OPEN PROBLEMS

1ST KOREAN WORKSHOP ON GRAPH THEORY

1. Seog-Jin Kim

Question 1 (Gyárfás, 1997)**.** *If the vertices of every path in a graph G induce a 3-colorable subgraph, then is G 4-colorable? Is there a constant C such that G is C-colorable?*

For an n-vertex graph G, it is known that $\chi(G) \leq 3 \log n$ because $V(G)$ can be partitioned into $\log n$ paths and 3 colors can be used on each path. This is one of the 25 pretty graph coloring problems by Jensen and Toft.

2. Seog-Jin Kim

Given two graphs G_1 and G_2, the *Cartesian graph product* $G_1 \square G_2$ is the graph on the vertex set $V(G_1) \times V(G_2)$ where $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ in $G_1 \square G_2$ if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The *square* G^2 of a graph G is the graph on the same vertex set $V(G)$ where two vertices of G^2 are adjacent if and only if their distance in G is at most 2.

Let $G_n = C_n \square C_n \square C_n$. The following results are known:

- $\chi(G_7^2) = 7$,
- $\chi(G_n^2) \geq 7$ for all n,
- $\chi(G_n^2) > 7$ for all $n \geq 3$ that is not divisible by 7,
- $\chi(G_5^2) = 9$,
- $8 \leq \chi(G_n^2) \leq 9$ for all n that is divisible by 5, and
- $\chi(G_n^2) = 8$ for all n that is divisible by 4.

Question 2. *Is it true that* $\chi(G_n^2) \leq 9$ *for all* $n \geq 3$*?*

Question 3. *Is it true that* $\chi(G_{n+1}^2) \leq \chi(G_n^2) + 1$ *for all* n*?*

Note that if Question 2 is true, then we have $\chi(G_n^2) \leq 13$ for all n.

3. Sang-il Oum

The *intersection graph* of a family $\mathcal{F}$ of sets is the graph with vertex set $\mathcal{F}$ where two members of $\mathcal{F}$ are adjacent if and only if they have common elements. A *unit disk graph* is the intersection graph of a family of unit disks in the Euclidean plane.

Peeters [7] proved that the choromatic number of a unit disk graph with clique number ω is at most $3\omega - 2$.

Question 4. *Does a unit disk graph with clique number ω have chromatic number at most $3\omega - 3$?*

Question 5. *Is there a constant C such that the chromatic number of a unit disk graph with clique number ω is at most $2.99\omega - C$?*

Note that the coefficient of ω in Question 5 cannot be better than 1.5, see [6].

Remark. A graph G is *m-degenerate* if every subgraph H of G has a vertex of degree at most m in H. Kim, Kostochka, and Nakprasit [3] proved that the intersection graph of a family of convex sets obtained by translations of a fixed convex set in the plane with clique number ω is $(3\omega - 3)$-degenerate. This bound is sharp. Since every m-degenerate graph is $(m + 1)$-colorable for an integer m, we also have that the chromatic number of a unit disk graph with clique number ω is at most $3\omega - 2$. There is a survey paper [4] in this area by Kostochka.

4. Ilkyoo Choi

Let $\mathcal{C}$ be a set of circles in the plane. The circles may have different radii. No three circles in $\mathcal{C}$ are tangent to each other at a single point. Let G be the graph with vertex set $\mathcal{C}$ where two vertices are adjacent if and only if the two corresponding circles are tangent to each other.

Question 6 (Ringel's Circle Problem)**.** *Is there a constant upper bound on the chromatic number of G?*

It is known that the upper bound is at least 5.

5. Jaehoon Kim

The following is conjectured by Carsten Thomassen.

Conjecture 5.1. *For all positive integers g and d, there exists an integer N such that every graph of average degree at least N contains a subgraph of average degree at least d and girth greater than g.*

We may assume that the graphs are bipartite and C_4-free. See [5].

6. Joonkyung Lee

A *strong tree decomposition* of a graph H is a tree decomposition $(T, \{B_v\}_{v \in V(T)})$ of H satisfying the following two extra conditions:

- The induced subgraphs $H[B_x]$ of H on B_x are edge-disjoint trees for all bags.
- For every edge xy in T, there is an isomorphism between the minimum spanning subtrees containing $B_x \cap B_y$ in $H[B_x]$ and in $H[B_y]$ that fixes $B_x \cap B_y$.

A graph H is *strongly tree decomposable* if it admits a strong tree decomposition. One can easily verify that every strongly tree decomposable graph is bipartite, and if a bipartite graph with a partition (A, B) contains a vertex v satisfying $N(v) = A$, then it is strongly tree decomposable.

Question 7. *Let H be a bipartite graph with m edges. Find the minimum c such that adding c edges to H makes H strongly tree decomposable.*

It is known that $c \leq \frac{|V(H)|}{2} -$ (max degree on one side).

This questions is related to Sidorenko's Conjecture, which states that for all bipartite graphs H, the following holds for every graph G:

$$|\mathrm{Hom}(H, G)| \geq |V(G)|^{|V(H)|} \left(\frac{2|E(G)|}{|V(G)|^2} \right)^{|E(H)|}, \tag{1}$$

where $\mathrm{Hom}(H, G)$ denotes the set of all homomorphisms from H to G.

We say that a graph H has Sidorenko's property if (1) holds for every graph G. Colon, Kim, Lee, and Lee [2] proved that if H is strongly tree decomposable, then H has Sidorenko's property. If we find c in Question 7, then the following can be proven. For all strongly tree decomposable graphs H, the following holds for every graph G:

$$|\mathrm{Hom}(H, G)| \geq |V(G)|^{|V(H)|} \left(\frac{2|E(G)|}{|V(G)|^2} \right)^{|E(H)|+c}. \tag{2}$$

7. Seog-Jin Kim

For integers n and k satisfying $k \geq 1$ and $n \geq 2k+1$, the *Kneser graph* $K(n,k)$ has all k-element subsets of $\{1, 2, \ldots, n\}$ as vertices, and two vertices are adjacent if and only if the two corresponding sets are disjoint.

Question 8. *Does $K(2k+1, k)$ have a Hamiltonian cycle for all $k \neq 2$?*

Note that for $k = 2$, since $K(5,2)$ is the Peterssn graph, it has no Hamiltonian cycle. Chen [1] proved that for any $k \geq 1$ and $n \geq 2.62k + 1$, the Kneser graph $K(n,k)$ has a Hamiltonian cycle.

8. Seungsang Oh

The *Ramsey number* is the minimum number $R(m,n)$ such that all graphs having $R(m,n)$ vertices contain either a clique of size m or an independent set of size n.

The determination of the value of $\lim R(k,k)^{1/k}$, if it exists, is one of the major open problems in Ramsey theory. It is known that

$$\sqrt{2} \leq \liminf R(k,k)^{1/k} \leq \limsup R(k,k)^{1/k} \leq 4.$$

This bound will be improved using $R(3,2) = 3$ if one can answer YES for the following question.

Question 9. *Is it true that*

$$(R(m,n)-1)(R(3,2)-1) \leq R(m+2,n+1)-1?$$

References

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