Lecture Note on Well-posedness Theory for the Navier-Stokes Equations*

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*The note is based on a series of lectures at the KAIST branch of CMC(Center for Mathematical Challenges) from August 3, 2015 to August 7, 2015. Hyunseok Kim would like to express his sincere gratitude to the program organizers, especially Professors Jaeyoung Byeon and Soonsik Kwon at KAIST for their kind invitation and warm encouragement.

Introduction

The motion of an incompressible viscous Newtonian fluid is governed by a nonlinear system of partial differential equations, called the Navier-Stokes equations. The purpose of this note is to introduce a classical theory of the well-posedness for the Navier-Stokes equations in three-dimensional domains.

The note consists of three sections. The first section, Section 1, is devoted to proving some preliminary results which have been essential tools to the mathematical theory of fluid mechanics. Of particular importance to the (incompressible) Navier-Stokes equations is the De Rham theorem in Sobolev spaces of arbitrary order which characterizes the gradient of a scalar field; such a result allows us to deduce the existence of a pressure from the weak formulation for the Navier-Stokes equations. Among several approaches to the De Rham theorem, we follow a duality approach based on the solvability of the equation $\operatorname{div} u = g$ with u being an unknown vector field. As a byproduct, we prove the so-called Helmholtz-Weyl decomposition of vector fields in Lebesgue spaces. The section ends with discussing the well-posedness for the stationary Stokes equations and introducing the Stokes operator.

The goal of Section 2 is to develop the existence theory of global weak solutions due to Leray. Weak solutions of the Navier-Stokes equations may be defined, as usual, by multiplying the equations by divergence-free test functions and integrating by parts. We derive several equivalent definitions of weak solutions, which are indeed necessary to prove the existence and regularity of weak solutions. On the other hand, by virtue of the nonlinear character of the problem, any existence proof of weak solutions should rely on suitable compactness results in vector-valued Lebesgue spaces. The so-called Aubin-Lions compactness lemma is proved in a quite self-contained manner. Finally, the global existence of weak solutions is established, by applying the standard Faedo-Galerkin method.

The uniqueness and regularity of weak solutions have been the most outstanding open questions in the mathematical fluid mechanics and are closely related to one of the seven Clay Millennium Problems: the so-called "Navier-Stokes existence and smoothness problem". Some partial answers are given in Section 3. First, we establish weak-strong uniqueness results which show that weak solutions are unique if a strong or smooth solution exists. It is also shown that a strong or smooth solution exists at least for a short time interval and globally in time if the data is sufficiently small. The regularity of weak solutions is then studied in the last part of the section. We prove the Leray structure theorem for the singular time set of a weak solution. A refinement due to Scheffer is also discussed. Finally, further regularity of weak solutions is deduced from the classical maximal regularity results for the linear Stokes equations.

1 Preliminaries

1.1 The equation $\operatorname{div} u = g$

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$. Consider the problem of finding a vector field $u = (u^1, ..., u^n) : \Omega \to \mathbb{R}^n$ such that

$$\operatorname{div} u = g \quad \text{in } \Omega, \tag{1}$$

where $g : \Omega \to \mathbb{R}$ is a given scalar function.

A solution of (1) can be constructed by a classical potential technique, provided that g is sufficiently regular. Let Γ be the fundamental solution of the Laplace equation, defined by

$$\Gamma(x) = \begin{cases} \frac{1}{n(2-n)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3 \\ \frac{1}{2\pi} \log |x| & \text{if } n = 2, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$:

$$\omega_n = |B_1(0)| = \frac{2\pi^{n/2}}{n\Gamma(n/2)}.$$

Assume that $g \in C_0^\infty(\Omega)$, and let G be the *Newtonian potential* of g:

$$G(x) = \int_\Omega \Gamma(x-y) g(y)\, dy \quad (x \in \mathbb{R}^n).$$

Then by a straightforward calculation,

$$G \in C^\infty(\mathbb{R}^n) \quad \text{and} \quad \Delta G = g \quad \text{in } \Omega.$$

Moreover, by the Calderón-Zygmund theory,

$$\|\nabla^{m+2} G\|_{q;\mathbb{R}^n} \leq C(m,q,n) \|\nabla^m g\|_{q;\Omega}$$

for every $m \in \mathbb{N} \cup \{0\}$ and $1 < q < \infty$, where

$$\|\nabla^m g\|_{q;\Omega}^q = \sum_{|\alpha|=m} \|D^\alpha g\|_{L^q(\Omega)}^q.$$

Hence the vector field

$$u(x) = \int_\Omega \nabla\Gamma(x-y) g(y)\, dy = \int_\Omega \frac{1}{n\omega_n} \frac{(x-y)}{|x-y|^n} g(y)\, dy \tag{2}$$

satisfies

$$u \in C^\infty(\mathbb{R}^n)^n := C^\infty(\mathbb{R}^n;\mathbb{R}^n), \quad \operatorname{div} u = g \quad \text{in } \Omega,$$

and

$$\|\nabla^{m+1}u\|_{q;\mathbb{R}^n} \le C(m,q,n)\|\nabla^m g\|_{q;\Omega}$$

for every $m \in \mathbb{N}\cup\{0\}$ and $1 < q < \infty$. Denote by $W_0^{m,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,q}(\Omega)$; thus $W_0^{0,q}(\Omega) = L^q(\Omega)$. Then by a standard continuity argument, we deduce that for each $g \in W_0^{m,q}(\Omega)$ there exists at least one $u \in W^{m+1,q}(\Omega)^n$ such that

$$\operatorname{div} u = g \quad \text{in } \Omega$$

and

$$\|\nabla u\|_{m,q;\Omega} \le C(m,q,n)\|g\|_{m,q;\Omega},$$

where

$$\|g\|_{m,q;\Omega}^q = \sum_{j=0}^m \|\nabla^j g\|_{q;\Omega}^q.$$

But for various applications to fluid mechanics, the vector field u should satisfy the additional property that

$$u \in W_0^{m+1,q}(\Omega)^n.$$

To find such a vector field u is a fundamental problem in mathematical fluid mechanics, which has been resolved by many researchers for quite general domains Ω including bounded Lipschitz domains. By a *bounded Lipschitz domain* in $\mathbb{R}^n$, we mean a bounded, open, and connected subset Ω of $\mathbb{R}^n$ such that for each $x_0 \in \partial\Omega$ there exist a number $r > 0$ and a Lipschitz continuous function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n < \phi(x_1, ..., x_{n-1})\}$$

and

$$\partial\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : x_n = \phi(x_1, ..., x_{n-1})\}$$

in some coordinate system $\{x_1, ..., x_n\}$ with the origin at x_0[1]. Here $B_r(x_0)$ denotes the open ball of radius r centered at x_0: $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

Following a classical approach due to Bogovskiĭ [1], we shall prove the following results.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and $\zeta \in C_0^\infty(\Omega)$ a fixed function with $\int_\Omega \zeta\, dx = 1$. Then there exists a linear operator*

$$\mathcal{B}_\Omega : C_0^\infty(\Omega) \to C_0^\infty(\Omega)^n$$

[1] Ω is said to be of class C^k if ϕ is a C^k-function

such that for each $g \in C_0^\infty(\Omega)$, the vector field $u = \mathcal{B}_\Omega[g]$ satisfies

$$\operatorname{div} u = g - \left(\int_\Omega g\,dx\right)\zeta \quad \textit{in } \Omega$$

and

$$\|u\|_{m+1,q;\Omega} \le C(m,q,n,\Omega)\|g\|_{m,q;\Omega}$$

for every $m \in \mathbb{N}\cup\{0\}$ and $1 < q < \infty$. Moreover, by continuity, $\mathcal{B}_\Omega$ can be extended uniquely to a bounded linear operator from $W_0^{m,q}(\Omega)$ into $W_0^{m+1,q}(\Omega)^n$, called the Bogovskiĭ operator and denoted again by $\mathcal{B}_\Omega$.

Corollary 1.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $m \in \mathbb{N}\cup\{0\}$ and $1 < q < \infty$. Then for each $g \in W_0^{m,q}(\Omega)$ with $\int_\Omega g\,dx = 0$, there exists $u \in W_0^{m+1,q}(\Omega)^n$ such that*

$$\operatorname{div} u = g \quad \textit{in } \Omega$$

and

$$\|u\|_{m+1,q;\Omega} \le C(m,q,n,\Omega)\|g\|_{m,q;\Omega}.$$

Remark 1.3. *The so-called compatibility condition $\int_\Omega g\,dx = 0$ is necessary for the existence of $u \in W_0^{1,q}(\Omega)^n$ such that $\operatorname{div} u = g$ in Ω; indeed by the divergence theorem,*

$$\int_\Omega g\,dx = \lim_{k\to\infty}\int_\Omega \operatorname{div} u_k\,dx = \lim_{k\to\infty}\int_{\partial\Omega} u_k\cdot\nu\,d\sigma = 0,$$

where $\{u_k\}$ is any sequence in $C_0^\infty(\Omega)^n$ with $u_k \to u$ in $W^{1,q}(\Omega)^n$.

Theorem 1.1 can be proved by using an explicit representation formula due to Bogovskiĭ [1], if the domain Ω is star-shaped with respect to a ball $B \subset \Omega$; that is, $\lambda x_0 + (1-\lambda)x \in \Omega$ for every $x_0 \in B$, $x \in \Omega$, and $\lambda \in [0,1]$. To state his result, we fix a function $\eta \in C_0^\infty(B_1(0);[0,1])$ with $\int_{B_1(0)} \eta\,dx = 1$ and define $\eta_R(x) = R^{-n}\eta(x/R)$. Note that $\eta_R \in C_0^\infty(B_R(0))$ and $\int_{B_R(0)} \eta_R\,dx = 1$.

Lemma 1.4 (Bogovskiĭ [1])**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ that is star-shaped with respect to an open ball $B = B_R(0)$ with $\overline{B} \subset \Omega$. For each $g \in C_0^\infty(\Omega)$, we define*

$$\begin{aligned} u(x) &= \mathcal{B}_\Omega[g](x) \\ &= \int_\Omega g(y)\left[\frac{(x-y)}{|x-y|^n}\int_{|x-y|}^\infty \eta_R\left(y + r\frac{x-y}{|x-y|}\right) r^{n-1}\,dr\right] dy \end{aligned}$$

for all $x \in \Omega$. Then

$$u \in C_0^\infty(\Omega)^n, \quad \operatorname{div} u = g - \left(\int_\Omega g\,dx\right)\eta_R \quad \textit{in } \Omega,$$

and

$$\|\nabla u\|_{m,q;\Omega} \le C(m,q,n,\delta(\Omega)/R)\|g\|_{m,q;\Omega}$$

for every $m \in \mathbb{N} \cup \{0\}$ *and* $1 < q < \infty$*, where* $\delta(\Omega)$ *is the diameter of* Ω*.*

Proof. **Step 1**. Note first that if $y \in \Omega$ and $y \neq x$, then

$$\begin{aligned}\int_{|x-y|}^{\infty} \eta_R\left(y + r\frac{x-y}{|x-y|}\right) r^{n-1}\, dr &\le \int_{|x-y|}^{R+|y|} \frac{1}{R^n} r^{n-1}\, dr \\ &\le C(n,\delta(\Omega)/R)).\end{aligned}$$

Hence $u(x)$ is well-defined for all $x \in \Omega$. By a change of variables, we have

$$u(x) = \int_\Omega g(y)\left[(x-y)\int_1^\infty \eta_R\left(y + r(x-y)\right) r^{n-1}\, dr\right] dy$$

for $x \in \Omega$. This implies, in particular, that u is supported in the compact set

$$E = \{\lambda x_0 + (1-\lambda)y : x_0 \in \operatorname{supp}(\eta_R),\ y \in \operatorname{supp}(g),\ \lambda \in [0,1]\}.$$

Since $\operatorname{supp}(\eta_R) \subset B$ and Ω is star-shaped with respect to B, it follows that $E \subset \Omega$. Hence u has compact support in Ω. Moreover, noting that

$$u(x) = \int_{\mathbb{R}^n} g(x-z)\left[z\int_0^\infty \eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right] dz,$$

we deduce that $u \in C_0^\infty(\Omega)$; for instance,

$$\begin{aligned}D_j u^i(x) &= \int_{\mathbb{R}^n} D_j g(x-z)\left[z_i\int_0^\infty \eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right] dz \\ &\quad + \int_{\mathbb{R}^n} g(x-z)\left[z_i\int_0^\infty D_j\eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right] dz\end{aligned}$$

for $i, j = 1, ..., n$.

Step 2. Fix $i, j = 1, ..., n$. Then for $x \in \Omega$, we have

$$\begin{aligned}D_j u^i(x) &= \lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)^c} \frac{\partial}{\partial z_j}\left[-g(x-z)\right]\left[z_i\int_0^\infty \eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right] dz \\ &\quad + \lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)^c} g(x-z)\left[z_i\int_0^\infty D_j\eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right] dz.\end{aligned}$$

Using the integration by parts, we thus obtain

$$\begin{aligned}D_j u^i(x) &= \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(0)} g(x-z)\left[z_i\int_0^\infty \eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right]\frac{z_j}{|z|}\, d\sigma(z) \\ &\quad + \lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)^c} g(x-z)\left[\delta_{ij}\int_0^\infty \eta_R\left(x + rz\right)(r+1)^{n-1}\, dr\right. \\ &\qquad\qquad \left. + z_i\int_0^\infty D_j\eta_R\left(x + rz\right)(r+1)^{n}\, dr\right] dz \\ &= I_1^{ij}(x) + I_2^{ij}(x).\end{aligned}$$

It is easy to compute $I_1^{ij}(x)$:

$$\begin{aligned}
I_1^{ij}(x) &= \lim_{\varepsilon\to 0}\int_{\partial B_1(0)} g(x-\varepsilon z)\left[\varepsilon z_i z_j\int_0^\infty \eta_R\left(x+\varepsilon r z\right)(r+1)^{n-1}\,dr\right]\varepsilon^{n-1}d\sigma(z)\\
&= \lim_{\varepsilon\to 0}\int_{\partial B_1(0)} g(x-\varepsilon z)\left[z_i z_j\int_0^\infty \eta_R\left(x+rz\right)(r+\varepsilon)^{n-1}\,dr\right]d\sigma(z)\\
&= g(x)\int_{\partial B_1(0)} z_i z_j\left[\int_0^\infty \eta_R\left(x+rz\right)r^{n-1}\,dr\right]d\sigma(z)\\
&= g(x)\int_{\mathbb{R}^n}\frac{y_i y_j}{|y|^2}\eta_R(x+y)\,dy.
\end{aligned}$$

Hence

$$|I_1^{ij}(x)|\le |g(x)|\int_{\mathbb{R}^n}\eta_R(x+y)\,dy=|g(x)|$$

and

$$\sum_{i=1}^n I_1^{ii}(x)=g(x)\int_{\mathbb{R}^n}\eta_R(x+y)\,dy=g(x).$$

It is easier to compute $\sum_{i=1}^n I_2^{ii}(x)$:

$$\begin{aligned}
\sum_{i=1}^n I_2^{ii}(x) &= \lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)^c} g(x-z)\left[n\int_0^\infty \eta_R\left(x+rz\right)(r+1)^{n-1}\,dr\right.\\
&\qquad\qquad\left.+\int_0^\infty \frac{d}{dr}\left\{\eta_R\left(x+rz\right)\right\}(r+1)^n\,dr\right]dz\\
&= \lim_{\varepsilon\to 0}\int_{B_\varepsilon(0)^c} g(x-z)\left[-\eta_R(x)\right]dz\\
&= -\left(\int_\Omega g\,dx\right)\eta_R(x).
\end{aligned}$$

Step 3. To complete the proof for the case $m=0$, it remains to show that

$$\|I_2^{ij}\|_{q;\Omega}\le C(n,q,\delta(\Omega)/R)\|g\|_{q;\Omega}.$$

To show this, we write

$$I_2^{ij}(x)=\lim_{\varepsilon\to 0}\int_{\Omega\setminus B_\varepsilon(x)} g(y)K^{ij}(x,x-y)\,dy,$$

where

$$K^{ij}(x,z)=\delta_{ij}\int_0^\infty \eta_R\left(x+rz\right)(r+1)^{n-1}\,dr+z_i\int_0^\infty D_j\eta_R\left(x+rz\right)(r+1)^n\,dr$$

for $z\ne 0$. Define

$$K_s^{ij}(x,z)=\delta_{ij}\int_0^\infty \eta_R\left(x+rz\right)r^{n-1}\,dr+z_i\int_0^\infty D_j\eta_R\left(x+rz\right)r^n\,dr$$

and

$$K^{ij}_{ws}(x,z) = K^{ij}(x,z) - K^{ij}_s(x,z).$$

Note that if $x \in \Omega$ and $0 < |z| < \delta(\Omega)$, then

$$\begin{aligned}|K^{ij}_{ws}(x,z)| &\le C\int_0^\infty \left[\eta_R\,(x+rz)\,(r+1)^{n-2} + |z||D\eta_R\,(x+rz)\,|(r+1)^{n-1}\right]\,dr \\ &\le \frac{C(n,\delta(\Omega)/R)}{|z|^{n-1}} \in L^1(\Omega).\end{aligned}$$

Hence by Young's convolution inequality, we obtain

$$\left\|\int_\Omega g(y)K^{ij}_{ws}(\cdot,\cdot - y)\,dy\right\|_{q;\Omega} \le C\|g\|_{q;\Omega}.$$

On the other hand, the singular kernel K^{ij}_s can be written as

$$K^{ij}_s(x,z) = \frac{k^{ij}_s(x,z/|z|)}{|z|^n},$$

where k^{ij}_s is a smooth function on $\mathbb{R}^n \times \partial B_1(0)$ defined by

$$k^{ij}_s(x,\omega) = \delta_{ij}\int_0^\infty \eta_R\,(x+r\omega)\,r^{n-1}\,dr + \omega_i\int_0^\infty D_j\eta_R\,(x+r\omega)\,r^n\,dr.$$

Note that

$$\int_{\partial B_1(0)} k^{ij}_s(x,\omega)\,d\sigma(\omega) = \int_{\mathbb{R}^n}\left[\delta_{ij}\eta_R(x+z) + z_iD_j\eta(x+z)\right]\,dz = 0.$$

Therefore, by the Calderón-Zygumnd theory (see [3, Theorem 2]),

$$\left\|\int_\Omega g(y)K^{ij}_s(\cdot,\cdot - y)\,dy\right\|_{q;\Omega} \le C\|g\|_{q;\Omega}.$$

This completes the proof for the most important case $m = 0$. The above argument can be adapted to prove the lemma for the case $m \ge 1$ but its details is omitted. □

Proof of Theorem 1.1. Here we provide only a sketch of the proof. A more detailed proof is given in Galdi's book [6].

