SCHOOL & WORKSHOP ON GEOMETRIC ANALYSIS

7 – 11 DECEMBER 2015 KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL

7 (Monday) – Morning Session

Chair: Jaigyoung Choe

10:00 – 10:50 **Haizhong Li** (Tsinghua University, Beijing)
A new characterization of the Clifford torus as a Lagrangian self-shrinker

11:10 – 12:00 **Keomkyo Seo** (Sookmyung Women's University)
Characterizations of Clifford hypersurfaces in a unit sphere

7 (Monday) – Afternoon Session

Chair: Benjamin Andrews

2:00 – 2:50 **Qi Ding** (Shanghai Center for Mathematical Sciences, Fudan University)
Minimal cones and self-expanding solutions to mean curvature flows

3:10 – 4:00 **Otis Chodosh** (University of Cambridge)
Area minimizing surfaces in asymptotically flat three-manifolds

Chair: Juncheol Pyo

4:20 – 5:10 **Jingyi Chen** (University of British Columbia)
Radially symmetric solutions to the Willmore surface equation

8 (Tuesday) – Morning Session

Chair: Sung-Hong Min

10:00 – 10:50 **Benjamin Andrews** (Australian National University)
Multi-point maximum principles in geometry I

11:10 – 12:00 **Nicolaos Kapouleas** (Brown University)
Gluing constructions in differential geometry I

8 (Tuesday) – Afternoon Session

Chair: Jingyi Chen

2:00 – 2:50 **Martin Li** (The Chinese University of Hong Kong)
Constructions of free boundary minimal surfaces – old and new

3:10 – 4:00 **Tobias Lamm** (Karlsruhe Institute of Technology)
Conformal Willmore tori in $\mathbb{R}^4$

Chair: Sung-Ho Park

4:20 – 5:10 **Mario Micallef** (University of Warwick)
Limits of α*-harmonic maps*

9 (Wednesday) – Morning Session

Chair: Haizhong Li

10:00 – 10:50 **Benjamin Andrews** (Australian National University)
Multi-point maximum principles in geometry II

11:10 – 12:00 **Nicolaos Kapouleas** (Brown University)
Gluing constructions in differential geometry II

10 (Thursday) – Morning Session

Chair: Ahmad El Soufi

10:00 – 10:50 **Benjamin Andrews** (Australian National University)
Multi-point maximum principles in geometry III

11:10 – 12:00 **Nicolaos Kapouleas** (Brown University)
Gluing constructions in differential geometry III

10 (Thursday) – Afternoon Session

Chair: Nicolaos Kapouleas

2:00 – 2:50 **David Wiygul** (University of California, Irvine)
Minimal surfaces in the 3*-sphere by stacking tori*

3:10 – 4:00 **Xiang Ma** (Peking University)
A Fenchel-type inequality for closed spacelike curves in 3*-dimensional Lorentz space*

Chair: Keomkyo Seo

4:20 – 5:10 **Ahmad El Soufi** (Université François Rabelais de Tours)
Geometry of manifolds with large eigenvalues

11 (Friday) – Morning Session

Chair: Mu-Tao Wang

10:00 – 10:50 **Tommaso Pacini** (Scuola Normale Superiore, Pisa)
Complexified diffeomorphism groups and the space of totally real submanifolds

11:10 – 12:00 **Jason Lotay** (University College London)
The Laplacian flow in G_2 geometry

11 (Fri) – Afternoon Session

Chair: Mario Micallef

2:00 – 2:50 **Marcos Cavalcante** (Federal University of Alagoas)
Uniqueness theorems for fully nonlinear conformal equations on subdomains of the sphere

3:10 – 4:00 **Qing-Ming Cheng** (Fukuoka University)
Geometry of critical points of weighted area functional

Chair: Seong-Deog Yang

4:20 – 5:10 **Mu-Tao Wang** (Columbia University)
Quasi-local mass and isometric embedding

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1. Mini-courses

1.1. Multi-point maximum principles in geometry.

Benjamin Andrews (Australian National University)

In these talks I will illustrate the application of maximum principle arguments for functions involving several points to control the global behaviour of solutions of geometric flows. The three talks apply these ideas in quite different contexts, unified by the underlying methods.

