Electron and muon g-2: Recent developments

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Outline

- Predicting the muon g-2 in the SM: recent progress
- Limiting 2HDMs with the muon g-2
- Testing the SM with the electron g-2
Predicting the muon g-2 in the SM: recent progress
The anomalous magnetic moment: the basics

- The Dirac theory predicts for a lepton \( l = e, \mu, \tau \):
  \[
  \vec{\mu}_l = g_l \left( \frac{e}{2m_l c} \right) \vec{s} \quad g_l = 2
  \]

- QFT predicts deviations from the Dirac value:
  \[
  g_l = 2 \left( 1 + a_l \right)
  \]

- Study the photon–lepton vertex:
  \[
  \bar{u}(p') \Gamma_\mu u(p) = \bar{u}(p') \left[ \gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m} F_2(q^2) + \ldots \right] u(p)
  \]

\[
F_1(0) = 1 \quad F_2(0) = a_l
\]

A pure “quantum correction” effect!
The muon g-2: the experimental result

Today: $a_\mu^{\text{EXP}} = (116592089 \pm 54_{\text{stat}} \pm 33_{\text{sys}}) \times 10^{-11} [0.5 \text{ppm}]$.

Future: new muon g-2 experiments proposed at:
- Fermilab E989, aiming at $\pm 16 \times 10^{-11}$, ie $0.14 \text{ppm}$
- J-PARC aiming at $0.1 \text{ ppm}$

Are theorists ready for this (amazing) precision? No(t yet)
The muon g-2: the QED contribution

\[ a_{\mu}^{\text{QED}} = \frac{1}{2}(\alpha/\pi) \]

Schwinger 1948

+ 0.765857426 (16) \((\alpha/\pi)^2\)

Sommerfield; Petermann; Suura&Wichmann '57; Elend '66; MP '04

+ 24.05050988 (28) \((\alpha/\pi)^3\)

Remiddi, Laporta, Barbieri ... ; Czarnecki, Skrzypek; MP '04; Friot, Greynat & de Rafael '05, Mohr, Taylor & Newell 2012

+ 130.8796 (63) \((\alpha/\pi)^4\)


+ 753.29 (1.04) \((\alpha/\pi)^5\) COMPLETED!

Kinoshita et al. '00, Yelkhovsky, Milstein, Starshenko, Laporta, Karshenboim,..., Kataev, Kinoshita & Nio '06, Kinoshita et al. 2012

Adding up, we get:

\[ a_{\mu}^{\text{QED}} = 116584718.951 (22)(77) \times 10^{-11} \]

from coeffs, mainly from 4-loop unc \[\downarrow\]

from \(\delta\alpha(\text{Rb})\)

with \(\alpha=1/137.035999049(90) [0.66 \text{ ppb}]\)
The muon g-2: the electroweak contribution

**One-loop term:**

\[ a_{\mu}^{EW} (1\text{-loop}) = \frac{5G_{\mu}m_{\mu}^2}{24\sqrt{2}\pi^2} \left[ 1 + \frac{1}{5} \left( 1 - 4\sin^2\theta_W \right)^2 + O \left( \frac{m_{\mu}^2}{M_{Z,W,H}^2} \right) \right] \approx 195 \times 10^{-11} \]

1972: Jackiw, Weinberg; Bars, Yoshimura; Altarelli, Cabibbo, Maiani; Bardeen, Gastmans, Lautrup; Fujikawa, Lee, Sanda; Studenikin et al. '80s

**One-loop plus higher-order terms:**

\[ a_{\mu}^{EW} = 153.6 (1) \times 10^{-11} \]

with \( M_{\text{Higgs}} = 125.6 (1.5) \text{ GeV} \)

Hadronic loop uncertainties and 3-loop nonleading logs.

Kukhto et al. '92; Czarnecki, Krause, Marciano '95; Knecht, Peris, Perrottet, de Rafael '02; Czarnecki, Marciano and Vainshtein '02; Degrassi and Giudice '98; Heinemeyer, Stockinger, Weiglein '04; Gribouk and Czarnecki '05; Vainshtein '03; Gnendiger, Stockinger, Stockinger-Kim 2013.
The muon g-2: the hadronic LO contribution (HLO)

\[ K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(s/m^2)} \]

\[ a_{\mu}^{\text{HLO}} = \frac{1}{4\pi^3} \int_{4m^2_\pi}^{\infty} ds K(s) \sigma^{(0)}(s) = \frac{\alpha^2}{3\pi^2} \int_{4m^2_\pi}^{\infty} \frac{ds}{s} K(s) R(s) \]

\[ a_{\mu}^{\text{HLO}} = 6903 (53)_{\text{tot}} \times 10^{-11} \]

\[ = 6923 (42)_{\text{tot}} \times 10^{-11} \]

\[ = 6949 (37)_{\text{exp}} (21)_{\text{rad}} \times 10^{-11} \]
HNLO: Vacuum Polarization

O(\(\alpha^3\)) contributions of diagrams containing hadronic vacuum polarization insertions:

\[ a_\mu^{\text{HNLO (vp)}} = -98 \ (1) \times 10^{-11} \]

Krause ’96, Alemany et al. ’98, Hagiwara et al. 2011
The muon g-2: the hadronic NLO contributions (HNLO) - LBL

- **HNLO: Light-by-light contribution**

  Unlike the HLO term, the hadronic l-b-l term relies at present on theoretical approaches.