Step 1. First we recall that the bounded Lipschitz domain Ω can be written as the union of a finite number of star-shaped domains; more precisely, there exist a finite number of open sets $G_1, ..., G_m, G_{m+1}, ..., G_{m+k}$ such that

(i) $\overline{\Omega} \subset \cup_{i=1}^{m+k} G_i$, $\partial\Omega \subset \cup_{i=1}^m G_i$;

(ii) $\Omega_i = \Omega \cap G_i$ is star-shaped with respect to an open ball B_i with $\overline{B}_i \subset \Omega_i$ for each $i = 1, ..., m$;

(iii) $\Omega_{m+j} = G_{m+j}$ is an open ball with $\overline{G}_{m+j} \subset \Omega$ for each $j = 1, ..., k$.

Step 2. We next need to prove that if $g \in C_0^\infty(\Omega)$ and $\int_\Omega g\,dx = 0$, then

$$g = \sum_{i=1}^{m+k} g_i$$

for some functions $g_i \in C_0^\infty(\Omega_i)$ with $\int_\Omega g_i\,dx = 0$ given by

$$g_i = \zeta_i g + \sum_{j=1}^{m_i} \left(\int_\Omega \phi_j g\,dx \right) \theta_j,$$

where $m_i \in \mathbb{N}$, $\zeta_i, \theta_j \in C_0^\infty(\Omega_i)$, and $\phi_j \in C^\infty(\mathbb{R}^n)$. Note also that

$$\sum_{i=1}^{m+k} \|g_i\|_{m,q;\Omega} \leq C(m,q,n,\Omega)\|g\|_{m,q;\Omega} \quad (m \geq 0,\ 1 < q < \infty).$$

To show this, we choose functions $\psi_1, ..., \psi_{m+k}$ such that

$$\psi_i \in C_0^\infty(G_i) \quad \text{for } 1 \leq i \leq m+k \quad \text{and} \quad \sum_{i=1}^{m+k} \psi_i = 1 \quad \text{in } \Omega.$$

Then the functions g_i may be defined by

$$g_1 = \psi_1 g - \left(\int_\Omega \psi_1 g\,dx \right) \chi_1, \quad h_1 = g - g_1,$$

$$g_2 = \psi_2 h_1 - \left(\int_\Omega \psi_2 h_1\,dx \right) \chi_2, \quad h_2 = g_1 - g_2,$$

$$\vdots$$

$$g_{m+k} = \psi_{m+k} h_{m+k-1} - \left(\int_\Omega \psi_{m+k} h_{m+k-1}\,dx \right) \chi_{m+k},$$

for some suitable functions $\chi_i \in C_0^\infty(\Omega_i)$ with $\int_\Omega \chi_i\,dx = 0$.

Step 3. Let $g \in C_0^\infty(\Omega)$ be given. Suppose first that $\int_\Omega g\,dx = 0$. Then using Lemma 1.4 together with the decomposition

$$g = \sum_{i=1}^{m+k} g_i$$

from Step 2, we define

$$\mathcal{B}_\Omega[g] = \sum_{i=1}^{m+k} \mathcal{B}_{\Omega_i}[g_i].$$

By Lemma 1.4 and Step 2, we have

$$\mathcal{B}_\Omega[g] \in C_0^\infty(\Omega),$$

$$\operatorname{div} \mathcal{B}_\Omega[g] = \sum_{i=1}^{m+k} \operatorname{div} \mathcal{B}_{\Omega_i}[g_i] = \sum_{i=1}^{m+k} g_i = g \quad \text{in } \Omega,$$

and

$$\begin{aligned} \|\mathcal{B}_\Omega[g]\|_{m+1,q;\Omega} &\le \sum_{i=1}^{m+k} \|\mathcal{B}_{\Omega_i}[g_i]\|_{m+1,q;\Omega_i} \\ &\le C \sum_{i=1}^{m+k} \|g_i\|_{m,q;\Omega} \le C\|g\|_{m,q;\Omega}. \end{aligned}$$

Finally, for general $g \in C_0^\infty(\Omega)$, we define

$$\mathcal{B}_\Omega[g] = \mathcal{B}_\Omega \left[g - \left(\int_\Omega g\, dx \right) \zeta \right].$$

This completes the proof of Theorem 1.1. $\square$

1.2 The Helmholtz-Weyl decomposition

Theorem 1.1 enables us to deduce several important results in fluid mechanics by some elementary arguments. First of all, we prove a quite general result which characterizes the gradient in Sobolev spaces of arbitrary orders.

For $m \ge 1$ and $1 < q < \infty$, we denote by $W^{-m,q}(\Omega)$ the dual space of $W_0^{m,q'}(\Omega)$, where $q' = q/(q-1)$ is the Hölder conjugate of q. The norm on $W^{-m,q}(\Omega)$ is denoted by $\|\cdot\|_{-m,q;\Omega}$:

$$\|f\|_{-m,q;\Omega} = \sup \left\{ < f, \phi > \, : \, \phi \in W_0^{m,q'}(\Omega), \ \|\phi\|_{m,q';\Omega} \le 1 \right\}.$$

Let $C_{0,\sigma}^\infty(\Omega)$ be the space of all divergence-free (or solenoidal) test functions on Ω:

$$C_{0,\sigma}^\infty(\Omega) = \left\{ u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega \right\}.$$

Theorem 1.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $m \in \mathbb{Z}$ and $1 < q < \infty$. If $f \in W^{m,q}(\Omega)^n$ satisfies*

$$\langle f, \Phi \rangle = 0 \quad \textit{for all } \Phi \in C_{0,\sigma}^\infty(\Omega), \tag{3}$$

then there exists $\psi \in W^{m+1,q}(\Omega)$ such that

$$f = \nabla \psi \quad \textit{in } \Omega$$

and

$$\|\psi\|_{m+1,q;\Omega} \le C\|f\|_{m,q;\Omega}$$

for some constant $C = C(m, q, \Omega)$.

Proof. Assume first that $m \leq -1$. Then f is a bounded linear functional on $W_0^{-m,q'}(\Omega)^n$. Since the Bogovskiĭ operator $\mathcal{B} = \mathcal{B}_\Omega$ maps $W_0^{-m-1,q'}(\Omega)$ into $W_0^{-m,q'}(\Omega)^n$ boundedly, it follows that

$$\begin{aligned} < f, \mathcal{B}[g] > \; &\leq ||f||_{m,q;\Omega} ||\mathcal{B}[g]||_{-m,q';\Omega} \\ &\leq C||f||_{m,q;\Omega}||g||_{-m-1,q';\Omega} \end{aligned}$$

for all $g \in W_0^{-m-1,q'}(\Omega)$. Hence if we define ψ by

$$< \psi, g > \; = \; - < f, \mathcal{B}[g] > \quad \text{for all } g \in W_0^{-m-1,q'}(\Omega), \tag{4}$$

then

$$\psi \in W^{m+1,q}(\Omega) \quad \text{and} \quad \|\psi\|_{m+1,q;\Omega} \leq C\|f\|_{m,q;\Omega};$$

if $m = -1$, then $\psi \in L^q(\Omega)$ by the Riesz representation theorem.

Let $\Phi \in C_0^\infty(\Omega)^n$ be given. Then taking $g = \operatorname{div} \Phi$, we have

$$\operatorname{div} \mathcal{B}[g] = \operatorname{div} \Phi - \left(\int_\Omega \operatorname{div} \Phi \, dx \right) \zeta = \operatorname{div} \Phi,$$

which implies that $\mathcal{B}[g] - \Phi \in C_{0,\sigma}^\infty(\Omega)$. Hence it follows from (4) and (3) that

$$- < \psi, \operatorname{div} \Phi > \; = \; - < \psi, g > \; = \; < f, \mathcal{B}[g] > \; = \; < f, \Phi > .$$

This completes the proof for the case that $m \leq -1$. We next assume that $m \geq 0$. Then since $f \in W^{-1,q}(\Omega)^n$ in particular, it follows that $f = \nabla \psi$ for some scalar $\psi \in L^q(\Omega)$. It is obvious that $\psi \in W^{m+1,q}(\Omega)$. The proof is complete. □

Corollary 1.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and let $m \in \mathbb{Z}$ and $1 < q < \infty$. If ψ is a distribution on Ω such that $\nabla \psi \in W^{m,q}(\Omega)^n$, then*

$$\psi \in W^{m+1,q}(\Omega).$$

Moreover, if $m = -1$, then

$$\left\| \psi - \frac{1}{|\Omega|} \int_\Omega \psi \, dx \right\|_{q;\Omega} \leq C(n, q, \Omega) \|\nabla \psi\|_{-1,q;\Omega}.$$

It can be shown (not so difficult) that any domain in $\mathbb{R}^n$ is the union of an increasing sequence of bounded Lipschitz domains. Hence from Theorem 1.5, we can deduce the following result.

Theorem 1.7. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ and let $1 < q < \infty$. If $f \in L^q_{loc}(\Omega)^n$ satisfies*

$$\int_\Omega f \cdot \Phi \, dx = 0 \quad \textit{for all } \Phi \in C_{0,\sigma}^\infty(\Omega),$$

then there exists $\psi \in W^{1,q}_{loc}(\Omega)$ such that

$$f = \nabla \psi \quad \textit{in } \Omega.$$

We next prove the so-called *Helmholtz-Weyl decomposition* of $L^q(\Omega)^n$. For $1 < q < \infty$, we denote by $L^q_\sigma(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^q(\Omega)^n$. We also denote by $D^{1,q}(\Omega)$ the *homogeneous Sobolev space* consisting of all $\psi \in L^q_{loc}(\Omega)$ such that $\nabla\psi \in L^q(\Omega)^n$.

Lemma 1.8. *Let Ω be an arbitrary domain in $\mathbb{R}^n$ and let $1 < q < \infty$. Then*

$$L^q_\sigma(\Omega) = \left\{ u \in L^q(\Omega)^n : \int_\Omega u \cdot \nabla\psi \, dx = 0 \quad \text{for all } \psi \in D^{1,q'}(\Omega) \right\}.$$

Hence $L^q_\sigma(\Omega)$ consists of all $u \in L^q(\Omega)^n$ such that

$$\operatorname{div} u = 0 \quad \text{in } \Omega \quad \text{and} \quad u \cdot \nu = 0 \quad \text{on } \partial\Omega$$

in some weak sense.

Proof. By a density argument, we easily deduce that if $u \in L^q_\sigma(\Omega)$, then

$$\int_\Omega u \cdot \nabla\psi \, dx = 0 \quad \text{for all } \psi \in D^{1,q'}(\Omega),$$

which proves one inclusion. To prove the reverse inclusion, we argue by contraposition. Let $u \in L^q(\Omega)^n \setminus L^q_\sigma(\Omega)$ be given. Then since $L^q_\sigma(\Omega)$ is a closed subspace of $L^q(\Omega)^n$, it follows from the hyperplane separation theorem in Banach spaces (a consequence of the Hahn-Banach theorem) that there is $v \in L^{q'}(\Omega)^n$ such that

$$\int_\Omega u \cdot v \, dx \neq 0 \quad \text{and} \quad \int_\Omega w \cdot v \, dx = 0 \quad \text{for all } w \in L^q_\sigma(\Omega).$$

By Theorem 1.7, there is $\psi \in D^{1,q'}(\Omega)$ such that $v = \nabla\psi$ in Ω. But this implies that

$$\int_\Omega u \cdot \nabla\psi \, dx = \int_\Omega u \cdot v \, dx \neq 0.$$

The proof is complete. $\square$

Theorem 1.9. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Then*

$$L^2(\Omega)^n = L^2_\sigma(\Omega) \oplus \nabla \left[D^{1,2}(\Omega) \right];$$

that is, for each $u \in L^2(\Omega)^n$ there exists a unique pair $(v, \nabla p)$ such that

$$u = v + \nabla p, \quad v \in L^2_\sigma(\Omega) \quad \text{and} \quad p \in D^{1,2}(\Omega).$$

Moreover, by the orthogonality, we have

$$\|v\|^2_{2;\Omega} + \|\nabla p\|^2_{2;\Omega} = \|u\|^2_{2;\Omega}.$$

Therefore, the mapping $u \mapsto v$ defines a bounded linear operator P from $L^2(\Omega)^n$ onto $L^2_\sigma(\Omega)$, called the Helmholtz projection.

Proof. By Theorem 1.7 and Lemma 1.8, we have

$$\left[L^2_\sigma(\Omega)\right]^\perp = \nabla\left[D^{1,2}(\Omega)\right].$$

Hence the theorem follows immediately from the projection theorem in Hilbert space theory. $\square$

Let $u \in L^2(\Omega)^n$ be given. Then by Theorem 1.9, there exists $p \in D^{1,2}(\Omega)$, unique up to additive constants, such that

$$u - \nabla p \in L^2_\sigma(\Omega).$$

By Lemma 1.8, we have

$$\int_\Omega (\nabla p - u) \cdot \nabla\psi \, dx = 0 \quad \text{for all } \psi \in D^{1,2}(\Omega);$$

that is, p is a *weak solution* of the Neumann problem

$$(N) \qquad \begin{cases} \Delta p = \operatorname{div} u & \text{in } \Omega \\ \frac{\partial p}{\partial \nu} = u \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Conversely, if $p \in D^{1,2}(\Omega)$ is a weak solution of (N), then $u - \nabla p \in L^2_\sigma(\Omega)$. We have proved

Corollary 1.10. *Let Ω be an arbitrary domain in $\mathbb{R}^n$. Then for each $u \in L^2(\Omega)^n$ there exists a weak solution $p \in D^{1,2}(\Omega)$ of the Neumann problem (N), unique up to additive constants. Moreover, we have*

$$\|\nabla p\|_{2;\Omega} \le \|u\|_{2;\Omega}$$

The Helmholtz-Weyl decomposition of $L^q(\Omega)^n$ also holds for $1 < q < \infty$, if Ω is sufficiently smooth.

Theorem 1.11. *Let Ω be a bounded smooth (say, C^1-) domain in $\mathbb{R}^n$ and let $1 < q < \infty$. Then*

$$L^q(\Omega)^n = L^q_\sigma(\Omega) \oplus \nabla\left[D^{1,q}(\Omega)\right];$$

that is, for each $u \in L^q(\Omega)^n$ there exists a unique pair $(v, \nabla p)$ such that

$$u = v + \nabla p, \quad v \in L^q_\sigma(\Omega) \quad \textit{and} \quad p \in D^{1,q}(\Omega).$$

Moreover, the mapping $u \mapsto v$ defines a bounded linear operator P_q from $L^q(\Omega)^n$ onto $L^q_\sigma(\Omega)$, called the Helmholtz projection.