Lecture 1: Isoperimetric profiles and their application in Ricci flow and curve shortening flow

I will show how ideas of Hamilton can be strengthened to give very strong control on the isoperimetric profile of a solution of Ricci flow on the two-dimensional sphere, directly implying exponential convergence to a constant curvature metric. Similarly, previous estimates of Hamilton and Huisken can be improved to give a remarkably direct proof of Grayson's theorem, that embedded convex curves in the plane become circular in shape as they contract to points under the curve-shortening flow.

Lecture 2: Moduli of continuity, moduli of convexity and the fundamental gap conjecture

In this lecture I will use similar arguments (in particular, developing maximum principles for functions involving several points) to control the modulus of continuity for solutions of the heat equation in various contexts, deriving sharp inequalities on eigenvalues as a consequence. In particular I will describe my proof with Julie Clutterbuck of the fundamental gap conjecture for convex Euclidean domains.

Lecture 3: Non-collapsing for mean curvature flow and the Lawson and Pinkall-Sterling conjectures

In this lecture I will show how similar multi-point maximum principles can be applied to the mean curvature flow of hypersurfaces, and prove a *non-collapsing* estimate for mean-convex solutions. I will briefly describe some of the applications of this estimate in the analysis of mean curvature flow, and perhaps some extensions to a wider family of evolution equations, and will outline how the argument is used in the proof of the Lawson conjecture (for minimal tori in the three-sphere) and the Pinkall-Sterling conjecture (for CMC tori).

1.2. **Gluing constructions in differential geometry.**

Nicolaos Kapouleas (Brown University)

In these talks I will discuss various gluing constructions which roughly follow the ideas in [R. Schoen: CPAM 1988] and [N. Kapouleas: Annals 1990]. I will also discuss the various developments and refinements in methodology which were employed to make these constructions possible. I will roughly order the presentation historically presenting the older constructions first.

Lecture 1: Gluing constructions for constant mean curvature surfaces

I will outline the general framework for the gluing constructions and then concentrate on the main ideas in [Kapouleas: Annals 1990]. I will then discuss the extensions and modifications in [Kapouleas: JDG 1992] and [Kapouleas: Inventiones 1995] and explain the main ideas involved including what I call *Geometric Principle'*. Finally I will briefly discuss two recent papers [Breiner-Kapouleas: Math. Annalen 2014] and [Breiner-Kapouleas: Preprint close to completion].

Lecture 2: Gluing constructions for minimal surfaces

I will first discuss desingularization constructions for minimal surfaces, starting with the $O(2)$-invariant initial configuration case as in [Kapouleas: JDG 1997]. I will then briefly discuss some recent constructions with more symmetry in various settings (see also Martin Li's talk). Finally I will outline and discuss extensions to less symmetric settings or settings without any symmetries as presented in [Kapouleas: Clay proceedings, Vol.2, 2005], [Kapouleas: ALM, vol 20, 2011], and further ongoing work.

I will then start discussing doubling constructions for minimal surfaces which I will finish in the beginning of my third talk. For doubling constructions I will first discuss the general motivation and framework and I will mention briefly the earlier work in [Kapouleas-Yang: 2010] (see also David Wiygul's talk). I will then concentrate on the ideas in [Kapouleas: arXiv:1409.0226, 2014] and in particular I will present in some detail the Linearized Doubling methodology.

Lecture 3: Other gluing constructions

I will first finish the discussion of the doubling constructions and then I will discuss constructions for Einstein four-manifolds and some related ancient solutions for the Ricci-flow as in [Brendle-Kapouleas: arXiv:1405.0056, 2014]. Finally to the extent that time permits I will discuss gluing constructions for Special Lagrangian cones as in [Haskins-Kapouleas: Inventiones 2007] and [Haskins-Kapouleas: ALM, vol.7, 2008].

