Segre stratifications on the moduli spaces of G-bundles over a curve

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Segre–Nagata's bound

X: smooth irreducible curve of genus $g \geq 2$ over $\mathbb{C}$.

S: ruled surface over X.

Theorem (Segre 1887–89, Nagata 1970)

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Note When $S = \mathbb{P}V$ for a rank 2 bundle V and σ corresponds to a line subbundle $L \subset V$,

$$\sigma^2 = \deg V - 2\deg L.$$

Rank 2 bundles

V: vector bundle over X of rank 2 and degree d.

The Segre invariant of V is defined by

$$s(V) := \min_{L \subset V} \{d - 2\deg L \mid L : \text{line subbundles}\}.$$

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Theorem (Gunning, Stuhler, Lange–Narasimhan)

(1) $s(V) \leq g$.

(2) If $d - g$ is even/odd, $s(V) = g/g - 1$ for a general V.

(3) If $s(V) = g$, there are 1-dimensional family of line subbundles of (maximal) degree $\frac{1}{2}(d - g)$.

(4) A general V with $s(V) < g$ has finite number of line subbundles of maximal degree.

Generalized Segre invariant

V: principal G-bundle over X

$P \subset G$: parabolic subgroup $\rightsquigarrow$ V/P: induced G/P-bundle

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(r-th Segre invariant) For a vector bundle V of rank n and degree d, and for each $r = 1, 2, \cdots, n-1$,

$$s_r(V) := \min_{E \subset V} \{rd - n\deg(E) \mid \mathrm{rk}(E) = r\}.$$

Upper bound

(Holla–Narasimhan 2001) For any G-bundle V,

$$s(V; P) \leq \dim(G/P) \cdot g.$$

(Mukai–Sakai 1985) $s_r(V) \leq r(n-r)g.$

(Hirschowitz 1986) For a vector bundle V of rank n,

$$s_r(V) \leq r(n-r)(g-1) + n - 1.$$

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Naïve Proof Consider the extensions

$$0 \to E^{(r,e)} \to V^{(n,d)} \to Q^{(n-r,d-e)} \to 0.$$

Find the largest e such that

$$\dim M(r,e) + \dim M(n-r,d-e) + \dim \mathbb{P}H^1(X, Q^* \otimes E) \geq \dim M(n,d).$$

Segre stratification (vector bundle case)

$$\boxed{s_r(V) \ := \ \min_{E \subset V} \{rd - n \deg(E) \mid \mathrm{rk}(E) = r\}}$$

In the case of $M(n, 0)$ with respect to s_r, define

$$M(n, 0; s) \ := \ \{V \in M(n, 0) \mid s_r(V) = s\}.$$

Then, $$M(n, 0) \ = \ \bigsqcup_{0 < k \leq k_0} M(n, 0; s = kn).$$

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(Brambila-Paz–Lange 1998, Russo–Teixidor i Bigas 1999)

(1) For $0 < k \leq k_0$, $M(n,0;kn)$ is nonempty and irreducible.

(2) $\dim M(n,0;(k+1)n) - \dim M(n,0;kn) \ = \ n$.

(3) $M(n,0;kn) \ \subset \ \overline{M(n,0;(k+1)n)}$.

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Note The key-point is the non-emptyness.

Lange–Narasimhan's approach (rank 2 case)

Every vector bundle V of rank 2 and degree $d \gg 0$ fits into

$$0 \to \mathcal{O}_X \to V \to L \to 0,$$

hence $[V] \in \mathbb{P}H^1(X, L^{-1}) \cong \mathbb{P}H^0(X, K_X L)^\vee$.

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(Atiyah, Lange–Narasimhan) The Segre stratification matches with the stratification given by the higher secant varieties of X:

$$s(V) \leq d - 2m \iff [V] \in Sec^{d-m} X.$$

For example, this reproves the Nagata's bound ($s(V) \leq g$).

Proof

If V has a subbundle M of degree $m > 0$, then we get

$$\begin{tikzcd}
 & & \tau \arrow[r, equal] & \tau & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & V \arrow[r] \arrow[u] & L \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u, equal] & \mathcal{O}_X \oplus M \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Hence

$$[V] \in \mathbb{P}Ker\left[H^1(X, L^{-1}) \longrightarrow H^1(X, M^{-1}) = H^1(X, L^{-1}(\tau))\right]$$

for some torsion sheaf τ of degree $d - m$.