Proof. Let $u \in L^q(\Omega)^n$ be given. Then by a classical elliptic PDE theory, there exists a weak solution p in $D^{1,q}(\Omega)$ (called a *q-weak solution*) of (N), unique up to additive constants. This p satisfies

$$\|\nabla p\|_{q;\Omega} \le C(n, q, \Omega)\|u\|_{q;\Omega}.$$

Moreover, it follows from Lemma 1.8 that

$$v = P_q u = u - \nabla p \in L^q_\sigma(\Omega).$$

This proves the existence of a decomposition of u. To prove the uniqueness, suppose that $u = v_1 + \nabla p_1$ is another decomposition of u. Then since $u - \nabla p_1 \in L^q_\sigma(\Omega)$, it follows from Lemma 1.8 again that p_1 is a q-weak solution of (N). Hence by the uniqueness of q-weak solutions of (N)

$$\nabla p_1 = \nabla p \quad \text{and so} \quad v = v_1.$$

The linearity of P_q follows easily from the uniqueness of the Helmholtz-Weyl decomposition. This completes the proof. □

1.3 The stationary Stokes equations

Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 2$. Consider the following Dirichlet problem for the stationary Stokes equations:

$$(S) \qquad \begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Suppose that $f \in W^{-1,2}(\Omega)^n$, $g \in L^2(\Omega)$ and $\int_\Omega g\, dx = 0$. Then a vector field $u : \Omega \to \mathbb{R}^n$ is called a *weak solution* of (S) if

$$u \in W^{1,2}_0(\Omega)^n, \quad \operatorname{div} u = g \quad \text{a.e. in } \Omega,$$

and

$$\int_\Omega \nabla u : \nabla \Phi\, dx = < f, \Phi > \quad \text{for all } \Phi \in C^\infty_{0,\sigma}(\Omega).$$

To prove the uniqueness of weak solutions of (S), we need a density result. For $1 < q < \infty$, we denote by $W^{1,q}_{0,\sigma}(\Omega)$ the closure of $C^\infty_{0,\sigma}(\Omega)$ in $W^{1,q}(\Omega)^n$.

Lemma 1.12. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then*

$$W^{1,q}_{0,\sigma}(\Omega) = \left\{ u \in W^{1,q}_0(\Omega)^n : \operatorname{div} u = 0 \textit{ in } \Omega \right\}.$$

Proof. Suppose that $u \in W^{1,q}_0(\Omega)^n$ and $\operatorname{div} u = 0$ in Ω. Let $\{v_k\}$ be a sequence in $C^\infty_0(\Omega)^n$ such that $v_k \to u$ in $W^{1,q}(\Omega)^n$. For each k, define

$$u_k = v_k + \mathcal{B}_\Omega[-\operatorname{div} v_k].$$

Then since $v_k \in C^\infty_0(\Omega)^n$ and $\int_\Omega (-\operatorname{div} v_k)\, dx = 0$, we have

$$u_k \in C^\infty_0(\Omega)^n \quad \text{and} \quad \operatorname{div} u_k = 0 \quad \text{in } \Omega.$$

Moreover since $\operatorname{div} v_k \to \operatorname{div} u = 0$ in $L^q(\Omega)$, we have

$$\begin{aligned}\|u_k - u\|_{1,q;\Omega} &\le \|\mathcal{B}_\Omega[-\operatorname{div} v_k]\|_{1,q;\Omega} + \|v_k - u\|_{1,q;\Omega} \\ &\le C\|\operatorname{div} v_k\|_{q;\Omega} + \|v_k - u\|_{1,q;\Omega} \to 0.\end{aligned}$$

Hence it follows that $u \in W^{1,q}_{0,\sigma}(\Omega)$. This proves one inclusion, but the reverse one is trivial. $\square$

Theorem 1.13. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then for each $f \in W^{-1,2}(\Omega)^n$ and $g \in L^2(\Omega)$ with $\int_\Omega g\,dx = 0$, there exists a unique weak solution u of (S). Moreover, there exists a unique $p \in L^2(\Omega)$ with $\int_\Omega p\,dx = 0$ such that*

$$\int_\Omega \nabla u : \nabla \Phi\,dx - \int_\Omega p \operatorname{div} \Phi\,dx = < f, \Phi > \quad \textit{for all } \Phi \in C_0^\infty(\Omega)^n.$$

Finally, we have

$$\|u\|_{1,2;\Omega} + \|p\|_{2;\Omega} \le C(n,\Omega)\left(\|f\|_{-1,2;\Omega} + \|g\|_{2;\Omega}\right).$$

Proof. To prove the uniqueness assertion, suppose that u_1, u_2 are weak solutions of (S). Then $u = u_1 - u_2$ satisfies

$$u \in W_0^{1,2}(\Omega)^n, \quad \operatorname{div} u = 0 \quad \text{a.e. in } \Omega$$

and

$$\int_\Omega \nabla u : \nabla \Phi\,dx = 0 \quad \text{for all } \Phi \in C^\infty_{0,\sigma}(\Omega).$$

It follows from Lemma 1.12 that $u \in W^{1,2}_{0,\sigma}(\Omega)$. Hence choosing $\Phi_k \in C^\infty_{0,\sigma}(\Omega)$ with $\Phi_k \to u$ in $W^{1,2}(\Omega)^n$, we have

$$\int_\Omega |\nabla u|^2\,dx = \lim_{k\to\infty} \int_\Omega \nabla u : \nabla \Phi_k\,dx = 0,$$

which implies that $u = 0$ in Ω.

To prove the existence assertion, let us consider the Hilbert space

$$H = \left\{ u \in W_0^{1,2}(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega \right\}$$

equipped with the inner product[2]

$$(u,v)_H = \int_\Omega \nabla u : \nabla v\,dx.$$

[2] $(\cdot,\cdot)_H$ being a complete inner product is an easy consequence of the Poincaré inequality.

Suppose now that $f \in W^{-1,2}(\Omega)^n$, $g \in L^2(\Omega)$ and $\int_\Omega g\,dx = 0$. Then the functional

$$\Phi \mapsto < f, \Phi > - \int_\Omega \nabla \mathcal{B}_\Omega[g] \cdot \nabla \Phi \, dx$$

is linear and bounded on H. Hence by the Riesz representation theorem, there exists a unique $v \in H$ such that

$$\int_\Omega \nabla v : \nabla \Phi \, dx = < f, \Phi > - \int_\Omega \nabla \mathcal{B}_\Omega[g] \cdot \nabla \Phi \, dx$$

for all $\Phi \in H$. Obviously, $u = v + \mathcal{B}_\Omega[g]$ is a weak solution of (S). Moreover, taking $\Phi = v$, we have

$$\begin{aligned}
\|u\|_{1,2;\Omega} &\le \|v\|_{1,2;\Omega} + \|\mathcal{B}_\Omega[g]\|_{1,2;\Omega} \\
&\le C(n,\Omega) \left(\|f\|_{-1,2;\Omega} + \|\mathcal{B}_\Omega[g]\|_{1,2;\Omega} \right) \\
&\le C(n,\Omega) \left(\|f\|_{-1,2;\Omega} + \|g\|_{2;\Omega} \right).
\end{aligned}$$

Finally, the existence and estimate of p follow from Theorem 1.5. □

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then since $L^2_\sigma(\Omega) \hookrightarrow W^{-1,2}(\Omega)^n$, it follows from Theorem 1.13 that for each $f \in L^2_\sigma(\Omega)$ there exists a unique $u = Sf \in W^{1,2}_{0,\sigma}(\Omega)$ such that

$$(\nabla u, \nabla \Phi) = (f, \Phi) \quad \text{for all } \Phi \in C^\infty_{0,\sigma}(\Omega).$$

where $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ denotes the inner product on $L^2(\Omega)^n$ or $L^2(\Omega)^{n^2}$ (or sometimes $L^2(\Omega)$):

$$(f, \Phi) = \int_\Omega f \cdot \Phi \, dx.$$

The solution operator $S : L^2_\sigma(\Omega) \to W^{1,2}_{0,\sigma}(\Omega)$ is linear, bounded, and injective. The inverse of S is called the *Stokes operator* (in $L^2_\sigma(\Omega)$) and denoted by A. The domain $\mathcal{D}(A)$ of A is dense in $L^2_\sigma(\Omega)$ because

$$C^\infty_{0,\sigma}(\Omega) \subset \mathcal{D}(A) \subset W^{1,2}_{0,\sigma}(\Omega);$$

indeed, if $u \in C^\infty_{0,\sigma}(\Omega)$, then $f = -\Delta u \in L^2_\sigma(\Omega)$ and $u = Sf \in \mathcal{D}(A)$. Hence the Stokes operator A is an unbounded operator in $L^2_\sigma(\Omega)$ with dense domain. Let $I : W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ be the natural embedding, which is compact by the Rellich-Kondrachov compactness theorem (see [5] e.g.). Then the composition $K = I \circ S$ is a compact linear operator on $L^2_\sigma(\Omega)$. Moreover, K is symmetric and positive: for every $f, g \in L^2_\sigma(\Omega)$, we have

$$(Kf, g) = (\nabla u, \nabla v) = (f, Kg)$$

and

$$(Kf, f) = \|\nabla u\|^2_{2;\Omega} > 0 \quad \text{if } f \neq 0,$$

where $u = Sf$ and $v = Sg$. Therefore, by the spectral theory of symmetric compact operators (see [5, Appendix D] or [9, Chapter 5]), we conclude that

(i) the spectrum $\sigma(A)$ of the Stokes operator A consists entirely of its positive eigenvalues with finite multiplicity;

(ii) if we repeat each eigenvalue of A according to its multiplicity, then

$$\sigma(A) = \{\lambda_k\}_{k=1}^{\infty},$$

where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ and $\lambda_k \to \infty$;

(iii) there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2_\sigma(\Omega)$, where each $w_k \in \mathcal{D}(A)$ is an eigenfunction of A corresponding to λ_k. Hence for each $u \in L^2_\sigma(\Omega)$, we have

$$u = \sum_{k=1}^{\infty} \hat{u}_k w_k \quad \text{in } L^2(\Omega)^n \quad \text{and} \quad \|u\|^2_{2;\Omega} = \sum_{k=1}^{\infty} |\hat{u}_k|^2 < \infty,$$

where

$$\hat{u}_k = (u, w_k) \quad (k = 1, 2, ...);$$

(iv) [3] $\{\lambda_k^{-1/2} w_k\}_{k=1}^{\infty}$ is an orthonormal basis of the Hilbert space $W^{1,2}_{0,\sigma}(\Omega)$ equipped with the inner production $(\nabla u, \nabla v)_{L^2(\Omega)^{n^2}}$. Hence, for each $u \in W^{1,2}_{0,\sigma}(\Omega)$, we have

$$u = \sum_{k=1}^{\infty} \hat{u}_k w_k \quad \text{in } W^{1,2}(\Omega)^n \quad \text{and} \quad \|\nabla u\|^2_{2;\Omega} = \sum_{k=1}^{\infty} |\hat{u}_k|^2 \lambda_k < \infty;$$

(v) [4] for each $f \in \left[W^{1,2}_{0,\sigma}(\Omega)\right]'$, we have

$$\|f\|^2_{\left[W^{1,2}_{0,\sigma}(\Omega)\right]'} = \sum_{k=1}^{\infty} \frac{|\hat{f}_k|^2}{\lambda_k}, \quad \text{where } \hat{f}_k = < f, w_k > .$$

Assume in addition that Ω is sufficiently smooth (say, of class C^2). Then it can be shown by a standard method of difference quotients (see [16, Section III.1.5] for a detailed proof) that $S : L^2_\sigma(\Omega) \to W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^n$ and so $\mathcal{D}(A) \subset W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^n$. Let $u \in W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^n$ be given. Then for all $\Phi \in C^\infty_{0,\sigma}(\Omega)$, we have

$$(\nabla u, \nabla \Phi) = -(\Delta u, \Phi) = (-P\Delta u, \Phi),$$

[3](iv) is deduced easily from (iii) by observing that $(\nabla u, \nabla w_k)_{L^2(\Omega)^{n^2}} = \lambda_k (u, w_k)_{L^2(\Omega)^n}$ for all $u \in W^{1,2}_{0,\sigma}(\Omega)$.

[4](v) follows from (iv) by using the Riesz representation theorem.

where P is the Helmholtz projection of $L^2(\Omega)^n$ onto $L^2_\sigma(\Omega)$. Hence by the definition of A, we conclude that

$$\mathcal{D}(A) = W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^n \quad \text{and} \quad Au = -P\Delta u, \quad u \in \mathcal{D}(A).$$

In addition to (i)-(v), there holds the following property for $\{w_k\}_{k=1}^\infty$:

(vi) for each $u \in W^{1,2}_{0,\sigma}(\Omega) \cap W^{2,2}(\Omega)^n$, we have

$$u = \sum_{k=1}^\infty \hat{u}_k w_k \quad \text{in } W^{2,2}(\Omega)^n \quad \text{and} \quad \|Au\|^2_{2;\Omega} = \sum_{k=1}^\infty |\hat{u}_k|^2 \lambda_k^2 < \infty.$$

Theorem 1.13 was extended to q-weak solutions for $1 < q < \infty$ by Cattabriga [2]. By a *q-weak solution*, we mean a vector field $u : \Omega \to \mathbb{R}^n$ such that

$$u \in W^{1,q}_0(\Omega)^n, \quad \operatorname{div} u = g \quad \text{a.e. in } \Omega,$$

and

$$\int_\Omega \nabla u \cdot \nabla \Phi \, dx = < f, \Phi > \quad \text{for all } \Phi \in C^\infty_{0,\sigma}(\Omega),$$

provided that $f \in W^{-1,q}(\Omega)^n$, $g \in L^q(\Omega)$ and $\int_\Omega g\, dx = 0$. A complete proof of the following result can be found in Galdi's book [6].

Theorem 1.14. *Let Ω be a bounded smooth (say, C^2-) domain in $\mathbb{R}^n$ and let $1 < q < \infty$. Then for each $f \in W^{-1,q}(\Omega)^n$ and $g \in L^q(\Omega)$ with $\int_\Omega g\, dx = 0$, there exists a unique q-weak solution u of (S). Moreover, we have*

$$\|u\|_{1,q;\Omega} \le C(n, q, \Omega) \left(\|f\|_{-1,q;\Omega} + \|g\|_{q;\Omega} \right).$$

In addition, if $f \in L^q(\Omega)^n$ and $g \in W^{1,q}(\Omega)$, then

$$u \in W^{2,q}(\Omega)^n \quad \textit{and} \quad \|u\|_{2,q;\Omega} \le C(n, q, \Omega) \left(\|f\|_{q;\Omega} + \|g\|_{1,q;\Omega} \right).$$

Let Ω be a bounded smooth domain in $\mathbb{R}^n$. For $1 < q < \infty$, the *Stokes operator* A_q (in $L^q_\sigma(\Omega)$) is an unbounded operator in $L^q_\sigma(\Omega)$ defined by

$$A_q u = -P_q \Delta u \quad \text{for all } u \in \mathcal{D}(A_q) = W^{1,q}_{0,\sigma}(\Omega) \cap W^{2,q}(\Omega)^n,$$

where P_q is the Helmholtz projection of $L^q(\Omega)^n$ onto $L^q_\sigma(\Omega)$.

2 Existence of weak solutions

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^3$ and $0 < T < \infty$ a finite time. Consider the following initial boundary value problem for the Navier-Stokes equations:

$$(NS) \qquad \begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla p = f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u = u_0 & \text{on } \{t = 0\} \times \Omega, \end{cases}$$

where $u = (u^1, u^2, u^3)$ and p are unknown velocity and pressure, respectively, of a viscous incompressible fluid (with viscosity equal to 1, only for simplicity).