References

[1] Schoen, Richard M. *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. 41 (1988), no. 3, 317–392.

[2] Kapouleas, Nicolaos *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2) 131 (1990), no. 2, 239–330.

[3] Kapouleas, Nicolaos *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geom. 33 (1991), no. 3, 683–715.

[4] Kapouleas, Nikolaos *Constant mean curvature surfaces constructed by fusing Wente tori*, Invent. Math. 119 (1995), no. 3, 443–518.

[5] Kapouleas, Nikolaos *Complete embedded minimal surfaces of finite total curvature*. J. Differential Geom. 47 (1997), no. 1, 95?169.

[6] Kapouleas, Nikolaos *Doubling and desingularization constructions for minimal surfaces. Surveys in geometric analysis and relativity*, 281–325, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, 2011.

[7] Kapouleas, Nikolaos *Constructions of minimal surfaces by gluing minimal immersions*. Global theory of minimal surfaces, 489–524, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI, 2005.

[8] Haskins, Mark; Kapouleas, Nikolaos *Special Lagrangian cones with higher genus links*, Invent. Math. 167 (2007), no. 2, 223–294.

[9] Haskins, Mark; Kapouleas, Nikolaos *Gluing constructions of special Lagrangian cones. Handbook of geometric analysis*, No. 1, 77–145, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

[10] Kapouleas, Nikolaos; Yang, Seong-Deog *Minimal surfaces in the three-sphere by doubling the Clifford torus*, Amer. J. Math. 132 (2010), no. 2, 257?295.

[11] Kapouleas, Nikolaos *Minimal surfaces in the round three-sphere by doubling the equatorial two-sphere, I.*, arXiv preprint arXiv:1409.0226 (2014).

[12] Breiner, Christine; Kapouleas, Nikolaos *Embedded constant mean curvature surfaces in Euclidean three-space*, Math. Ann. 360 (2014), no. 3-4, 1041–1108.

[13] Brendle, Simon; Kapouleas, Nikolaos *Gluing Eguchi-Hanson metrics and a question of Page*, arXiv preprint arXiv:1405.0056 (2014).

2. Talks

2.1. Uniqueness theorems for fully nonlinear conformal equations on subdomains of the sphere.

Marcos Cavalcante (Federal University of Alagoas)

In this talk we extend Escobar's classification result [1] to elliptic fully nonlinear conformal equations on certain subdomains of the sphere with prescribed constant mean curvature on its boundary. Such subdomains are the hemisphere with prescribed constant mean curvature on its boundary, and annular domains with minimal boundary. This is joint work with **José Espinar** (IMPA).

References

[1] José F. Escobar, *Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate*, Comm. Pure Appl. Math. **43** (1990), no. 7, 857–883.

2.2. Radially symmetric solutions to the Willmore surface equation.

Jingyi Chen (University of British Columbia)

We show that a smooth radially symmetric solution u to the graphic Willmore surface equation is either a constant or the defining function of a half sphere in $\mathbb{R}^3$. In particular, radially symmetric entire Willmore graphs in R^3 must be flat. When u is a smooth radial solution over a punctured disk $D(r) - \{0\}$ and is in $C^1(D(r))$, we show that there exist a constant λ and a function β in $C^0(D(r))$ such that $u''(r) = \frac{\lambda}{2}\log r + \beta(r)$; moreover, the graph of u is contained in a graphical region of an inverted catenoid which is uniquely determined by λ and $\beta(0)$. It is also shown that a radial solution on the punctured disk extends to a C^1 function on the disk when the mean curvature is square integrable. This is joint work with **Yuxiang Li**.

2.3. **Geometry of critical points of weighted area functional.**

Qing-Ming Cheng (Fukuoka University)

In this talk, we talk about weighted area functional for the weighted volume-preserving variations. We consider critical points of the weighted area functional for the weighted volume-preserving variations, which are called λ-hypersurface of weighted volume-preserving mean curvature flow. Examples of compact embedded λ-hypersurfaces are constructed. Complete λ-hypersurfaces are studied.