Therefore, $[V] \in Sec^{d-m}X$. And vice versa. $\square$

Goal

(1) Generalize Lange–Narasimhan's idea to bundles of higher rank to reprove the Hirschowitz' bound.

(2) Study the Segre stratifications on the moduli spaces of symplectic and orthogonal bundles.

Joint works with George H. Hitching:

- Secant varieties and Hirschowitz bound on vector bundles over a curve, Manuscr. Math. (2010)
- Lagrangian subbundles of symplectic bundles over a curve, Math. Proc. Camb. Phil. Soc. (2012)
- A stratification on the moduli spaces of symplectic and orthogonal bundles over a curve, Internat. J. Math. (2014)
- Lagrangian subbundles of orthogonal bundles of odd rank over a curve, arXiv: 1402.2816
- Non-defectivity of Grassmannian bundles over a curve

General picture

$$0 \longrightarrow \mathcal{O}_X \longrightarrow V \to L \longrightarrow 0$$

$$0 \longrightarrow E^r \longrightarrow V^n \longrightarrow F^{n-r} \longrightarrow 0$$

$$0 \longrightarrow E^n \longrightarrow V^{2n} \to E^* \longrightarrow 0$$

rank 2	X	$\mathbb{P}H^1(L^*)$
higher rank	$\bigcup_{x\in X}(\mathbb{P}E_x \times \mathbb{P}F_x^*) \subset \mathbb{P}(E \otimes F^*)$	$\mathbb{P}H^1(E \otimes F^*)$
symplectic	$\mathbb{P}E \subset \mathbb{P}(Sym^2 E)$	$\mathbb{P}H^1(Sym^2 E)$
orthogonal	$Gr(2, E) \subset \mathbb{P}(\wedge^2 E)$	$\mathbb{P}H^1(\wedge^2 E)$
ortho. (odd rank)		

(1) Vector bundle case

For fixed E^r and F^{n-r}, the extensions

$$0 \to E \to V \to F \to 0$$

are classified by the space

$$\mathbb{P}H^1(X, Hom(F, E)) \;\cong\; \mathbb{P}H^0(\mathbb{P}Hom(F, E), \pi^* K_X \otimes \mathcal{O}_X(1))^\vee.$$

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For stable E and F with $\mu(F) - \mu(E) \gg 0$, we get a scroll

$$\mathbb{P}Hom(F,E) \subset \mathbb{P}H^1(X, Hom(F,E)),$$

which contains the Segre bundle

$$\mathcal{S}_{E,F} = \bigcup_{x \in X} (\mathbb{P}E_x \times \mathbb{P}F_x^*) \subset \mathbb{P}Hom(F,E).$$

Segre invariant and Segre bundle

$$\boxed{\mathcal{S}_{E,F} \;\subset\; \mathbb{P}Hom(F,E) \;\subset\; \mathbb{P}H^1(X, Hom(F,E)) =: \mathbb{P}^N}$$

Recall:

$$s_r(V) \;=\; \min_{H \subset V} \{rd - n\deg(H) \mid \mathrm{rk}(H) = r\}\,.$$

Theorem Let $E \in M_X(n-r, 0)$ and $F \in M_X(r,d)$ with $d \gg 0$. Given $[V] \in \mathbb{P}^N$, if $[V] \in Sec^k(\mathcal{S}_{E,F})$, then V has a subbundle H of rank r and degree $\geq d-k$, hence

$$s_r(V) \leq rd - n(d-k).$$

Proof of Hirschowitz bound

$$\boxed{\mathcal{S}_{E,F} \ \subset \ \mathbb{P}Hom(F,E) \ \subset \ \mathbb{P}H^1(X, Hom(F,E)) =: \mathbb{P}^N}$$

Since $\dim \mathcal{S}_{E,F} = (r-1) + (n-r-1) + 1 = n-1$,

$exp \dim(Sec^k\,(\mathcal{S}_{E,F})) = \min\{(n-1)k + (k-1) = nk - 1,\ \dim \mathbb{P}^N\}$.