We first derive basic a priori estimates for solutions of (NS). For simplicity, we write $\|\phi\|_q = \|\phi\|_{q;\Omega}$ for the norm of ϕ in $L^q(\Omega)$ or $L^q(\Omega)^n$ or even $L^q(\Omega)^{n^2}$. Let (u,p) be a smooth solution of (NS). Then multiplying the first equation in (NS) by u and integrating over Ω, we obtain

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2\,dx + \int_\Omega \left[(u\cdot\nabla)u\cdot u - \Delta u\cdot u + \nabla p\cdot u\right]\,dx = \int_\Omega f\cdot u\,dx. \tag{5}$$

Recall that $\operatorname{div} u = 0$ and $u = 0$ on $(0,T) \times \partial\Omega$. Hence by the divergence theorem,

$$-\int_\Omega \Delta u\cdot u\,dx = \int_\Omega |\nabla u|^2\,dx,$$

$$\int_\Omega \nabla p\cdot u\,dx = -\int_\Omega p\operatorname{div} u\,dx = 0,$$

and

$$\int_\Omega (u\cdot\nabla)u\cdot u\,dx = \int_\Omega u\cdot\nabla\left(\frac{1}{2}|u|^2\right)\,dx = 0.$$

Hence from (5), we derive the *(differential) energy equality*

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|u|^2\,dx + \int_\Omega |\nabla u|^2\,dx = \int_\Omega f\cdot u\,dx. \tag{6}$$

Integrating this over $[0,t]$, we also derive the *(integral) energy equality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\,dt = \frac{1}{2}\|u_0\|_2^2 + \int_0^t (f(s), u(s))\,dt \tag{7}$$

for all $t \in [0,T]$. Moreover, by the Cauchy-Schwartz inequality, we deduce from (6) that

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 \le \|f(t)\|_2^2 + \|u(t)\|_2^2.$$

This differential inequality can be solved by the method of integrating factors. That is, multiplying by e^{-t}, we obtain

$$\frac{d}{dt}\left(e^{-t}\|u(t)\|_2^2\right) + 2e^{-t}\|\nabla u(t)\|_2^2 \le e^{-t}\|f(t)\|_2^2$$

and so

$$\|u(t)\|_2^2 + 2\int_0^t e^{t-s}\|\nabla u(s)\|_2^2\,ds \le e^t\|u_0\|_2^2 + \int_0^t e^{t-s}\|f(s)\|_2^2\,ds$$

for all $t \in [0,T]$. Since $1 \le e^{t-s} \le e^T$ for all $0 \le s \le t \le T$, we finally derive the following a priori estimate:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2\,ds \le e^T\left(\|u_0\|_2^2 + \int_0^t \|f(s)\|_2^2\,ds\right)$$

for all $t \in [0,T]$. This motivates us to introduce the function space

$$L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$$

for *weak solutions* of (NS). Here, for a Banach space X and $1 \le r \le \infty$, $L^r(0,T;X)$ denotes the Banach space of all Bochner-integrable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^r(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^r\,dt\right)^{1/r} & \text{if } 1 \le r < \infty \\ \sup\limits_{0\le t\le T} \|u(t)\|_X & \text{if } r = \infty \end{cases}$$

is finite. We also denote by $W^{1,r}(0,T;X)$ the Banach space of all $u \in L^r(0,T;X)$ having weak derivatives in $L^r(0,T;X)$; that is, $u \in W^{1,r}(0,T;X)$ if and only if

$$u \in L^r(0,T;X)$$

and there exists $v \in L^r(0,T;X)$ such that

$$\int_0^T \eta'(t) < f, u(t) > \,dt = -\int_0^T \eta(t) < f, v(t) > \,dt$$

for all $\eta \in C_0^\infty(0,T)$ and all $f \in X'$; hence for each $f \in X'$ the scalar function $< f, u(\cdot) >$ is weakly differentiable on $[0,T]$ and its weak derivative is $< f, v(\cdot) >$. Such a function v, which is unique a.e. on $[0,T]$, is called the *weak derivative* of u and denoted by u' or du/dt. It can be shown (see [5, Chapter 5] e.g.) that every $u \in W^{1,r}(0,T;X)$ can be redefined on a subset of $[0,T]$ with measure zero so that

$$u \in C([0,T];X) \quad \text{and} \quad \max_{0\le t\le T}\|u(t)\|_X \le C\|u\|_{W^{1,r}(0,T;X)},$$

which verifies the continuous embedding

$$W^{1,r}(0,T;X) \hookrightarrow C([0,T];X).$$

2.1 Definitions of weak solutions

Suppose that $f \in L^2(0,T;L^2(\Omega)^3)$ and $u_0 \in L^2_\sigma(\Omega)$. Then by a *weak solution* of (NS), we mean a vector field

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$$

such that

$$\begin{aligned} \int_0^T \left[-(u(t),v_t(t)) + (\nabla u(t),\nabla v(t)) + ((u(t)\cdot\nabla)u(t),v(t))\right]\,dt \\ = (u_0,v(0)) + \int_0^T (f(t),v(t))\,dt \end{aligned} \tag{8}$$

for all $v \in C_0^\infty([0,T)\times\Omega)^3$ with $\operatorname{div} v = 0$. It should be noted that each term in (8) makes sense due to Hölder's inequality: for instance,

$$\begin{aligned} \int_0^T |((u(t)\cdot\nabla)u(t),v(t))|\,dt &\le \int_0^T \|u(t)\|_2\|\nabla u(t)\|_2\|v(t)\|_\infty\,dt \\ &\le \|u\|^2_{L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))}\|v\|_{L^\infty((0,T)\times\Omega)^3}. \end{aligned}$$

A refined estimate can be derived by using Sobolev's inequality. Since $u(t) \in W^{1,2}_{0,\sigma}(\Omega)$ for a.e. t, it follows from the Hölder and Sobolev inequalities that

$$\begin{aligned} |((u(t)\cdot\nabla)u(t),v(t))| &\le \|u(t)\|_3\|\nabla u(t)\|_2\|v(t)\|_6 \\ &\le \|u(t)\|_2^{1/2}\|u(t)\|_6^{1/2}\|\nabla u(t)\|_2\|v(t)\|_6 \\ &\le C\|u(t)\|_2^{1/2}\|\nabla u(t)\|_2^{3/2}\|\nabla v(t)\|_2 \end{aligned}$$

for a.e. $t \in [0,T]$. Using Hölder's inequality again, we derive a basic estimate

$$\begin{aligned} &\int_0^T |((u(t)\cdot\nabla)u(t),v(t))|\,dt \\ &\qquad \le C\|u\|^{1/2}_{L^\infty(0,T;L^2_\sigma(\Omega))}\|u\|^{3/2}_{L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))}\|v\|_{L^4(0,T;W^{1,2}_{0,\sigma})}, \end{aligned} \tag{9}$$

where C is an absolute constant. From now on, we will use the following notations

$$H = L^2_\sigma(\Omega) \quad \text{and} \quad V = W^{1,2}_{0,\sigma}(\Omega)$$

for the sake of simplicity. Hence weak solutions of (NS) belong to

$$L^\infty(0,T;H) \cap L^2(0,T;V).$$

Equivalent definitions of weak solutions are provided by the following result.

Lemma 2.1. *Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Then the following three statements are all equivalent:*

(i) u is a weak solution of (NS).

(ii) The identity (8) holds for all $v \in L^4(0,T;V) \cap W^{1,1}(0,T;H)$ with $v(T) = 0$.

(iii) u satisfies the identity

$$\begin{aligned} -\int_0^T \eta'(t)\,(u(t),\Phi)\,dt + \int_0^T \eta(t)\,[(\nabla u(t),\nabla\Phi) + ((u(t)\cdot\nabla)u(t),\Phi)]\,dt \\ = \eta(0)\,(u_0,\Phi) + \int_0^T \eta(t)\,(f(t),\Phi)\,dt \end{aligned} \tag{10}$$

for all $\Phi \in V$ and all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$.

Proof. The implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii)$ are trivial.

To prove $(i) \Rightarrow (iii)$, let u be a weak solution of (NS). Fix any $\Phi \in C^\infty_{0,\sigma}(\Omega)$. Then taking $v = \eta(t)\Phi(x)$, we deduce that (10) holds for all $\eta \in C^\infty_0([0,T))$. Since $C^\infty_0([0,T))$ is dense in $\{\phi \in W^{1,1}([0,T]) : \phi(T) = 0\}$, we also deduce that (10) also holds for all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$.

Now, fix any $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Given $\Phi \in V$, let $\{\Phi_k\}$ be a sequence in $C^\infty_{0,\sigma}(\Omega)$ such that $\Phi_k \to \Phi$ in V. Then by (9), we have

$$\begin{aligned} &\int_0^T |\eta(t)|\,|(u(t)\cdot\nabla)u(t),\Phi_k) - (u(t)\cdot\nabla)u(t),\Phi)|\,dt \\ &\qquad \le \int_0^T |\eta(t)| \int_\Omega |u(t)||\nabla u(t)||\Phi_k - \Phi|\,dx\,dt \\ &\qquad \le C(\Omega,u,\eta)\|\Phi_k - \Phi\|_{1,2} \to 0. \end{aligned}$$

Hence by a standard density argument, we deduce that (10) also holds for all $\Phi \in V$.

Next, to prove $(iii) \Rightarrow (ii)$, assume that (10) holds for all $\Phi \in V$ and $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Fix $v \in L^4(0,T;V) \cap W^{1,1}(0,T;H)$ with $v(T) = 0$. Let $\{w_k\}$ be an orthonormal basis of H consisting of eigenvectors of the Stokes operator A in H. For each $k \in \mathbb{N}$, define

$$v_k = v_k(t) = \sum_{j=1}^k \eta_j(t) w_j,$$

where

$$\eta_j(t) = (v(t), w_j)\,.$$

Since

$$\eta_j \in W^{1,1}([0,T]), \quad \eta_j'(t) = (\partial_t v(t), w_j) \quad \text{and} \quad \eta_j(T) = 0,$$

it follows, by linearity, that (8) holds for $v = v_k$. Moreover, since $\{w_k\}$ is an orthonormal basis of H, we have

$$\|\partial_t v_k(t)\|_2 \le \|\partial_t v(t)\|_2 \quad \text{and} \quad \lim_{k\to\infty} \|\partial_t v_k(t) - \partial_t v(t)\|_2 = 0$$

for a.a. $t \in [0,T]$. Recall that $\{\lambda_k^{-1/2} w_k\}$ is an orthonormal basis of V, where λ_k is the eigenvalue corresponding to w_k. Hence for a.a. $t \in [0,T]$, we have

$$\|\nabla v_k(t)\|_2 \le \|\nabla v(t)\|_2 \quad \text{and} \quad \lim_{k\to\infty} \|\nabla v_k(t) - \nabla v(t)\|_2 = 0.$$

Hence by the dominated convergence theorem, we deduce that

$$\lim_{k\to\infty} \int_0^T \left(\|\nabla v_k(t) - \nabla v(t)\|_2^4 + \|\partial_t v_k(t) - \partial_t v(t)\|_2 \right) dt = 0.$$

Using Hölder's inequality and the estimate (9), we have

$$\begin{aligned}\int_0^T |(u(t), \partial_t v_k(t)) - (u(t), \partial_t v(t))| \, dt &\le \int_0^T \|u(t)\|_2 \|\partial_t v_k(t) - \partial_t v(t)\|_2 \, dt \\ &\le C(u) \int_0^T \|\partial_t v_k(t) - \partial_t v(t)\|_2 \, dt\end{aligned}$$

and

$$\begin{aligned}&\int_0^T |((u(t)\cdot\nabla)u(t), v_k(t)) - ((u(t)\cdot\nabla)u(t), v(t))| \, dt \\ &\qquad\qquad \le C(u) \left(\int_0^T \|v_k(t) - v(t)\|_{1,2}^4 \, dt \right)^{1/4},\end{aligned}$$

where $C(u)$ is a constant depending only on the norm of u. Therefore, passing to the limit as $k \to \infty$, we easily obtain (8). This proves $(iii) \Rightarrow (ii)$. The proof is complete. $\square$

We provide another equivalent definition of weak solutions of (NS), which indeed establishes the important regularity property of weak solutions - the weak continuity in H.

Lemma 2.2. *Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Then u is a weak solution of (NS) if and only if u can be redefined on a subset of $[0,T]$ with measure zero so that*

(i) $u(t) \in H$ for all $t \in [0,T]$;

(ii) u satisfies the identity

$$\begin{aligned}(u(t), \Phi) + \int_0^t [(\nabla u(s), \nabla \Phi) + ((u(s)\cdot\nabla)u(s), \Phi)] \, ds \\ = (u_0, \Phi) + \int_0^t (f(s), \Phi) \, ds\end{aligned} \tag{11}$$

for all $\Phi \in V$ and all $t \in [0,T]$;

(ii) u is weakly continuous in $L^2(\Omega)^3$ on $[0,T)$, that is,

$$\lim_{t\to t_0} (u(t),\Phi) = (u(t_0),\Phi) \qquad (\Phi \in L^2(\Omega)^3)$$

for all $t_0 \in [0,T]$.

Proof. Let u be a weak solution of (NS). For each $t \in [0,T]$, let us define $F(t) : W_0^{1,2}(\Omega)^3 \to \mathbb{R}$ by

$$< F(t), \Phi > = (f(t),\Phi) - (\nabla u(t), \nabla\Phi) - ((u(t)\cdot\nabla)u(t),\Phi)\,.$$

Then it follows from (9) that

$$F \in L^{4/3}(0,T; W^{-1,2}(\Omega)^3). \tag{12}$$

By Lemma 2.1, we have

$$-\int_0^T \eta'(t)\,(u(t),\Phi)\;dt = \eta(0)\,(u_0,\Phi) + \int_0^T \eta(t) < F(t),\Phi >\;dt \tag{13}$$

for all $\Phi \in V$ and $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Given $0 \le t < T$ and $0 < h < T-t$, we take $\eta = \eta_{t,h}$ in (13), where

$$\eta_{t,h}(s) = \begin{cases} 1 & \text{if } 0 \le s \le t \\ 1 - \frac{s-t}{h} & \text{if } t \le s \le t+h \\ 0 & \text{if } t+h \le s \le T. \end{cases}$$

Then since $F \in L^{4/3}(0,T;V')$, we deduce that

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} (u(s),\Phi)\;ds = (u_0,\Phi) + \int_0^t < F(s),\Phi >\;ds \tag{14}$$

for all $\Phi \in V$ and $t \in [0,T)$. Taking $\eta = \eta_{T,h}$ in (13), where

$$\eta_{T,h}(s) = \begin{cases} 1 & \text{if } 0 \le s \le T-h \\ \frac{T-s}{h} & \text{if } T-h \le s \le T, \end{cases}$$

we also deduce that

$$\lim_{h\to 0^+} \frac{1}{h}\int_{T-h}^{T} (u(s),\Phi)\;ds = (u_0,\Phi) + \int_0^T < F(s),\Phi >\;ds \tag{15}$$

for all $\Phi \in V$.

Given $t \in [0,T]$, let $g_t : V \to \mathbb{R}$ be defined by

$$< g_t, \Phi > = (u_0,\Phi) + \int_0^t < F(s),\Phi >\;ds$$

for all $\Phi \in V$. Then by (14) and (15), we have

$$|< g_t, \Phi >| \leq \|u\|_{L^\infty(0,T;H)} \|\Phi\|_2$$

for all $\Phi \in V$. Since V is dense in H, g_t can be extended uniquely to a bounded linear functional on H. Hence by the Riesz representation theorem, there exists a unique $u^*(t) \in H$ such that

$$\|u^*(t)\|_2 \leq \|u\|_{L^\infty(0,T;H)} \tag{16}$$

and

$$(u^*(t), \Phi) = (u_0, \Phi) + \int_0^t < F(s), \Phi > \, ds \quad \text{for all } \Phi \in V. \tag{17}$$

We now show that u^* is weakly continuous in $L^2(\Omega)^3$ on $[0, T]$. Let $t_0 \in [0, T]$ and $\Phi \in H$ be fixed. Given $\varepsilon > 0$, we choose $\Psi \in V$ such that

$$2\|u\|_{L^\infty(0,T;H)} \|\Psi - \Phi\|_2 \leq \varepsilon.$$

Then by (16) and (17), we have

$$\begin{aligned}
|(u^*(t) - u^*(t_0), \Phi)| &\leq |(u^*(t) - u^*(t_0), \Psi)| + |(u^*(t) - u^*(t_0), \Phi - \Psi)| \\
&\leq \left| \int_{t_0}^t < F(s), \Psi > \, ds \right| + \|u^*(t) - u^*(t_0)\|_2 \|\Phi - \Psi\|_2 \\
&\leq \left| \int_{t_0}^t \|F(s)\|_{V'} \, ds \right| \|\Psi\|_V + \varepsilon.
\end{aligned}$$

for all $t \in [0, T]$. Since $F \in L^{4/3}(0, T; V')$, we deduce that

$$\lim_{t \to t_0} (u^*(t), \Phi) = (u^*(t_0), \Phi).$$

More generally, using the Helmholtz projection P, we have

$$\lim_{t \to t_0} (u^*(t), \Phi) = \lim_{t \to t_0} (u^*(t), P\Phi) = (u^*(t_0), P\Phi) = (u^*(t_0), \Phi)$$

for all $\Phi \in L^2(\Omega)^3$.

Finally, to show that $u = u^*$ a.e. on $[0, T]$, observe that $u \in L^1(0, T; H)$. Hence by the Lebesgue differentiation theorem (see [4, Section II.2] for a simple proof), there is a subset $N \subset (0, T)$ with measure zero such that

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{t-h}^{t+h} \|u(s) - u(t)\|_2 \, ds = 0$$

for all $t \in (0, T) \setminus N$. Let $t \in (0, T) \setminus N$ be fixed. Then by (14), (15), and (17), we obtain

$$(u(t), \Phi) = (u^*(t), \Phi) \quad \text{for all } \Phi \in V, \tag{18}$$

which implies that $u(t) = u^*(t)$. This completes the proof of (i), (ii), and (iii).