2.4. **Area minimizing surfaces in asymptotically flat three-manifolds.**

Otis Chodosh (University of Cambridge)

I will sketch the proof of the following result: If an asymptotically flat three manifold with non-negative scalar curvature contains an unbounded area minimizing surface, then the ambient manifold is flat. Time permitting, I will discuss applications to the isoperimetric problem and classification of static asymptotically flat three manifolds. This is joint work **M. Eichmair**.

2.5. **Minimal cones and self-expanding solutions to mean curvature flows.**

Qi Ding (Shanghai Center for Mathematical Sciences, Fudan University)

In this talk, I shall talk about the theory of self-expanding solutions to mean curvature flows in Euclidean space such as the existence and uniqueness. They can be seen as natural perturbation of minimal cones. In particular, an embedded mean convex (but not area-minimizing) cone can be foliated by mean curvature flow whose every leaf is a smooth self-expanding hypersurface with positive mean curvature.

2.6. Geometry of manifolds with large eigenvalues.

Ahmad El Soufi (Université François Rabelais de Tours)

The sequence of eigenvalues of the Dirichlet Laplacian on a bounded Euclidean domain satisfies several restrictive conditions such as Faber-Krahn isoperimetric inequality, Li-Yau inequality, Payne-Plya-Weinberger universal inequalities, etc. The situation changes completely as soon as Euclidean domains are replaced by closed hypersurfaces. For example, any compact hypersurface of dimension $n \geq 3$ of the Euclidean space can be deformed so that its first positive eigenvalue tends to infinity while its measure remains constant.

In this talk, I will discuss the effect of the geometry on the eigenvalues and present a set of results that allow us to understand what geometric situations can lead to the existence of large eigenvalues.

2.7. Conformal Willmore tori in $\mathbb{R}^4$.

Tobias Lamm (Karlsruhe Institute of Technology)

In this talk I am going to present recent existence and non-existence results for conformal Willmore Tori in $\mathbb{R}^4$ which were obtained in a collaboration with **Reiner M. Schätzle** (Tübingen).

2.8. A new characterization of the Clifford torus as a Lagrangian self-shrinker.

Haizhong Li (Tsinghua University, Beijing)

Self-shrinkers play an important role in the study of the mean curvature flow. Not only they correspond to self-shrinking solutions to the mean curvature flow, but also they describe all possible Type I blow ups at a given singularity of the mean curvature flow.

In this talk, we will show that the Clifford torus is the unique compact Lagrangian self-shrinker in 2-dimensional complex plane with $|A|^2 \leq 2$, which gives an affirmative answer to Castro-Lerma's conjecture. This is joint work with **Xianfeng Wang**.

2.9. **Constructions of free boundary minimal surfaces – Old and New.**

Martin Li (The Chinese University of Hong Kong)

Free boundary minimal surfaces are minimal surfaces which meet a constraint surface orthogonally along its boundaries. They arise as critical points to the area functional for surfaces whose boundaries are allowed to move *freely* on the constraint surface. In this talk, we will give a brief survey about some classical and recent constructions of free boundary minimal surfaces.

2.10. **The Laplacian flow in G_2 geometry.**

Jason Lotay (University College London)

A key challenge in Riemannian geometry is to find Ricci-flat metrics on compact manifolds, which has led to fundamental breakthroughs, particularly using geometric analysis methods. All non-trivial examples of such metrics have special holonomy, and the only special holonomy metrics which can occur in odd dimensions must be in dimension 7 and have holonomy G_2. I will describe recent progress on a proposed geometric flow method for finding metrics with holonomy G_2, called the Laplacian flow. This is joint work with **Yong Wei**.