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Since $\dim \mathbb{P}^N = (n-r)(d + r(g-1)) - 1$,

$$[V] \in Sec^m(\mathcal{S}_{E,F}) \quad \text{for } m = \left\lceil \frac{n-r}{n}(d + r(g-1)) \right\rceil .$$

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Hence by the previous Theorem,

$$s_r(V) \le rd - n(d-m).$$

Non-defectiveness of $Sec^k(\mathcal{S}_{E,F})$

Terracini Lemma $\dim(Sec^k Z) = \dim < \mathbb{T}_{z_1} Z, \cdots, \mathbb{T}_{z_k} Z >$

Hence it suffices to show that for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}$,

$$\dim < \mathbb{T}_{[\mu_1 \otimes e_1]}\mathcal{S}, \cdots, \mathbb{T}_{[\mu_m \otimes e_k]}\mathcal{S} > = \min\{nk-1, \dim \mathbb{P}^N\}.$$

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Lemma For $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}|_x = \mathbb{P}F_x^* \times \mathbb{P}E_x$, consider

$$0 \longrightarrow F^* \xrightarrow{\mu_i} \widehat{F_i^*} \longrightarrow \mathbb{C}_x \longrightarrow 0,$$

$$0 \longrightarrow E \xrightarrow{e_i} \widehat{E}_i \longrightarrow \mathbb{C}_x \longrightarrow 0.$$

For the elementary transformation $F^* \otimes E \subset \widehat{F_i^*} \otimes \widehat{E}_i$,

$$\mathbb{T}_{[\mu_i \otimes e_i]}\mathcal{S}_{E,F} = \mathbb{P}\ker\left[H^1(X, F^* \otimes E) \longrightarrow H^1(X, \widehat{F_i^*} \otimes \widehat{E}_i)\right].$$

Therefore, for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F},\ 1 \leq i \leq k$,

$$
\begin{aligned}
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where $\widehat{F^*}$ (resp. $\widehat{E}$) are the elementary transformations of F^* (resp. E) at $\mu_1, \cdots, \mu_k$ (resp. $e_1, \cdots, e_k$).

Therefore, for general points $[\mu_i \otimes e_i] \in \mathcal{S}_{E,F}, \; 1 \leq i \leq k$,

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Finally, as desired,

$$\dim(Sec^k(\mathcal{S}_{E,F})) \;=\; \dim\mathbb{P}\ker \;=\; \min\{nk-1, \dim\mathbb{P}^N\}$$

by **Hirschowitz' lemma**: For general E and F^*,

$$\mu(F^* \otimes E) \leq g - 1 \implies H^0(X, F^* \otimes E) = 0.$$

$\square$

(2) Symplectic/orthogonal bundles of even rank

- A symplectic(orthogonal) bundle over X is a vector bundle V equipped with an isomorphism $\omega : V \cong V^*$, which is anti-symmetric(symmetric): ${}^t\omega = -\omega$ (${}^t\omega = \omega$).

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- A subbundle E of V is called isotropic if $\omega(E)|_E \equiv 0$, Lagrangian if it is isotropic and $\mathrm{rk}(E) = \mathrm{rk}(V)/2$.
- For a Lagrangian subbundle $E \subset V$,

$$0 \longrightarrow E \longrightarrow V \longrightarrow (E^\perp)^* \cong E^* \longrightarrow 0.$$

- (Hitching) The subspace $H^1(X, Sym^2 E)$ and $H^1(X, \wedge^2 E)$ parameterizes the symplectic(orthogonal) extensions inside $H^1(X, E \otimes E)$, with respect to which E is Lagrangian.

Segre stratification for symplectic bundles

For a symplectic/orthogonal bundle V of rank $2n$,

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We get a stratification on the moduli space $Sp_X(2n)$:

$$Sp(2n; t) := \{[V] \in Sp_X(2n) \mid t(V) = t\}.$$

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Theorem

- For any symplectic bundle V of rank $2n$, $t(V) \leq n(g-1)+1$.
- For each even t with $2 \leq t \leq n(g-1)+1$,
  the strara $Sp(2n;t)$ is nonempty and irreducible.
- $Sp(2n;t) \subset \overline{Sp(2n;t+2)}$ of codimension $n+1$.

Segre stratification for orthogonal bundles

The moduli space has two components up to $w_2(V) \in \{0, 1\}$:

$$O_X(2n) \;=\; O_X(2n)^+ \bigsqcup O_X(2n)^-.$$

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Theorem

- $w_2(V) = 0/1 \iff t(V) = 0/2 \mod 4.$
- For any orthogonal bundle V of rank $2n$, $t(V) \leq n(g-1)+3$.
- For each even t with $2 \leq t \leq n(g-1)+3$, the strata $O(2n; t) := \{[V] \in O_X(2n) \mid t(V) = t\}$ is non-empty, irred.
- $O(2n; t) \ \subset \ \overline{O(2n; t+4)}$ of codimension $2(n-1)$.
- For the orthogonal bundles with $t(V) \geq n(g-1)+2$, all the maximal subbundles of rank n are not Lagrangian.