Suppose conversely that u satisfies (i), (ii), and (iii). Given $\Phi \in V$, let $\phi : [0,T] \to \mathbb{R}$ be defined by

$$\phi(t) = (u(t), \Phi)\,.$$

Then since

$$\phi(t) = (u_0, \Phi) + \int_0^t < F(s), \Phi > \, ds \quad \text{for all } t \in [0,T],$$

it follows that

$$\phi \in W^{1,4/3}([0,T]), \quad \phi(0) = (u_0, \Phi)\,, \quad \text{and} \quad \phi'(t) = < F(t), \Phi > .$$

Hence using the integration by parts, we deduce that

$$-\int_0^T \eta'(t)\,(u(t), \Phi)\; dt = \eta(0)\,(u_0, \Phi) + \int_0^T \eta(t) < F(t), \Phi > \, dt$$

for all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Since $\Phi \in V$ is arbitrary, it follows from Lemma 2.1 that u is a weak solution of (NS). The proof is complete. $\square$

Remark 2.3 (the time-derivative and pressure)**.** *Let u be a weak solution of (NS). Then it follows from (12) and (13) that*

$$u \in W^{1,4/3}\,(0,T;V') \quad \textit{and} \quad u' = F\big|_V.$$

But this does not imply the better regularity of u:

$$u \in W^{1,4/3}\left(0,T; W^{-1,2}(\Omega)^3\right),$$

which is in fact wrong! It should be noted here that

$$V = W^{1,2}_{0,\sigma}(\Omega) \subsetneq W^{1,2}_0(\Omega)^3 \quad \textit{and} \quad W^{-1,2}(\Omega)^3 \subsetneq V'.$$

Next, for each $t \in [0,T]$, we define $l_t : W^{1,2}_0(\Omega)^3 \to \mathbb{R}$ by

$$< l_t, \Phi > = (u(t) - u_0, \Phi) - \int_0^t < F(s), \Phi > \, ds$$

for all $\Phi \in W^{1,2}_0(\Omega)^3$. Then l_t belongs to $W^{-1,2}(\Omega)^3$ and satisfies

$$< l_t, \Phi > = 0 \quad \textit{for all } \Phi \in C^\infty_{0,\sigma}(\Omega).$$

Hence by Theorem 1.5, there exists a unique $Q(t) \in L^2(\Omega)$ with $\int_\Omega Q(t)\,dx = 0$ such that

$$(Q(t), \operatorname{div}\Phi) = (u(t) - u_0, \Phi) - \int_0^t < F(s), \Phi > \, ds \tag{19}$$

for all $\Phi \in W_0^{1,2}(\Omega)^3$*. We shall show that the function*

$$Q : [0,T] \to L^2(\Omega)$$

is weakly continuous and bounded. For all $t, t_0 \in [0,T]$*, we have*

$$(Q(t) - Q(t_0), \operatorname{div} \Phi) = (u(t) - u(t_0), \Phi) - \int_{t_0}^{t} < F(s), \Phi > \, ds$$

for all $\Phi \in W_0^{1,2}(\Omega)^3$*. Given* $g \in L^2(\Omega)$*, we take* $\Phi = \mathcal{B}_\Omega\,[\overline{g}]$*, where*

$$\overline{g} = g - \frac{1}{|\Omega|} \int_\Omega g\, dx.$$

Then since $\int_\Omega (Q(t) - Q(t_0))\, dx = 0$*, we have*

$$(Q(t) - Q(t_0), g) = (u(t) - u(t_0), \mathcal{B}_\Omega\,[\overline{g}]) - \int_{t_0}^{t} < F(s), \mathcal{B}_\Omega\,[\overline{g}] > \, ds,$$

which proves the weak continuity of Q*. The boundedness of* Q *is proved in the same way. Hence using the integration by parts, we deduce from (19) that*

$$\begin{aligned} - \int_0^T \eta'(t)\,(u(t), \Phi)\, dt + \int_0^T \eta'(t)\,(Q(t), \operatorname{div} \Phi)\, dt \\ = \eta(0)\,(u_0, \Phi) + \int_0^T \eta(t) < F(t), \Phi > \, dt \end{aligned} \tag{20}$$

for all $\Phi \in W_0^{1,2}(\Omega)^3$ *and* $\eta \in W^{1,1}([0,T])$ *with* $\eta(T) = 0$*. Adapting the proof of Lemma 2.1, we also obtain*

$$\begin{aligned} \int_0^T \left[- (u, v_t) + (\nabla u, \nabla v) + ((u \cdot \nabla) u, v) + (Q, \operatorname{div} v_t)\right] dt \\ = (u_0, v(0)) + \int_0^T (f, v)\, dt \end{aligned} \tag{21}$$

for all $v \in W^{1,1}(0,T; W_0^{1,2}(\Omega))$ *with* $v(T) = 0$*. This proves the existence of an pressure* $p = -dQ/dt$ *which is the distributional derivative of* Q *with respect to time. Moreover, given* $w \in W_0^{1,1}((0,T); L^2(\Omega))$*, we can take* $v = \mathcal{B}_\Omega\,[\overline{w}]$ *in (21) to obtain*

$$\begin{aligned} < p, w > \; &= \int_0^T (Q, w_t)\, dt \\ &= \int_0^T \left[(u, v_t) - (\nabla u, \nabla v) - ((u \cdot \nabla) u, v)\right] dt + \int_0^T (f, v)\, dt. \end{aligned}$$

Hence it follows that

$$p \in W^{-1,\infty}(0,T; L^2(\Omega)).$$

2.2 Compactness results in $L^r(0,T;X)$

Existence questions for nonlinear problems can be very often resolved by means of suitable compactness results. In this subsection, we prove the so-called Aubin-Lions lemma, which is one of the simplest but still useful compactness results for nonlinear evolution problems.

Let us first recall some definitions and facts from Functional Analysis. For two Banach spaces X and Y, the intersection $X \cap Y$ is a Banach space equipped with the norm $\|u\|_{X\cap Y} = \|u\|_X + \|u\|_Y$. Suppose further that $X \subset Y$. Then we say that X is *continuously embedded* into Y and write $X \hookrightarrow Y$ if the embedding $u \mapsto u$ is bounded, that is, there is a constant $C > 0$ such that $\|u\|_Y \le C\|u\|_X$ for all $u \in X$. In addition, if the embedding $u \mapsto u$ is compact, that is, every bounded sequence $\{u_k\}$ in X has a subsequence which converges in Y, then we say that X is *compactly embedded* into Y and write $X \hookrightarrow\hookrightarrow Y$. Recall the following familiar results for weak/weak-star convergence and compactness (see [11] e.g.):

(i) Every weakly convergent sequence in X is bounded.

(ii) Every weakly-star convergent sequence in X' is bounded.

(iii) Every bounded sequence in X' has a weakly-star convergent subsequence.

(iv) Every bounded sequence in X has a weakly convergent sequence if X is reflexive.[5]

Hence $X \hookrightarrow\hookrightarrow Y$ holds for a reflexive space X if and only if $u_k \to u$ weakly in X implies $u_k \to u$ strongly in Y; we may assume further that $u = 0$.

Next, we briefly review the theory of Bochner integrals. Let X be a Banach space with norm $\|\cdot\| = \|\cdot\|_X$, X' its dual space, and $T > 0$ a finite number. The dual pairing of $f \in X'$ and $u \in X$ is denoted by $\langle f, u\rangle_{X',X}$ or simply $\langle f, u\rangle$. We say that a function $u : [0,T] \to X$ is *(Bochner-) integrable* if there is a sequence $\{u_k\}$ of simple functions on $[0,T]$ such that

$$\lim_{k\to\infty} \|u_k(t) - u(t)\| = 0 \quad \text{for a.e. } t \in [0,T]$$

and

$$\lim_{k\to\infty} \int_0^T \|u_k(t) - u(t)\|\, dt = 0.$$

In this case, the *(Bochner) integral* of u is defined by

$$\int_0^T u(t)\, dt = \lim_{k\to\infty} \int_0^T u_k(t)\, dt.$$

[5] X is reflexive if for every $F \in X''$ there is $u \in X$ such that $\langle F, f\rangle = \langle f, u\rangle$ for all $f \in X'$. Typical examples are Hilbert spaces and Lebesgue spaces L^q for $1 < q < \infty$.

It is quite standard to check that the limit indeed exists in X and is independent of the sequence $\{u_k\}$. It is also easy to show that

$$\left\| \int_0^T u(t)\, dt \right\| \le \int_0^T \|u(t)\|\, dt.$$

and

$$\left\langle f, \int_0^T u(t)\, dt \right\rangle = \int_0^T \langle f, u(t)\rangle\, dt \quad \text{for all } f \in X'.$$

For $1 \le r \le \infty$, let $L^r(0,T;X)$ be the Banach space of all integrable functions $u : [0,T] \to X$ such that

$$\|u\|_{L^r(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|^r\, dt\right)^{1/r} & \text{if } 1 \le r < \infty \\ \sup\limits_{0 \le t \le T} \|u(t)\| & \text{if } r = \infty \end{cases}$$

is finite. Obviously there holds the duality inequality

$$\int_0^T \langle f(t), u(t)\rangle\; dt \le \|u\|_{L^r(0,T;X)} \|f\|_{L^{r'}(0,T;X')}$$

for all $u \in L^r(0,T;X)$ and $f \in L^{r'}(0,T;X')$. However, the familiar equality

$$[L^r(0,T;X)]' = L^{r'}(0,T;X') \quad (1 \le r < \infty)$$

holds only if X has some additional property, for instance, the reflexivity. Hence $L^r(0,T;X)$ is reflexive if X is reflexive and $1 < r < \infty$; see [4, Chapter IV]) for more details. Given $u, v \in L^1(0,T;X)$, we say that v is the *weak derivative* of u and write $v = u'$ or $v = du/dt$, if

$$\int_0^T \eta'(t) u(t)\, dt = -\int_0^T \eta(t) v(t)\, dt \quad \text{for all } \eta \in C_0^\infty(0,T),$$

or equivalently

$$\int_0^T \eta'(t) < f, u(t) > \; dt = -\int_0^T \eta(t) < f, v(t) > \; dt$$

for all $\eta \in C_0^\infty(0,T)$ and all $f \in X'$. For $1 \le r \le \infty$, $W^{1,r}(0,T;X)$ denotes the Banach space of all $u \in L^r(0,T;X)$ such that u' exists and belongs to $L^r(0,T;X)$. By a standard method of mollification, we easily prove the continuous embedding $W^{1,r}(0,T;X) \hookrightarrow C([0,T];X)$ (see [5] e.g.). Note also that $W^{1,r}(0,T;X)$ is reflexive if X is reflexive and $1 < r < \infty$.

We are now ready to state and prove the Aubin-Lions compactness lemma. We first prove a preliminary result.

Lemma 2.4. *Let X_0, X and X_1 be three Banach spaces with*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1.$$

Then for each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\|u\|_X \le \varepsilon \|u\|_{X_0} + C_\varepsilon \|u\|_{X_1} \quad \text{for all } u \in X_0.$$

Proof. We argue by contradiction. If the statement were not true, then there would be a number $\varepsilon > 0$ and a sequence $\{u_k\}$ in X_0 such that

$$\|u_k\|_X > \varepsilon \|u_k\|_{X_0} + k\|u_k\|_{X_1} \quad \text{for all } k \in \mathbb{N}.$$

Note that $u_k \neq 0$ for each k. Defining $v_k = u_k / \|u_k\|_{X_0}$, we have

$$\|v_k\|_{X_0} = 1 \quad \text{and} \quad \|v_k\|_X > \varepsilon + k\|v_k\|_{X_1} \quad \text{for all } k \in \mathbb{N}.$$

Since $X_0 \hookrightarrow\hookrightarrow X$, we may assume that $v_k \to v$ in X for some $v \in X$. Moreover, since $X \hookrightarrow X_1$, it follows that $v_k \to v$ in X_1. Note however that

$$\|v_k\|_X > \varepsilon \quad \text{and} \quad \frac{1}{k}\|v_k\|_X > \|v_k\|_{X_1} \quad \text{for all } k \in \mathbb{N}.$$

Hence letting $k \to \infty$, we obtain

$$\|v\|_X \ge \varepsilon \quad \text{and} \quad \|v\|_{X_1} = 0,$$

which is a contradiction. $\square$

Theorem 2.5 (Aubin-Lions)**.** *Let X_0, X and X_1 be three Banach spaces with*

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1.$$

Suppose that X_0, X_1 are reflexive. Then for $0 < T < \infty$ and $1 < r, s < \infty$, we have

$$L^r(0,T;X_0) \cap W^{1,s}(0,T;X_1) \hookrightarrow\hookrightarrow L^r(0,T;X).$$

Proof. Since $L^r(0,T;X_0)$ and $W^{1,s}(0,T;X_1)$ are reflexive, it suffices to show that if $u_k \to 0$ weakly both in $L^r(0,T;X_0)$ and in $W^{1,s}(0,T;X_1)$, then $u_k \to 0$ strongly in $L^r(0,T;X)$. By Lemma 2.4, it suffices to show that $u_k \to 0$ strongly in $L^r(0,T;X_1)$. [6]

[6] Indeed, if $u_k \to 0$ strongly in $L^r(0,T;X_1)$, then for each $\varepsilon > 0$, we have

$$\int_0^T \|u_k(t)\|_X^r \, dt \le \varepsilon \int_0^T \|u_k(t)\|_{X_0}^r \, dt + C_\varepsilon \int_0^T \|u_k(t)\|_{X_1}^r \, dt$$

and so

$$\limsup_{k\to\infty} \int_0^T \|u_k(t)\|_X^r \, dt \le \varepsilon M,$$

where $M = \sup_k \int_0^T \|u_k(t)\|_{X_0}^r \, dt$ is finite due to the weak convergence of $\{u_k\}$.

Suppose thus that

$$u_k \to 0 \quad \text{weakly both in } L^r(0,T;X_0) \text{ and in } W^{1,s}(0,T;X_1).$$

Then since $u_k \to 0$ weakly in $W^{1,s}(0,T;X_1)$, $\{u_k\}$ is bounded in $W^{1,s}(0,T;X_1)$:

$$M \equiv \sup_k \|u_k\|_{W^{1,s}(0,T;X_1)} < \infty. \tag{22}$$

By the continuous embedding $W^{1,s}(0,T;X_1) \hookrightarrow C([0,T];X_1)$, we thus deduce that $\{u_k\}$ is bounded in $C([0,T];X_1)$. Hence to prove that

$$\int_0^T \|u_k(t)\|_{X_1}^r \, dt \to 0,$$

it suffices, by the dominated convergence theorem, to prove that

$$\lim_{k\to\infty} \|u_k(t)\|_{X_1} = 0 \quad \text{for a.a. } t \in [0,T].$$

Let $0 < t_0 < T$ be fixed. Then for all $t \in [t_0, T]$, we have

$$u_k(t_0) = u_k(t) - \int_{t_0}^t u_k'(\tau)\, d\tau.$$

Integrating this over $[t_0, t_1]$, we also have

$$(t_1 - t_0) u_k(t_0) = \int_{t_0}^{t_1} u_k(t)\, dt - \int_{t_0}^{t_1} \int_{t_0}^{t} u_k'(\tau)\, d\tau\, dt$$

and so

$$u_k(t_0) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} u_k(t)\, dt - \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (t_1 - \tau) u_k'(\tau)\, d\tau \tag{23}$$

for all $t_1 \in (t_0, T)$. By (22), we obtain

$$\begin{aligned}\left\| \int_{t_0}^{t_1} (t_1 - \tau) u_k'(\tau)\, d\tau \right\|_{X_1} &\le \left(\int_{t_0}^{t_1} (t_1 - \tau)^{s'}\, d\tau \right)^{1/s'} \|u_k'\|_{L^s(0,T;X_1)} \\ &\le M (t_1 - t_0)^{1+1/s'}.\end{aligned}$$

Hence given $\varepsilon > 0$, we can choose $t_1 \in (t_0, T)$ such that

$$\sup_k \left\| \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (t_1 - \tau) u_k'(\tau)\, d\tau \right\|_{X_1} < \varepsilon.$$

For each k, define

$$\Phi_k = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} u_k(t)\, dt.$$

Then since $u_k \to 0$ weakly in $L^r(0,T;X_0)$, it follows that $\Phi_k \to 0$ weakly in X_0. By the compact embedding of X_0 into X_1, we thus deduce that $\Phi_k \to 0$ strongly in X_1. Therefore, letting $k \to \infty$ in (23), we obtain

$$\limsup_{k\to\infty} \|u_k(t_0)\|_{X_1} \leq \varepsilon.$$

The proof is complete. $\square$

2.3 Global existence of weak solutions

We are now ready to state and prove the fundamental existence result due to Leray [12] and Hopf [10].

Theorem 2.6. *Suppose that $f \in L^2(0,T;L^2(\Omega)^3)$ and $u_0 \in H$. Then there exists at least one weak solution u of (NS) which satisfies the following additional properties:*

(i) u is weakly continuous in $L^2(\Omega)^3$ on $[0,T]$;

(ii) u satisfies the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\,ds \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t (f(s),u(s))\;ds$$

for all $t \in [0,T]$;

(iii) u satisfies even the strong energy inequality, that is, there is a subset N of $(0,T]$ with measure zero such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u(s)\|_2^2\,ds \leq \frac{1}{2}\|u(t_0)\|_2^2 + \int_{t_0}^t (f(s),u(s))\;ds$$

for all $t_0 \in [0,T] \setminus N$ and $t \in [t_0,T]$;

(iv) finally, u is strongly right-continuous in $L^2(\Omega)^3$ on $[0,T]\setminus N$:

$$\lim_{t\to t_0^+} \|u(t)-u(t_0)\|_2 = 0 \quad \textit{for all } t_0 \in [0,T]\setminus N.$$

Such a weak solution u will be called a Leray-Hopf weak solution of (NS).

We first prove a simple lemma.

