v_j) is p-equiintegrable and generates ν as well (because the difference between u_j and v_j converges to zero in norm). □

We here omit the extension to the proper strict orientation-preserving constraint $\det \nabla u > 0$ a.e. for reasons of space. However, this step is now not too difficult anymore and uses a refined geometry argument. Details can be found in Section 5 of [KRW13]. However, if we assume Theorem 7 to be fully established, we can record the following corollary, again from [KRW13]:

Corollary 13. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and $1 < p < d$. Let $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ be weakly orientation-preserving,*

$$\det \nabla u \geq 0 \quad a.e.$$

Then, there exists a sequence $(v_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ that is strictly orientation-preserving,

$$\det \nabla v_j > 0 \quad a.e. \qquad \text{for all } j \in \mathbb{N},$$

and such that $\|v_j - u\|_{1,p} \to 0$ as $j \to \infty$.

4. Extension to prescribed Jacobians

We finally record statements about extensions to *prescribed* Jacobians. Proofs for the following results can be found in [KRW15]:

Theorem 14. *Let $1 < p < d$. Suppose that $\Omega \subset \mathbb{R}^d$ is an open and bounded Lipschitz domain, and let $\nu = (\nu_x)_{x\in\Omega}$ be a p-Young measure. Moreover let $J_1 \colon \Omega \to [-\infty,+\infty)$, $J_2 \colon \Omega \to (-\infty,+\infty]$ be measurable and such that $J_1(x) \leq J_2(x)$ for a.e. $x \in \Omega$. Also, assume that for $i = 1,2$,*

$$\int_\Omega |J_1^+(x)|^{p/d}\,\mathrm{d}x < \infty \qquad \text{and} \qquad \int_\Omega |J_2^-(x)|^{p/d}\,\mathrm{d}x < \infty,$$

where $J_i^\pm$ denotes the positive or negative part of J_i, respectively. Then the following statements are equivalent:

(i) *There exists a sequence of gradients* $(\nabla u_j) \subset \mathrm{L}^p(\Omega;\mathbb{R}^{d\times d})$ *that generates* ν*, such that*

$$J_1(x) \le \det \nabla u_j(x) \le J_2(x) \quad \text{for all } j \in \mathbb{N} \text{ and a.e. } x \in \Omega.$$

(ii) *The following condition hold:*

(KP1)–(KP3) *from the Kinderlehrer–Pedregal Theorem.*

(SUPP) $\operatorname{supp} \nu_x \subset \{ A \in \mathbb{R}^{d\times d} \,:\, J_1(x) \le \det A \le J_2(x) \}$ *for a.e.* $x \in \Omega$,

Furthermore, in this case the sequence (u_j) *can be chosen so that* (∇u_j) *is p-equiintegrable and* $u_j \in \mathrm{W}^{1,p}_u(\Omega;\mathbb{R}^d)$*, where* $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ *is the deformation underlying* ν *(i.e. the function whose gradient is the barycenter of* ν*).*

As a consequence, we can solve Question Q3:

Corollary 15. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain,* $1 < p < d$*, and* $g \in \mathrm{W}^{1-1/p,p}(\partial\Omega;\mathbb{R}^d)$*. Then, there exists an incompressible map* $v \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ *with the given boundary data (in the sense of trace), i.e.*

$$\begin{cases} \det \nabla v(x) = 1 & \text{for a.e. } x \in \Omega, \\ v|_{\partial\Omega} = g & \text{in the sense of trace.} \end{cases}$$

Observe that no compatibility conditions on g are required. For instance, if $g(x) = 2x$, by the change of variables formula there can be no *smooth* solution of our problem. In the class $\mathrm{W}^{1,p}$, $p < d$, however, we may interpret our solutions as ones exhibiting "cavitation", cf. [Bal82].