2.11. **A Fenchel-type inequality for closed spacelike curves in 3-dimensional Lorentz space.**

Xiang Ma (Peking Univeristy)

In 3-dimensional Lorentz space we consider closed curves which are assumed to be strong spacelike (i.e. the tangent and normal directions are both spacelike at every point) with index 1 (i.e., it winds around some timelike axis with linking number 1). We show that the total curvature must be no more than 2π. Among several possible ways to prove this result, we will focus on the curve shortening flow. This is joint work with PhD candidate **Nan Ye**.

2.12. Limits of α-harmonic maps.

Mario Micallef (University of Warwick)

In a famous paper, Sacks and Uhlenbeck introduced a perturbation of the Dirichlet energy, the so-called α-energy E_α, $\alpha > 1$, to construct non-trivial harmonic maps of the two-sphere in manifolds with a non-contractible universal cover. The Dirichlet energy corresponds to $\alpha = 1$ and, as α decreases to 1, critical points of E_α are known to converge to harmonic maps in a suitable sense.

However, in a joint work with **Tobias Lamm** and **Andrea Malchiodi**, we show that not every harmonic map can be approximated by critical points of such perturbed energies. Indeed, we prove that constant maps and the rotations of $\mathbb{S}^2$ are the only critical points of E_α for maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ whose α-energy lies below some threshold, which is independent of α (sufficiently close to 1). In particular, nontrivial dilations (which are harmonic) cannot arise as strong limits of α-harmonic maps. We shall also discuss similar results for other perturbations of the Dirichlet energy.

2.13. Complexified diffeomorphism groups and the space of totally real submanifolds.

Tommaso Pacini (Scuola Normale Superiore, Pisa)

We will provide an overview of recent work, joint with **Jason Lotay**, concerning the geometry of totally real submanifolds. We will discuss relationships with the theory of Lagrangian submanifolds and mean curvature flow, and analogies with the Donaldson–Semmes viewpoint on extremal Kahler metrics. Time allowing, we will also present some open problems.

2.14. Characterizations of Clifford hypersurfaces in a unit sphere.

Keomkyo Seo (Sookmyung Women's University)

Let $\Sigma^{n\geq 3}$ be a compact embedded hypersurface in a unit sphere with constant mean curvature $H \geq 0$ and with two distinct principal curvatures λ and μ of multiplicity $n-1$ and 1, respectively. It is known that if $\lambda > \mu$, there exist many compact embedded constant mean curvature hypersurfaces.

In this talk, we prove that if $\mu > \lambda$, then Σ is congruent to a Clifford hypersurface. We also give a sharp curvature integral inequality for hypersurfaces in a unit sphere with constant m-th order mean curvature and with two distinct principal curvatures, which generalizes Simons' integral inequality and gives another characterization of Clifford hypersurfaces in a unit sphere. This is joint work with **Sung-Hong Min**.

2.15. Quasi-local mass and isometric embedding.

Mu-Tao Wang (Columbia University)

The positive mass theorem, a greatest accomplishment in the theory of general relativity, paves the way for a deep understanding of the notion of mass. Many important tools in geometric analysis such as minimal surfaces and the conformal method were brought in to study this fundamental yet subtle notion. Another important geometric PDE, the isometric embedding equation, arose naturally in the study of quasi-local mass recently. I shall review these new developments, and in particular discuss how the rigidity property of mass (when is mass equal to zero) is intimately related to the uniqueness of the isometric embedding problem. This talk is based on joint work with **Po-Ning Chen** and **Shing-Tung Yau**.

2.16. Minimal surfaces in the 3-sphere by stacking tori.

David Wiygul (University of California, Irvine)

Kapouleas and Yang have used gluing methods to construct sequences of minimal embeddings in the round 3-sphere converging to the Clifford torus counted with multiplicity 2. Each of their surfaces, which they call doublings of the Clifford torus, resembles a pair of nearby coaxial tori connected by small catenoidal tunnels and has symmetries exchanging the two tori. I will describe an extension of their work which yields doublings that admit no such symmetries as well as examples incorporating an arbitrary (finite) number of tori, that is Clifford torus triplings, quadruplings, and so on.