Lemma 2.7. *For all $u \in L^3_\sigma(\Omega)$ and $v, w \in W_0^{1,2}(\Omega)^3$, we have*

$$((u\cdot\nabla)v,w) = -\left((u\cdot\nabla)w,v\right).$$

In particular, taking $v = w$, we have

$$((u\cdot\nabla)v,v) = 0.$$

Proof. If $u \in C^\infty_{0,\sigma}(\Omega)$ and $v, w \in C^\infty_0(\Omega)^3$, then

$$
\begin{aligned}
((u\cdot\nabla)v, w) &= \sum_{i,j=1}^{3} \int_\Omega u^i \frac{\partial v^j}{\partial x_i} w^j \, dx \\
&= -\sum_{i,j=1}^{3} \int_\Omega \left(\frac{\partial u^i}{\partial x_i} v^j w^j + u^i v^j \frac{\partial w^j}{\partial x_i} \right) dx \\
&= -\left((u\cdot\nabla)w, v\right).
\end{aligned}
$$

The general case is proved by a simple density argument, because

$$((u\cdot\nabla)v, w) \le C\|u\|_3 \|\nabla v\|_2 \|\nabla w\|_2.$$

□

Proof. We prove the theorem, by applying the so-called Faedo-Galerkin method.

Step 1. Let $\{w_k\}$ be an orthonormal basis of H consisting of eigenvectors of the Stokes operator A in H. Recall from Subsection 1.3 that

$$w_k \in V \quad \text{and} \quad (\nabla w_k, \nabla\Phi) = \lambda_k \,(w_k, \Phi) \quad \text{for all } \Phi \in V,$$

where λ_k is the eigenvalue of S corresponding to w_k. For each $k \in \mathbb{N}$, let H_k be the k-dimensional subspace of H spanned by $\{w_1, ..., w_k\}$ and let P_k be the projection of H onto H_k defined by

$$P_k\Phi = \sum_{j=1}^{k} \hat{\Phi}_j w_j, \quad \text{where } \hat{\Phi}_j = (\Phi, w_j) \;\; \text{for } j = 1, ..., k.$$

Then it follows from the results in Subsection 1.3 that

$$\|P_k\Phi\|_2 \le \|\Phi\|_2, \quad \lim_{k\to\infty} \|P_k\Phi - \Phi\|_2 = 0 \quad \text{for all } \Phi \in H \tag{24}$$

and

$$\|\nabla P_k\Phi\|_2 \le \|\nabla\Phi\|_2, \quad \lim_{k\to\infty} \|\nabla P_k\Phi - \nabla\Phi\|_2 = 0 \quad \text{for all } \Phi \in V. \tag{25}$$

Step 2. For a fixed $k \in \mathbb{N}$, the Faedo-Galerkin scheme seeks for a function

$$u_k \in C([0,T]; H_k)$$

satisfying the identity

$$
\begin{aligned}
(u_k(t), \Phi) + \int_0^t \left[(\nabla u_k(s), \nabla\Phi) + ((u_k(s)\cdot\nabla)u_k(s), \Phi)\right] ds \\
= (u_0, \Phi) + \int_0^t (f(s), \Phi)\, ds
\end{aligned}
\tag{26}
$$

for all $\Phi \in H_k$ and all $t \in [0,T]$. Such a function u_k, if it exists, should satisfy

$$u_k \in W^{1,2}(0,T;H_k), \quad u_k(0) = P_k u_0,$$

and

$$(u_k'(t), \Phi) + (\nabla u_k(t), \nabla \Phi) + ((u_k(t) \cdot \nabla) u_k(t), \Phi) = (f(t), \Phi) \tag{27}$$

for all $\Phi \in H_k$ and almost all $t \in [0,T]$. Furthermore, taking $\Phi = w_j$ for each j, we derive the ODE system

$$\hat{u_k}_j'(t) + \lambda_j \hat{u_k}_j(t) + \sum_{i,l=1}^{k} ((w_i \cdot \nabla) w_l, w_j)\, \hat{u_k}_i \hat{u_k}_l = \hat{Pf}_j(t) \quad (1 \le j \le k) \tag{28}$$

for a.a. $t \in [0,T]$, where the coefficients $\hat{u_k}_j$ and $\hat{Pf}_j$ are defined, as usual, by

$$u_k(t) = \sum_{j=1}^{k} \hat{u_k}_j(t) w_j \quad \text{and} \quad Pf_k(t) := P_k[Pf(t)] = \sum_{j=1}^{k} \hat{Pf}_j(t) w_j.$$

Since the system (28) has a smooth but quadratic nonlinearity, it follows from the standard ODE theory that there exist a time $0 < T_* \le T$ and a unique function $u_k \in W^{1,2}(0,T_*;H_k)$ with $u_k(0) = P_k u_0$ satisfying the identity (27) for all $\Phi \in H_k$ and a.a. $t \in [0,T_*]$. In fact, the solution u_k exists globally up to T. To show this, we take $\Phi = u_k(t)$ in (27). Then by Lemma 2.7, we obtain the energy equality

$$\frac{1}{2}\frac{d}{dt}\|u_k(t)\|_2^2 + \|\nabla u_k(t)\|_2^2 = (f(t), u_k(t)) \quad \text{for a.a. } t \in [0,T_*]. \tag{29}$$

Hence by the method of integrating factors, we derive the estimate

$$\|u_k(t)\|_2^2 + 2\int_0^t \|\nabla u_k(s)\|_2^2\, ds \le e^T \|u_k(0)\|_2^2 + e^T \int_0^t \|f(s)\|_2^2\, ds \tag{30}$$

for all $t \in [0,T]$, which is independent of T_*. By the standard ODE theory, the solution u_k of (28) exists globally up to time T, i.e., $T_* = T$.

Step 3. Since $u_k(0) = P_k u_0$, it follows from (24) and (30) that the sequence $\{u_k\}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$. We can show that $\{u_k\}$ is also bounded in $W^{1,4/3}(0,T;V')$. Let $\Phi \in V$ be given. Then since $u_k'(t) \in H_k$ for a.a. t, it follows from (27), (25), and (24) that

$$\begin{aligned}
(u_k'(t), \Phi) &= (u_k'(t), P_k\Phi) \\
&= (f(t), P_k\Phi) - (\nabla u_k(t), \nabla P_k \Phi) - ((u_k(t)\cdot\nabla)u_k(t), P_k\Phi) \\
&\le C\left(\|f(t)\|_2 + \|\nabla u_k\|_2 + \|u_k(t)\|_2^{1/2}\|\nabla u_k\|_2^{3/2}\right)\|\Phi\|_{1,2}.
\end{aligned}$$

Hence $\{u_k\}$ is bounded in $W^{1,4/3}(0,T;V')$. Therefore, by the Aubin-Lions compactness lemma, we may assume that $u_k \to u$ strongly in $L^2(0,T;H)$. Moreover, it follows form the weak/weak-star compactness results that $u_k \to u$ weakly in $L^2(0,T;V)$ and $u_k \to u$ weakly-star in $L^\infty(0,T;H)$. To perform the limiting process, we observe that u_k also satisfies

$$\begin{aligned} -\int_0^T \eta'(t)\,(u_k(t),\Phi)\;dt + \int_0^T \eta(t)\,[(\nabla u_k(t),\nabla\Phi) + ((u_k(t)\cdot\nabla)u_k(t),\Phi)]\;dt \\ = \eta(0)\,(P_k u_0,\Phi) + \int_0^T \eta(t)\,(f(t),\Phi)\;dt \end{aligned}$$

for all $\Phi \in H_k$ and all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Then by viture of the weak and strong convergence $u_k \to u$, we easily show that for each $k \in \mathbb{N}$, the limit u satisfies (10) for all $\Phi \in H_k$ and all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Then the regularity of u allows us to prove that (10) holds even for all $\Phi \in V$. By Lemmas 2.1 and 2.2, we conclude that u is a weak solution of (NS) and weakly continuous in $L^2(\Omega)^3$ on $[0,T]$. On the other hand, by (29), we obtain

$$\frac{1}{2}\|u_k(t)\|_2^2 + \int_{t_0}^t \|\nabla u_k(s)\|_2^2\,ds = \frac{1}{2}\|u_k(t_0)\|_2^2 + \int_{t_0}^t (f(s),u_k(s))\;ds$$

for all $0 \le s \le t \le T$. Since $u_k \to u$ strongly in $L^2(0,T;H)$ and $u_k(0) = P_k u_0 \to u_0$ in H, there is a subset N of $(0,T]$ with measure zero such that $u_k(t) \to u(t)$ in H for all $t \in [0,T] \setminus N$. Moreover, since $u_k \to u$ weakly in $L^2(0,T;V)$, we obtain

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^t \|\nabla u(s)\|_2^2\,ds \le \frac{1}{2}\|u(t_0)\|_2^2 + \int_{t_0}^t (f(s),u(s))\;ds$$

for all $t_0 \in [0,T] \setminus N$ and for a.a. $t \in [t_0,T]$. This inequality indeed holds for all $t \in [t_0,T]$, due to the weak continuity of u in $L^2(\Omega)^3$. Finally, since

$$\|u(t) - u(t_0)\|_2^2 = \|u(t)\|_2^2 - 2\,(u(t),u(t_0)) + \|u(t_0)\|_2^2,$$

it follows from the strong energy inequality and weak continuity of u that

$$\limsup_{t\to t_0^+} \|u(t) - u(t_0)\|_2^2 = 0.$$

The proof is complete. □

3 Uniqueness and regularity of weak solutions

One of the main results in this section is the famous structure theorem due to Leray and Scheffer for Leray-Hope weak solutions of (NS). We shall first show that every weak solution of (NS) should coincide with a strong solution if the latter exists. We shall also prove local and global existence results for strong solutions.

3.1 Weak-strong uniqueness results

In this subsection, by a weak solution of (NS), we always mean a weak solution of (NS) that is weakly continuous in $L^2(\Omega)^3$.

Lemma 3.1. *Let u be a weak solution of (NS). Then*

$$\int_0^t [-(u(s), v_t(s)) + (\nabla u(s), \nabla v(s)) + ((u(s)\cdot\nabla)u(s), v(s))]\, ds$$
$$= -(u(t), v(t)) + (u_0, v(0)) + \int_0^t (f(s), v(s))\, ds.$$

for all $t \in [0,T]$ and all $v \in L^4(0,T;V) \cap W^{1,1}(0,T;H)$ with $v(T) = 0$.

Proof. Fix $t \in [0,T)$ and all $v \in L^4(0,T;V) \cap W^{1,1}(0,T;H)$ with $v(T) = 0$. Then taking $\eta_{t,h}(s)v(s,x)$ as a test function in (8), where

$$\eta_{t,h}(s) = \begin{cases} 1 & \text{if } 0 \le s \le t \\ 1 - \frac{s-t}{h} & \text{if } t \le s \le t+h \\ 0 & \text{if } t+h \le s \le T, \end{cases}$$

we obtain

$$\int_0^{t+h} \eta_{t,h}(s)\, [-(u(s), v_t(s)) + (\nabla u(s), \nabla v(s)) + ((u(s)\cdot\nabla)u(s), v(s))]\, ds$$
$$= -\frac{1}{h}\int_t^{t+h} (u(s), v(s))\, ds + (u_0, v(0)) + \int_0^{t+h} \eta_{t,h}(s)\,(f(s), v(s))\, ds.$$

The lemma follows by letting $h \to 0^+$, due to the strong regularity of v and weak continuity of u. ☐

Theorem 3.2 (Energy equality)**.** *Assume that $f \in L^2(0,T;L^2(\Omega)^3)$ and $u_0 \in H$. Let u be a weak solution of (NS) satisfying the additional property*

$$u \in L^4(0,T;L^4(\Omega)^3).$$

Then u satisfies the energy equality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\, ds = \frac{1}{2}\|u_0\|_2^2 + \int_0^t (f(s), u(s))\, ds$$

for all $t \in [0,T]$. Moreover, u is strongly continuous in $L^2(\Omega)^3$ on $[0,T]$.

Proof. Let $t \in (0, T]$ be fixed.

Step 1. Let $\rho \in C_0^\infty((-1,1);[0,1])$ be an even function with $\int_{-1}^1 \rho(s)\,ds = 1$. For $0 < \varepsilon < t$, we define

$$u_\varepsilon(s) = \int_0^t \frac{1}{\varepsilon}\rho\left(\frac{s-\tau}{\varepsilon}\right) u(\tau)\,d\tau \quad (-\infty < s < \infty).$$

Then by a standard argument(see [5] e.g.), we can deduce that

$$u_\varepsilon \in C^\infty([0,t];V) \quad \text{and} \quad u_\varepsilon \to u \quad \text{in } L^2(0,t;V).$$

Moreover, by the weak continuity of u on $[0,T]$,

$$(u_\varepsilon(t), \Phi) = \int_0^{t/\varepsilon} \rho(\tau)\,(u(t-\varepsilon\tau), \Phi)\ d\tau \to \frac{1}{2}\,(u(t), \Phi)$$

and

$$(u_\varepsilon(0), \Phi) = \int_0^{t/\varepsilon} \rho(\tau)\,(u(\varepsilon\tau), \Phi)\ d\tau \to \frac{1}{2}\,(u_0, \Phi)$$

as $\varepsilon \to 0^+$, for all $\Phi \in L^2(\Omega)^3$.

Step 2. Then by Step 1 and Lemma 3.1, we have

$$\begin{aligned}\int_0^t \left[-\left(u(s), (u_\varepsilon)'(s)\right) + \left(\nabla u(s), \nabla u_\varepsilon(s)\right) + \left((u(s)\cdot\nabla)u(s), u_\varepsilon(s)\right)\right]\,ds \\ = -\left(u(t), u_\varepsilon(t)\right) + \left(u_0, u_\varepsilon(0)\right) + \int_0^t \left(f(s), u_\varepsilon(s)\right)\,ds.\end{aligned}$$

Since η is even, it follows by a change of variables that

$$\int_0^t \left(u(s), (u_\varepsilon)'(s)\right)\,ds = \int_0^t \int_0^t \eta_\varepsilon'(s-s')\,(u(s), u(s'))\,ds'ds = 0.$$

Hence using the results in Step 1, we obtain a general identity

$$\begin{aligned}\int_0^t \|\nabla u(s)\|_2^2\,ds + \lim_{\varepsilon\to 0^+}\int_0^t \left((u(s)\cdot\nabla)u(s), u_\varepsilon(s)\right)\,ds \\ = -\frac{1}{2}\|u(t)\|_2^2 + \frac{1}{2}\|u_0\|_2^2 + \int_0^t \left(f(s), u(s)\right)\,ds\end{aligned}$$

Moreover, by Lemma 2.7 and Step 1 again,

$$\begin{aligned}&\left|\int_0^t \left((u(s)\cdot\nabla)u(s), u_\varepsilon(s)\right)\,ds\right| \\ &\quad = \left|-\int_0^t \left((u(s)\cdot\nabla)u_\varepsilon(s), u(s)\right)\,ds + \int_0^t \left((u(s)\cdot\nabla)u(s), u(s)\right)\,ds\right| \\ &\quad \le \int_0^t \|u(s)\|_4 \|\nabla u_\varepsilon(s) - \nabla u(s)\|_2 \|u(s)\|_4\,ds \to 0 \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

This completes the proof. $\square$

Theorem 3.3 (Weak-strong uniqueness)**.** *Assume that $f \in L^2(0,T;L^2(\Omega)^3)$ and $u_0 \in H$. Let u and v be weak solutions of (NS) satisfying the energy inequality. Suppose in addition that*

$$v \in L^r(0,T;L^q(\Omega)^3)$$

for some q and r satisfying

$$\frac{2}{r} + \frac{3}{q} = 1, \quad 3 < q \leq \infty, \quad \textit{and} \quad 2 \leq r < \infty.$$

Then

$$u(t) = v(t) \quad \textit{a.e. in } \Omega \quad \textit{for all } t \in [0,T].$$

Proof. Let $0 < t \leq T$ be fixed. Then by the hypotheses,

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\, ds \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t (f(s), u(s))\ ds \tag{31}$$

and

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v(s)\|_2^2\, ds \leq \frac{1}{2}\|u_0\|_2^2 + \int_0^t (f(s), v(s))\ ds. \tag{32}$$

In fact, the second inequality becomes equality because $v \in L^4(0,T;L^4(\Omega)^3)$ by the additional condition v. Adapting the proof of Theorem 3.2, we deduce from Lemma 3.1 that

$$\int_0^t [-(u(s), v_\varepsilon'(s)) + (\nabla u(s), \nabla v_\varepsilon(s)) + ((u(s)\cdot\nabla)u(s), v_\varepsilon(s))]\ ds$$
$$= -(u(t), v_\varepsilon(t)) + (u_0, v_\varepsilon(0)) + \int_0^t (f(s), v_\varepsilon(s))\ ds$$

and

$$\int_0^t [-(v(s), u_\varepsilon'(s)) + (\nabla v(s), \nabla u_\varepsilon(s)) + ((v(s)\cdot\nabla)v(s), u_\varepsilon(s))]\ ds$$
$$= -(v(t), u_\varepsilon(t)) + (u_0, u_\varepsilon(0)) + \int_0^t (f(s), u_\varepsilon(s))\ ds.$$

By the symmetry of ρ again,

$$\int_0^t (u(s), v_\varepsilon'(s))\ ds + \int_0^t (v(s), u_\varepsilon'(s))\ ds = 0.$$