Fiber bundles inside extension spaces

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Get a Veronese bundle and a Grassmannian bundle:

$$\mathbb{P}E \subset \mathbb{P}(Sym^2 E) \subset \mathbb{P}H^1(X, Sym^2 E),$$
$$Gr(2, E) \subset \mathbb{P}(\wedge^2 E) \subset \mathbb{P}H^1(X, \wedge^2 E).$$

Relation to $t(V)$

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Theorem Given $[V] \in \mathbb{P}^N$, if $[V] \in Sec^k(\mathbb{P}E)$, then V has a Lagrangian subbundle H of degree $\geq -(\deg E + k)$, hence $t(V) \leq 2(\deg E + k)$.

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(3) (Variant of Hirschowitz' lemma) For a general $E \in U_X(n, d)$, if $\mu(E \otimes E) \leq g - 1$, then $H^0(X, E \otimes E) = 0$. Therefore,

$$H^0(X, Sym^2 E) \ = \ 0 \ = \ H^0(X, \wedge^2 E).$$

(3) Orthogonal bundles of odd rank

Given an orthogonal bundle V of rank $2n+1$,

- An isotropic subbundle of rank n is called Lagrangian.
- For Lagrangians: $0 \to E \to E^{\perp} \to \det V \cong \mathcal{O}_X \to 0$.
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- $O(2n;t) \subset \overline{O(2n;t+4)}$ of codimension $2n$.

Orthogonal extensions of odd rank

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Orthogonal extensions of odd rank

Recall: For Lagrangian subbundles, get
$0 \to E \to V \to (E^{\perp})^* \to 0$, where $E^{\perp}/E \cong \mathcal{O}_X$.

Let $[V] \in H^1(X, F \otimes E)$ be an extension: $0 \to E \to V \to F^* \to 0$.
Lemma The bundle V has an orthogonal structure w. r. t. which E is Lagrangian if and only if
(i) there is an extension $[f]: \ 0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_X \longrightarrow 0$
such that
(ii) as an element of $H^1(X, F \otimes F)$, $[V] \in H^1(X, \wedge^2 F)$ and
(iii) as an element of $H^1(X, \wedge^2 F)$, $[V] \in q^{-1}[f]$, where

$$0 \longrightarrow H^1(X, \wedge^2 E) \longrightarrow H^1(X, \wedge^2 F) \xrightarrow{q} H^1(X, E) \longrightarrow 0.$$

Secant varieties vs. $t(V)$

$$\begin{tikzcd} H^1(X,E) & \\ H^1(X,\wedge^2 F) \arrow[u, "q"] \arrow[r, "\pi"'] & H^1(X,\wedge^2(F/H)) \end{tikzcd}$$

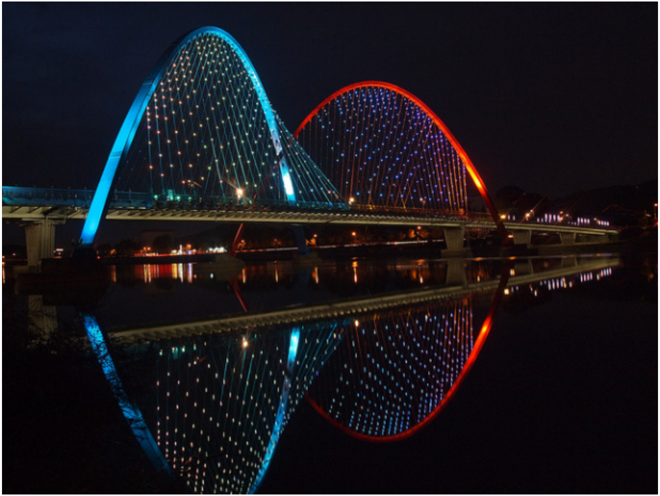
An orthogonal bundle corresponding to the extension $[V] \in q^{-1}[f]$ admits another Lagrangian subbundle $\widetilde{E}$ meeting E transversely

$$\Updownarrow$$

$W = V \perp \mathcal{O}_X$ admits two Lagrangian subbundles F and $\widetilde{F}$ meeting in H of rank 1

$$\Updownarrow$$

$[V]$ lies on $\pi^{-1}\left(Sec^k Gr(2, F/H)\right)$, $k = \frac{1}{2}(\widetilde{e} - e + 2h)$.



Thank you for attention!