Next we record another approximation result:

Corollary 16. *Let* $\Omega \subset \mathbb{R}^d$ *be open and bounded with* $|\partial\Omega| = 0$*. Suppose that* $1 < p < d$ *and* $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$*. Then, there exists a sequence* $(u_j) \subset \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$ *bounded such that for all* $j \in \mathbb{N}$*,* $u_j \in \mathrm{W}^{1,p}_u$*,*

$$\det \nabla u_j(x) \ge 0 \quad \text{for a.e. } x \in \Omega \qquad \text{and} \qquad u_j \rightharpoonup u \text{ in } \mathrm{W}^{1,p}(\Omega;\mathbb{R}^d) \quad \text{as } j \to \infty.$$

Finally, we present an application to the study of quasiconvexity: In the search for quasiconvexity conditions compatible with nonlinear elasticity, several different notions have been developed, see [Dac08, Ped97]. One of them is the so-called $\mathrm{W}^{1,p}$**-quasiconvexity** of a locally bounded Borel function $h\colon \mathbb{R}^{d\times d} \to \mathbb{R}$, that is,

$$h(A_0) \le \fint_{\mathbb{B}^d} h(\nabla v(x)) \,\mathrm{d}x \tag{4.1}$$

for all $A_0 \in \mathbb{R}^{d\times d}$ and all $v \in \mathrm{W}^{1,p}(\mathbb{B}^d;\mathbb{R}^d)$ with $v(x) = A_0 x$ on $\partial\mathbb{B}^d$ (in the sense of trace). As usual, the unit ball $\mathbb{B}^d$ can be replaced by any Lipschitz domain (and in fact also more general domains).

Another notion is that of **closed** $\mathrm{W}^{1,p}$**-quasiconvexity**, which entails that

$$h(A_0) \le \int h(A) \,\mathrm{d}\nu(A) \tag{4.2}$$

for every homogeneous gradient p-Young measure ν with barycenter $[\nu] = A_0$.

It is well known that if h satisfies the p-growth bound

$$|h(A)| \le M(1 + |A|^p) \tag{4.3}$$

then $\mathrm{W}^{1,p}$-quasiconvexity and closed $\mathrm{W}^{1,p}$-quasiconvexity are equivalent to the usual quasiconvexity; however, this growth condition is incompatible with the requirements of nonlinear elasticity, namely the **realistic growth condition**

$$h(A) \to \infty \quad \text{as } \det A \to 0^+ \qquad \text{and} \qquad h(A) = \infty \quad \text{for } \det A \le 0.$$

We want to show now that *even if we do not assume* (4.3), the aforementioned two flavors of quasiconvexity are in fact unsuitable for realistic problems of nonlinear elasticity in $\mathrm{W}^{1,p}$ with $p < d$ (which, for example, includes the prototypical quadratic $\mathrm{W}^{1,2}$ in three dimensions).

Lemma 17. *Let $1 < p < d$ and assume that h is $\mathrm{W}^{1,p}$-quasiconvex or closed $\mathrm{W}^{1,p}$-quasiconvex (no universal growth bounds are required). If there exists* one *$r > 0$ and a constant $M = M(r) \ge 0$ such that the very weak growth constraint*

$$h(A) \le M(1 + |A|^p) \qquad \text{for all } A \in \mathbb{R}^{d\times d} \text{ with } \det A = r$$

holds, then $h(A) < \infty$ for every $A \in \mathbb{R}^{d\times d}$.

Since the conclusion holds for matrices with negative determinant as well, (closed) $\mathrm{W}^{1,p}$-quasiconvexity therefore precludes the desirable constraint $h(A) = \infty$ for $\det A \le 0$. Notice that the growth bound mentioned in the lemma is rather weak indeed and for instance satisfied for the polyconvex integrand

$$h(A) := |A|^p + \frac{1}{\max\{\det A, 0\}}.$$

Proof. We only show this result in the case where h is $\mathrm{W}^{1,p}$-quasiconvex, the proof for closed $\mathrm{W}^{1,p}$-quasiconvexity is similar.

Let $A_0 \in \mathbb{R}^{d\times d}$ be arbitrary. The results of this work entail in particular that there exists a function $v \in \mathrm{W}^{1,p}(\mathbb{B}^d;\mathbb{R}^d)$ with $v(x) = A_0 x$ for $x \in \partial\mathbb{B}^d$ (in the sense of trace) and $\det \nabla v = r$ almost everywhere. Then,

$$h(A_0) \le \fint_{\mathbb{B}^d} h(\nabla v(x)) \,\mathrm{d}x \le M \fint_{\mathbb{B}^d} 1 + |\nabla v(x)|^p \,\mathrm{d}x < \infty.$$

This already implies the result. □

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