Hence summing the last two equalities and letting $\varepsilon \to 0$, we obtain

$$2\int_0^t (\nabla u(s), \nabla v(s))\ dt + \int_0^t [((u(s)\cdot\nabla)u(s), v(s)) + ((v(s)\cdot\nabla)v(s), u(s))]\ ds$$
$$= -(u(t), v(t)) + \|u_0\|_2^2 + \int_0^t (f(s), u(s))\ ds + \int_0^t (f(s), v(s))\ ds.$$

Combining this with (31) and (32), we have

$$\frac{1}{2}\|u(t)-v(t)\|_2^2 + \int_0^t \|\nabla u(s) - \nabla v(s)\|_2^2\, ds$$
$$\leq \int_0^t \left[((u(s)\cdot\nabla)u(s), v(s)) + ((v(s)\cdot\nabla)v(s), u(s))\right]\, ds.$$

Let $w = u - v$. Then using Lemma 2.7 several times, we obtain

$$\frac{1}{2}\|w(t)\|_2^2 + \int_0^t \|\nabla w(s)\|_2^2\, ds \leq \int_0^t ((w(s)\cdot\nabla)w(s), v(s))\, ds. \tag{33}$$

On the other hand, by the Hölder and Sobolev inequalities, one can show that

$$\int_0^t ((w(s)\cdot\nabla)w(s), v(s))$$
$$\leq C\left(\int_0^t \|\nabla w(s)\|_2^2\, ds\right)^{1-1/r}\left(\int_0^t \|v(s)\|_q^r \|w(s)\|_2^2\, ds\right)^r$$
$$\leq \frac{1}{2}\int_0^t \|\nabla w(s)\|_2^2\, ds + C\int_0^t \|v(s)\|_q^r \|w(s)\|_2^2\, ds.$$

Substituting this into (33) and using Gronwall's inequlaity, we deduce

$$\|w(t)\|_2^2 \leq \|w(0)\|_2^2 \exp\left(C\int_0^t \|v(s)\|_q^r\, ds\right)$$

for all $t \in [0,T]$. This complete the proof. $\square$

Remark 3.4. *Let u be a weak solution of (NS). Then since*

$$\|u(t)\|_4 \leq \|u(t)\|_2^{1/4}\|u(t)\|_6^{3/4} \leq C\|u(t)\|_2^{1/4}\|\nabla u(t)\|_2^{3/4},$$

we have

$$\int_0^T \|u(t)\|_4^{8/3}\, dt \leq C\|u\|_{L^\infty(0,T;H)}^{2/3}\|u\|_{L^2(0,T;V)}^{3/4} < \infty.$$

Hence u does not satisfy the additional integrability for the energy equality and uniqueness. In fact, it remains still open to prove the energy equality and uniqueness of weak solutions of (NS), which are closed related to the famous Navier-Stokes global regularity problem.

3.2 Existence of strong solutions

Throughout this subsection, let Ω be a bounded domain in $\mathbb{R}^3$ with smooth (say C^2-) boundary.

Theorem 3.5 (Local existence)**.** *Suppose that* $f \in L^2(0,T;L^2(\Omega)^3)$ *and* $u_0 \in V$*. Then there exist a time* $T_* \in (0,T]$ *and a unique vector field* $u : [0,T_*]\times\Omega \to \mathbb{R}^3$ *such that*

$$u \in C([0,T_*];V) \cap L^2(0,T_*;W^{2,2}(\Omega)^3) \cap W^{1,2}(0,T_*;H)$$

and

$$u_t + (u\cdot\nabla)u - \Delta u + \nabla p = f \quad a.e.\ in\ (0,T_*)\times\Omega$$

for some $p \in L^2(0,T_*;W^{1,2}(\Omega))$*. Moreover, the time* T_* *is bounded from below as follows:*

$$T_* \geq \min\left\{T,\ c(\Omega)\left(\|\nabla u_0\|_2^2 + \int_0^T \|f(t)\|_2^2\,dt\right)^{-2}\right\},$$

where $c(\Omega) > 0$ *is a constant depending only on the domain* Ω*. The vector field* u *or pair* (u,p) *will be called a strong solution of* (NS) *in* $[0,T_*]\times\Omega$*.*

Proof. **Step 1**. For each $k \in \mathbb{N}$, let $u_k \in W^{1,2}(0,T;H_k)$ be the unique solution of (27) with $u_k(0) = P_k u_0$. Recall from the results in Subsection 1.3 that

$$\mathcal{D}(A) = V \cap W^{2,2}(\Omega)^3, \quad Av = -P\Delta v \quad \text{for all } v \in \mathcal{D}(A),$$
$$c(\Omega)\|\nabla^2 v\|_2 \leq \|Av\|_2 \leq \|\nabla^2 v\|_2 \quad \text{for all } v \in \mathcal{D}(A), \tag{34}$$

and

$$w_k \in \mathcal{D}(A), \quad Aw_k = \lambda_k w_k, \quad A : H_k \to H_k.$$

This enables us to deduce from (27) that

$$(u_k'(t),\Phi) + (Au_k(t),\Phi) + ((u_k(t)\cdot\nabla)u_k(t),\Phi) = (f(t),\Phi) \tag{35}$$

for all $\Phi \in H_k$ and a.a. $t \in [0,T]$. Taking $\Phi = Au_k(t)$, we thus obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\nabla u_k(t)\|_2^2 + \|Au_k(t)\|_2^2 \\ &\leq \|f(t)\|_2\|Au_k(t)\|_2 + \|(u_k(t)\cdot\nabla)u_k(t)\|_2\|Au_k(t)\|_2\end{aligned}$$

and so

$$\frac{d}{dt}\|\nabla u_k(t)\|_2^2 + \|Au_k(t)\|_2^2 \leq 2\|f(t)\|_2^2 + 2\|(u_k(t)\cdot\nabla)u_k(t)\|_2^2 \tag{36}$$

for a.a. $t \in [0,T]$. Using the Hölder and Sobolev inequalities together with (34), we can estimate the nonlinear term as follows:

$$\begin{aligned}2\|(u_k(t)\cdot\nabla)u_k(t)\|_2^2 &\leq 2\|u_k(t)\|_6^2\|\nabla u_k(t)\|_3^2 \\ &\leq 2\|u_k(t)\|_6^2\|\nabla u_k(t)\|_2\|\nabla u_k(t)\|_6 \\ &\leq C(\Omega)\|\nabla u_k(t)\|_2^3\|Au_k(t)\|_2.\end{aligned}$$

Substituting this into (36), we obtain

$$\frac{d}{dt}\|\nabla u_k(t)\|_2^2 + \|Au_k(t)\|_2^2 \le 6\|f(t)\|_2^2 + C(\Omega)\|\nabla u_k(t)\|_2^6$$

and so

$$\begin{aligned} \|\nabla u_k(t)\|_2^2 &+ \int_0^t \|Au_k(s)\|_2^2\,ds \\ &\le \|\nabla u_0\|_2^2 + 6\int_0^t \|f(s)\|_2^2\,ds + C(\Omega)\int_0^t \|\nabla u_k(s)\|_2^6\,ds \end{aligned} \tag{37}$$

for all $t \in [0,T]$.

Step 2. Now, we prove the (local) uniform bound

$$\max_{0\le t\le T_*} \|\nabla u_k(t)\|_2^2 < M := 2\|\nabla u_0\|_2^2 + 12\int_0^T \|f(t)\|_2^2\,dt$$

for all $k \ge 1$, where $0 < T_* \le T$ is any time such that

$$C(\Omega)M^3T_* \le \frac{1}{2}M. \tag{38}$$

Suppose that the bound were not true. Then by the continuity of $\|\nabla u_k(\cdot)\|_2$, there should be a time $t_0 \in (0, T_*]$ such that

$$\|\nabla u_k(t_0)\|_2^2 = M \quad \text{and} \quad \|\nabla u_k(t)\|_2^2 < M \quad \text{for all } t \in [0, t_0).$$

But this would imply

$$\int_0^{t_0} \|\nabla u_k(t)\|_2^6\,dt < \int_0^{t_0} M^3\,dt \le M^3T_*.$$

Hence by (37) and (38), we would obtain

$$\|\nabla u_k(t_0)\|_2^2 < \|\nabla u_0\|_2^2 + 6\int_0^T \|f(t)\|_2^2\,dt + C(\Omega)M^3T_* \le M,$$

which is a contradiction. This proves the uniform bound. Using this bound, we deduce from (37) that

$$\max_{0\le t\le T_*} \|\nabla u_k(t)\|_2^2 + \int_0^{T_*} \|Au_k(t)\|_2^2\,dt \le M \tag{39}$$

for all $k \ge 1$. Furthermore, taking $\Phi = u_k'(t)$ in (35), we easily obtain

$$\int_0^{T_*} \|u_k'(t)\|_2^2\,dt \le CM \tag{40}$$

for all $k \geq 1$. Therefore, by the Aubin-Lions lemma, we may assume that $u_k \to u$ strongly in $L^2(0, T_*; V)$ and weakly in $L^2(0, T^*; W^{2,2}(\Omega)^3)$, and $u_k' \to u'$ weakly in $L^2(0, T_*; L^2(\Omega)^3)$. Using this convergence, we easily deduce from (35) that

$$u'(t) + P\left[(u(t) \cdot \nabla)u(t)\right] + Au(t) = Pf(t) \in H$$

for a.a. $t \in [0, T]$. Hence by Theorem 1.9, there is $p(t) \in W^{1,2}(\Omega)$ such that

$$u'(t) + (u(t) \cdot \nabla)u(t) - \Delta u(t) - f(t) = -\nabla p(t) \in L^2(\Omega)^3$$

for a.a. $t \in [0, T]$. From the regularity of u, it follows immediately that $p \in L^2(0, T; W^{1,2}(\Omega))$. Finally, since

$$u' - \Delta u = f - \nabla p - (u \cdot \nabla)u \in L^2(0, T_*; L^2(\Omega)^3),$$

it follows from the regularity theory of the heat equation that

$$u \in C([0, T_*]; V).$$

This proves the existence part of the theorem. The uniqueness proof is quite standard and omitted. □

Theorem 3.6 (Global existence when $f = 0$)**.** *Assume that $f = 0$. Then there is a constant $c(\Omega) > 0$, depending only on Ω, such that if $u_0 \in V$ satisfies*

$$\|u_0\|_2 \|\nabla u_0\|_2 \leq c(\Omega),$$

then there exists a unique strong solution u of (NS) in $[0, \infty) \times \Omega$. Moreover, we have

$$\|\nabla u(t)\|_2 \leq \|\nabla u_0\|_2 \exp(-Mt) \quad \text{for all } 0 < t < \infty,$$

where $M > 0$ is a constant depending only on Ω.

Proof. Since $f = 0$, it follows from (29) that

$$u_k \in W^{1,2}(0, T; H_k) \quad \text{for any } 0 < T < \infty$$

and

$$\|u_k(t)\|_2^2 + 2\int_0^t \|\nabla u_k(s)\|_2^2 \, ds \leq \|u_k(0)\|_2^2 \leq \|u_0\|_2^2 \tag{41}$$

for all $0 \leq t < \infty$. Hence the nonlinear term in (36) can be estimated as follows:

$$\begin{aligned} 2\|(u_k(t) \cdot \nabla)u_k(t)\|_2^2 &\leq 2\|u_k(t)\|_3^2 \|\nabla u_k(t)\|_6^2 \\ &\leq 2\|u_k(t)\|_2 \|u_k(t)\|_6 \|\nabla u_k(t)\|_6^2 \\ &\leq C(\Omega)\|u_0\|_2 \|\nabla u_k(t)\|_2 \|Au_k(t)\|_2^2. \end{aligned}$$

Substituting this into (36), we obtain

$$\frac{d}{dt}\|\nabla u_k(t)\|_2^2 + (1 - C(\Omega)\|u_0\|_2\|\nabla u_k(t)\|_2)\,\|Au_k(t)\|_2^2 \leq 0$$

and so

$$\|\nabla u_k(t)\|_2^2 + \int_{t_0}^{t} (1 - C(\Omega)\|u_0\|_2\|\nabla u_k(s)\|_2)\,\|Au_k(s)\|_2^2\,ds \leq \|\nabla u_k(t_0)\|_2^2 \quad (42)$$

for all $0 \leq t_0 < t < \infty$.

Now, assuming that the initial data u_0 satisfies

$$C(\Omega)\|u_0\|_2\|\nabla u_0\|_2 < \frac{1}{2}, \quad (43)$$

we prove the global uniform bound

$$C(\Omega)\|u_0\|_2\|\nabla u_k(t)\|_2 < \frac{1}{2} \qquad (0 < t < \infty,\ k \in \mathbb{N}).$$

Suppose that this bound were not true. Then by the continuity of $\|\nabla u_k(\cdot)\|_2$, there should be a time $t_1 \in (0, \infty)$ such that

$$C(\Omega)\|u_0\|_2\|\nabla u_k(t_1)\|_2 = \frac{1}{2}$$

and

$$C(\Omega)\|u_0\|_2\|\nabla u_k(t)\|_2 < \frac{1}{2} \quad \text{for all } t \in [0, t_1).$$

But this would imply, by virtue of (42) and (25), that

$$\|\nabla u_k(t_1)\|_2^2 + \frac{1}{2}\int_0^{t_1} \|Au_k(t)\|_2^2\,ds \leq \|\nabla u_k(0)\|_2^2 \leq \|\nabla u_0\|_2^2,$$

which is a contradiction. This proves the uniform bound. Hence from (42), we derive

$$\|\nabla u_k(t)\|_2^2 + \frac{1}{2}\int_{t_0}^{t} \|Au_k(s)\|_2^2\,ds \leq \|\nabla u_k(t_0)\|_2^2$$

for all $0 \leq t_0 < t < \infty$. Therefore, passing to the limit as $k \to \infty$, we deduce the existence of a global strong solution of (NS) satisfying

$$\|\nabla u(t)\|_2^2 + \frac{1}{2}\int_{t_0}^{t} \|Au(s)\|_2^2\,dt \leq \|\nabla u(t_0)\|_2^2$$

for all $0 \leq t_0 < t < \infty$. The exponential decay of u follows easily from these inequalities, by using the Poincaré inequality. The proof is complete. $\square$

3.3 The structure theorem

Throughout this subsection, let Ω be a bounded domain in $\mathbb{R}^3$ with smooth (say C^2-) boundary. Assume further that $f = 0$ and $0 \neq u_0 \in H$.

Let u be a Leray-Hopf weak solution of (NS) in $(0, \infty) \times \Omega$. Then u satisfies the strong energy inequality:

$$(SEI)_{t_0} \qquad \frac{1}{2}\|u(t)\|_2^2 + \int_{t_0}^{t} \|\nabla u(s)\|_2^2 \, ds \leq \frac{1}{2}\|u(t_0)\|_2^2 \quad (t \geq t_0)$$

for all $t_0 \in [0, \infty) \setminus N$, where N is a subset of $(0, \infty)$ with measure zero.

Denote by $\mathcal{R} = \mathcal{R}(u)$ the set of all times $t_0 \in (0, \infty)$ such that

$$u \in C((t_0 - \varepsilon, t_1 + \varepsilon); V) \quad \text{for some } \varepsilon > 0.$$

Then $\mathcal{R}$ is obviously open. The structure of the set $\mathcal{R}$ of regular times of a Leray-Hopf weak solution u has been studied by Leray [12] and Scheffer [14].

Theorem 3.7 (The Leray structure theorem)**.** *The complement* $(0, \infty) \setminus \mathcal{R}$ *of* $\mathcal{R}$ *has Lebesgue measure zero and there is a positive time* T^* *such that* $(T^*, \infty) \subset \mathcal{R}$. *Moreover, if* $u_0 \in V$ *in addition, then there is a positive time* T_* *such that* $(0, T_*) \subset \mathcal{R}$.

Proof. Denote by $\mathcal{R}_+ = \mathcal{R}_+(u)$ the set of all times $t_0 \in (0, \infty)$ such that

$$u(t_0) \in V \quad \text{and} \quad (SEI)_{t_0} \text{ holds.}$$

The complement $(0, \infty) \setminus \mathcal{R}_+$ has Lebesgue measure zero. Moreover, it is quite obvious that $\mathcal{R} \subset \mathcal{R}_+$. On the other hand, recalling the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 \, ds \leq \frac{1}{2}\|u_0\|_2^2 \quad (t > 0),$$

we deduce that

$$\inf_{t \in \mathcal{R}_+} \|\nabla u(t)\|_2 = 0$$

and there exists $T^* \in \mathcal{R}_+$ such that

$$\|u(T^*)\|_2 \|\nabla u(T^*)\|_2 \leq c(\Omega).$$

By Theorem 3.6, there exists a strong solution u^* of (NS) in $[T^*, \infty) \times \Omega$ with the initial data $u(T^*)$. Then by the weak-strong uniqueness theorem, Theorem 3.3, we deduce that $u(t) = u^*(t)$ for all $t \in [T^*, \infty)$. This implies that

$$u \in C([T^*, \infty); V) \quad \text{and so} \quad (T^*, \infty) \subset \mathcal{R} \subset \mathcal{R}_+.$$

We next show that $(0,\infty)\setminus\mathcal{R}$ has Lebesgue measure zero. To show this, we write the open set $\mathcal{R}\cap(0,T^*)$ as the union of an at most countable collection of disjoint open intervals:

$$\mathcal{R}\cap(0,T^*)=\bigcup_{k\in\Lambda}(s_k,\tau_k)\,,$$

where Λ is some subset of $\mathbb{N}$. Let $t_0\in\mathcal{R}_+\cap(0,T^*)$ be fixed. Then by Theorem 3.5, there exists a strong solution $\overline{u}$ of (NS) in $[t_0,t_1]\times\Omega$ with the initial data $u(t_0)$, where t_1 is a time with $t_0<t_1<T^*$. Moreover, since $(SEI)_{t_0}$ holds, it follows from the weak-strong uniqueness theorem, Theorem 3.3, that

$$u(t)=\overline{u}(t)\quad\text{for all } t\in[t_0,t_1].$$

This implies that

$$u\in C([t_0,t_1];V)\quad\text{and}\quad(t_0,t_1)\subset\mathcal{R}\cap(0,T^*).$$

Hence there is one and only one $k\in\Lambda$ such that $(t_0,t_1)\subset(s_k,\tau_k)$, that is, $s_k\le t_0<t_1\le\tau_k$. Note that if $s_k<t_0$, then $t_0\in\mathcal{R}$. Therefore, if $t_0\notin\mathcal{R}$, then we must have $t_0=s_k$. This proves that

$$(0,T^*)\cap(\mathcal{R}_+\setminus\mathcal{R})\subset\{s_k:k\in\Lambda\}\,.$$

Recalling that $\Lambda\subset\mathbb{N}$, we deduce that

$$\mathcal{R}_+\setminus\mathcal{R}=(0,T^*]\cap(\mathcal{R}_+\setminus\mathcal{R})$$

is at most countable. Moreover, since $(0,\infty)\setminus\mathcal{R}_+$ has Lebesgue measure zero and $\mathcal{R}\subset\mathcal{R}_+$, it follows that $(0,\infty)\setminus\mathcal{R}$ has Lebesgue measure zero too.

To complete the proof, suppose that $u_0\in V$. Then by Theorems 3.5 and 3.3, there exists a time $T_*>0$ such that $u\in C([0,T_*];V)$; hence it follows that $(0,T_*)\subset\mathcal{R}$. The proof is complete. $\square$

To analyze $\mathcal{R}$ further, we need an important result for blow-up times of u, due to Leray as well. Following Leray [12], we call a finite time $t_1>0$ an *epoch of irregularity* of u if

(i) there is $t_0\in(0,t_1)$ such that $u\in C((t_0,t_1);V)$, but

(ii) there is no $t_2>t_1$ such that $u\in C((t_0,t_2);V)$.

Theorem 3.8 (Leray)**.** *Let t_1 be an epoch of irregularity of u such that*

$$u\in C((t_0,t_1);V)$$

for some $t_0 \in (0, t_1)$*. Then*

$$\|\nabla u(t)\|_2^4 \geq \frac{c(\Omega)}{t_1 - t} \quad \textit{for all } t \in (t_0, t_1).$$

Consequently,

$$\lim_{t \to t_1^-} \|\nabla u(t)\|_2 = \int_{t_0}^{t_1} \|\nabla u(t)\|_2^4 \, dt = \infty$$

and

$$(t_1 - t_0)^{1/2} \leq C(\Omega) \int_{t_0}^{t_1} \|\nabla u(t)\|_2^2 \, dt.$$

Proof. Suppose to the contrary that

$$\|\nabla u(t_*)\|_2^4 < \frac{c(\Omega)}{t_1 - t_*}$$

for some $t_* \in (t_0, t_1)$. Then by Theorem 3.5, there is a time T_* with

$$T_* \geq \frac{c(\Omega)}{\|\nabla u(t_*)\|_2^4} > t_1 - t_*$$

such that there exists a strong solution $\overline{u}$ of (NS) in $[t_*, t_* + T_*] \times \Omega$ with the initial data $u(t_*)$. By the continuity of ∇u on $[t_0, t_1)$, we may assume that u satisfies the strong energy inequality at time t_*, that is, $(SEI)_{t_*}$ holds. Hence by the weak-strong uniqueness theorem, Theorem 3.3, we deduce that $u = \overline{u}$ on $[t_*, t_* + T_*]$. This contradicts that t_1 is an epoch of irregularity of u. We have thus shown that

$$\frac{c(\Omega)}{t_1 - t} \leq \|\nabla u(t)\|_2^4$$

for all $t \in (t_0, t_1)$. Integrating over (t_0, t_1), we have

$$\infty = \int_{t_0}^{t_1} \frac{c(\Omega)}{t_1 - t} \, dt \leq \int_0^{\infty} \|\nabla u(t)\|_2^4 \, dt.$$

Taking the square-root and integrating over (t_0, t_1), we also have

$$2c(\Omega)^{1/2}(t_1 - t_0)^{1/2} = \int_{t_0}^{t_1} \frac{c(\Omega)^{1/2}}{(t_1 - t)^{1/2}} \, dt \leq \int_{t_0}^{t_1} \|\nabla u(t)\|_2^2 \, dt.$$

This completes the proof. □

The Leray structure theorem was refined by a partial regularity theorem due to Scheffer (1976) in terms of the Hausdorff measure. Let E be a subset of $\mathbb{R}^n$ and α a positive number. The α-dimensional *Hausdorff measure* $\mathcal{H}^\alpha(E)$ of E is defined by

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha(E),$$

where

$$\mathcal{H}^\alpha_\delta(E) = \inf \sum_i (\operatorname{diam} B_i)^\alpha ,$$

the infimum being taken over all at most countable coverings $\{B_i\}$ of E constituted by closed balls B_i with $\operatorname{diam} B_i < \delta$. Note that if $\mathcal{H}^\alpha(E) < \infty$, then $\mathcal{H}^\beta(E) = 0$ for all $\beta > \alpha$. The number

$$\inf \left\{\beta > 0 : \mathcal{H}^\beta(E) = 0\right\} = \inf \left\{\alpha > 0 : \mathcal{H}^\alpha(E) < \infty\right\}$$

is called the *Hausdorff dimension* of E.

Theorem 3.9. *The $1/2$-dimensional Hausdorff measure of $(0, \infty) \setminus \mathcal{R}$ is equal to zero. Therefore, the Hausdorff dimension of $(0, \infty) \setminus \mathcal{R}$ is less than or equal to one half.*

Proof. From the proof of Theorem 3.7, we recall that

$$(0, \infty) \setminus \mathcal{R} = (0, T^*] \setminus \mathcal{R}$$

and

$$\mathcal{R} \cap (0, T_*) = \bigcup_{k \in \Lambda} (s_k, \tau_k) \qquad \text{(disjoint union)},$$

where Λ is some subset of $\mathbb{N}$. By the definition of $\mathcal{R}$, we have

$$u \in C((s_k, \tau_k); V) \quad \text{for each } k.$$

Moreover, since $\tau_k \notin \mathcal{R}$, there can not be $\tau'_k > \tau_k$ such that $u \in C((s_k, \tau'_k); V)$. Hence each τ_k is an epoch of irregularity of u. Hence by Theorem 3.8, we have

$$\begin{aligned}
\sum_{k \in \Lambda} (\tau_k - s_k)^{1/2} &\le C(\Omega) \sum_{k \in \Lambda} \int_{s_k}^{\tau_k} \|\nabla u(t)\|_2^2 \, dt \\
&\le C(\Omega) \int_0^\infty \|\nabla u(t)\|_2^2 \, dt \le C(\Omega) \|u_0\|_2^2.
\end{aligned}$$

Given $\delta > 0$, let Λ^δ be a finite subset of Λ such that

$$\sum_{k \notin \Lambda^\delta} (\tau_k - s_k) < \delta \quad \text{and} \quad \sum_{k \notin \Lambda^\delta} (\tau_k - s_k)^{1/2} < \delta$$

Then the closed set

$$[0, T^*] \setminus \bigcup_{k \in \Lambda^\delta} (s_k, \tau_k)$$

consists of a finite number of disjoint closed intervals $B_1, ..., B_N$. Note that

$$[0, T^*] \setminus \mathcal{R} \subset ([0, T^*] \setminus \mathcal{R}) \bigcup \left(\bigcup_{k \notin \Lambda^\delta} (s_k, \tau_k) \right) = \bigcup_{j=1}^N B_j$$

and for each $k \notin \Lambda^\delta$ there is one and only one j such that $(s_k, \tau_k) \subset B_j$. For each j, let Λ_j be the set of all $k \in \Lambda$ such that $(s_k, \tau_k) \subset B_j$. Then

$$\Lambda \setminus \Lambda^\delta = \bigcup_{j=1}^{N} \Lambda_j \qquad \text{(disjoint union)}$$

and

$$B_j = \Big[([0, T^*] \setminus \mathcal{R}) \bigcap B_j\Big] \bigcup \left(\bigcup_{k \in \Lambda_j} (s_k, \tau_k) \right)$$

Since $(0, T^*) \setminus \mathcal{R}$ has Lebesgue measure zero, we have

$$\operatorname{diam} B_j = \sum_{k \in \Lambda_j} (\tau_k - s_k) \quad \text{for each } j.$$

Note finally that

$$\operatorname{diam} B_j \leq \sum_{k \notin \Lambda^\delta} (\tau_k - s_k) < \delta$$

and

$$\begin{aligned} \sum_{j=1}^{N} (\operatorname{diam} B_j)^{1/2} &\leq \sum_{j=1}^{N} \left(\sum_{k \in \Lambda_j} (\tau_k - s_k) \right)^{1/2} \\ &< \sum_{j=1}^{N} \sum_{k \in \Lambda_j} (\tau_k - s_k)^{1/2} \\ &= \sum_{k \notin \Lambda_\delta} (\tau_k - s_k)^{1/2} < \delta. \end{aligned}$$

This completes the proof. □

3.4 Further regularity of weak solutions

Further regularity of Leray-Hopf solutions of (NS) can be deduced from the classical maximal regularity results for the linear Stokes equations. Let Ω be a bounded smooth domain of $\mathbb{R}^3$ and $T > 0$ a finite time.

The following fundamental result was established by Giga and Sohr [8] applying an abstract perturbation theorem (see Maremonti and Solonnikov [13] for a more elementary approach).

Theorem 3.10. *Let $1 < q, s < \infty$. Then for each $f \in L^s(0, T; L^q(\Omega)^3)$ there exists a unique pair (u, p) such that*

$$u \in L^s(0, T; W^{1,q}_{0,\sigma}(\Omega) \cap W^{2,q}(\Omega)^3) \cap W^{1,s}(0, T; L^q_\sigma(\Omega)), \quad u(0) = 0,$$

$$p \in L^2(0,T;W^{1,q}(\Omega)), \quad \int_\Omega p(t,x)\,dx = 0 \quad \textit{for a.a. } t \in (0,T),$$

and

$$u_t - \Delta u + \nabla p = f \quad \textit{a.e. in } (0,T)\times\Omega.$$

Moreover, we have

$$\|u\|_{L^s(0,T;W^{2,q}(\Omega)^3)} + \|u_t\|_{L^2(0,T;L^q(\Omega)^3)} + \|\nabla p\|_{L^s(0,T;L^q(\Omega)^3)} \\ \le C(q,s,\Omega)\|f\|_{L^s(0,T;L^q(\Omega)^3)}.$$

As a consequence of Theorem 3.10, we can obtain further regularity results for weak solutions of (NS). To do this, we need to prove two preliminary lemmas. The first one is easily proved by using the Hölder and Sobolev inequalities; we omit its details.

Lemma 3.11. *Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$.*

(i) For all (s,q) satisfying

$$\frac{2}{s} + \frac{3}{q} = \frac{3}{2}, \quad 2 \le s \le \infty, \quad \textit{and} \quad 2 \le q \le 6,$$

we have

$$u \in L^s(0,T;L^q(\Omega)^3).$$

(ii) For all (s,q) satisfying

$$\frac{2}{s} + \frac{3}{q} = 4, \quad 1 < s < 2, \quad \textit{and} \quad 1 < q < \frac{3}{2},$$

we have

$$(u\cdot\nabla)u \in L^s(0,T;L^q(\Omega)^3).$$

Secondly, following Seregin [15], we prove a uniqueness result.

Lemma 3.12. *Suppose that $u \in L^1(0,T;L^2_\sigma(\Omega))$ satisfies*

$$\int_0^T \int_\Omega u\cdot(v_t + \Delta v)\; dxdt = 0$$

for all $v \in C_0^\infty([0,T)\times\Omega)^3$ with $\operatorname{div} v = 0$. Then

$$u = 0 \quad \textit{in } (0,T)\times\Omega.$$

Proof. By a simple density argument, we have

$$\int_0^T [\eta'(t)\,(u(t),\Phi) - \eta(t)\,(u(t),A\Phi)]\; dt = 0 \tag{44}$$

for all $\Phi \in V \cap W^{2,2}(\Omega)^3$ and $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$, where A is the Stokes operator in H. Let $\{w_k\}$ be an orthonormal basis of H consisting of eigenvectors of A. Then

$$u(t) = \sum_{k=1}^{\infty} \hat{u}_k w_k \quad \text{in } H,$$

where

$$\hat{u}_k(t) = (u(t), w_k)\,.$$

Let $k \in \mathbb{N}$ be fixed. Then taking $\Phi = w_k$ in (44), we have

$$\int_0^T (\eta'(t)\hat{u}_k(t) - \lambda_k \hat{u}_k(t)\eta(t))\, dt = 0$$

for all $\eta \in W^{1,1}([0,T])$ with $\eta(T) = 0$. Hence it follows that

$$\hat{u}_k \in W^{1,1}([0,T]), \quad \hat{u}_k(0) = 0 \quad \text{and} \quad \hat{u}_k'(t) + \lambda_k \hat{u}_k(t) = 0 \quad (0 < t < T).$$

Multiplying the ODE by $e^{\lambda_k t}$, we derive

$$\frac{d}{dt}\left[e^{\lambda_k t}\hat{u}_k(t)\right] = 0$$

and so

$$\hat{u}_k(t) = \hat{u}_k(0)e^{-\lambda_k t} = 0 \quad \text{for all } t \in [0,T].$$

This completes the proof. $\square$

Theorem 3.13. *Suppose that $f \in L^2(0,T; L^2(\Omega)^3)$ and $u_0 \in H$. Let u be a weak solution of (NS) with p being an associated pressure. Then for every $0 < \delta < T$, we have*

$$u \in L^s(\delta, T; W^{1,q}_{0,\sigma}(\Omega) \cap W^{2,q}(\Omega)^3) \cap W^{1,s}(\delta, T; L^q_\sigma(\Omega))$$

and

$$p \in L^s(\delta, T; W^{1,q}(\Omega)) \cap L^s(\delta, T; L^{3q/(3-q)}(\Omega))$$

for all (s,q) satisfying

$$\frac{2}{s} + \frac{3}{q} = 4, \quad 1 < s < 2, \quad \textit{and} \quad 1 < q < \frac{3}{2}.$$

Proof. Fixing $\eta = \eta(t) \in C_0^\infty((0,T])$, we define

$$\overline{u}(t,x) = \eta(t)u(t,x).$$

Then from the weak formulation (8) of (NS), we easily deduce that

$$-\int_0^T (\overline{u}(t), v_t(t) + \Delta v(t))\, dt = \int_0^T \left(\overline{f}(t), v(t)\right)\, dt$$

for all $v \in C_0^\infty([0,T) \times \Omega)^3$ with $\operatorname{div} v = 0$, where

$$\overline{f}(t) = \eta(t) f(t) + \eta'(t) u(t) - \eta(t)(u(t) \cdot \nabla) u(t).$$

Let (s, q) be any pair satisfying

$$\frac{2}{s} + \frac{3}{q} = 4, \quad 1 < s < 2, \quad \text{and} \quad 1 < q < \frac{3}{2}.$$

Then by Lemma 3.11,

$$\overline{f}(t) \in L^s(0, T; L^q(\Omega)^3).$$

Hence it follows from Theorem 3.10 that there exists a pair (u^*, p^*) such that

$$u^* \in L^s(0, T; W^{1,q}_{0,\sigma}(\Omega) \cap W^{2,q}(\Omega)^3) \cap W^{1,s}(0, T; L^q_\sigma(\Omega)),$$

$$p^* \in L^2(0, T; W^{1,q}(\Omega)),$$

and

$$-\int_0^T (u^*(t), v_t(t) + \Delta v(t))\, dt - \int_0^T (p^*(t), \operatorname{div} v(t))\, dt = \int_0^T \left(\overline{f}(t), v(t)\right)\, dt$$

for all $v \in C_0^\infty([0,T) \times \Omega)^3$. Note that

$$\overline{u} - u^* \in L^1(0, T; L^2_\sigma(\Omega)).$$

Hence by Lemma 3.12, we conclude that

$$\overline{u} = u^* \quad \text{in } (0, T) \times \Omega.$$

This competes the proof. $\square$

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