  This term had a troubled life! Latest values:

  \[
  a_\mu^{\text{HNLO}(\text{lbl})} = +80 \ (40) \times 10^{-11} \quad \text{Knecht & Nyffeler '02}
  
  a_\mu^{\text{HNLO}(\text{lbl})} = +136 \ (25) \times 10^{-11} \quad \text{Melnikov & Vainshtein '03}
  
  a_\mu^{\text{HNLO}(\text{lbl})} = +105 \ (26) \times 10^{-11} \quad \text{Prades, de Rafael, Vainshtein '09}
  
  a_\mu^{\text{HHO}(\text{lbl})} = +116 \ (39) \times 10^{-11} \quad \text{Jegerlehner & Nyffeler '09}
  \]

  Results based also on Hayakawa, Kinoshita '98 & '02; Bijnens, Pallante, Prades '96 & '02

  “Bound” \( a_\mu^{\text{HNLO}(\text{lbl})} < \sim 160 \times 10^{-11} \)  

  Erler, Sanchez '06, Pivovarov '02; also Boughezal, Melnikov '11

  Pion exch. in holographic QCD agrees.  

  D.K.Hong, D.Kim '09; Cappiello, Catà, D'Ambrosio '11

  Lattice? Very hard but promising  

  Tom Blum @ Quark Confinement 2014

  Dispersive approach recently proposed  

  Colangelo, Hoferichter, Procura, Stoffer 1402.7081
The muon g-2: the hadronic NNLO contributions (HNNLO)

- **HNNLO: Vacuum Polarization**

  \[ a_\mu^{\text{HNNLO}(vp)} = 12.4 \ (1) \times 10^{-11} \]


- **HNNLO: Light-by-light**

  \[ a_\mu^{\text{HNNLO}(lbl)} = 3 \ (2) \times 10^{-11} \]

Adding up all contributions, we get the following SM predictions and comparisons with the measured value:

\[ a_{\mu}^{\text{EXP}} = 116592091 (63) \times 10^{-11} \]

<table>
<thead>
<tr>
<th>( a_{\mu}^{\text{SM}} \times 10^{11} )</th>
<th>( \Delta a_{\mu} = a_{\mu}^{\text{EXP}} - a_{\mu}^{\text{SM}} )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>116 591 809 (66)</td>
<td>282 (91) \times 10^{-11}</td>
<td>3.1 [1]</td>
</tr>
<tr>
<td>116 591 829 (57)</td>
<td>262 (85) \times 10^{-11}</td>
<td>3.1 [2]</td>
</tr>
<tr>
<td>116 591 855 (58)</td>
<td>236 (86) \times 10^{-11}</td>
<td>2.8 [3]</td>
</tr>
</tbody>
</table>

with the “conservative” \( a_{\mu}^{\text{HNLO(lbl)}} = 116 (39) \times 10^{-11} \) and the LO hadronic from:


Note that the th. error is now about the same as the exp. one
Limiting two-Higgs-doublet models

A. Broggio, E.J. Chun, MP, K. Patel, S. Vempati

2HDMs with Natural Flavor Conservation: $a_\mu$

- One-loop contribution:

$$\delta a^{2\text{HDM}}_\mu(1\text{loop}) = \frac{G_F m_\mu^2}{4\pi^2 \sqrt{2}} \sum_{j=h,H,A,H^\pm} (y^j_\mu)^2 r^j_\mu f_j(r^j_\mu)$$

For $r^j_\mu = \frac{m^2_\mu}{M^2_j} \ll 1$:

$$f_{h,H}(r) \sim - \ln r - 7/6 + O(r) > 0$$

$$f_A(r) \sim + \ln r + 11/6 + O(r) < 0$$

$$f_{H^\pm}(r) \sim - 1/6 + O(r) < 0$$

- Two-loop Barr-Zee type diagrams:

$$\delta a^{2\text{HDM}}_\mu(2\text{loop }- \text{BZ}) = \frac{G_F m_\mu^2}{4\pi^2 \sqrt{2}} \frac{\alpha_{\text{em}}}{\pi} \sum_{f; i=h,H,A} N^c_f Q^2_f y^i_\mu y^i_f r^i_f \ g_i(r^i_f)$$

$$g_{h,H}(r) < 0$$

$$g_A(r) > 0$$

Type II, $M_A \geq 3\text{GeV}$ & $\tan \beta \geq 5$: (2loop)$_A > (1\text{loop})_A$. Similar for X.
The 1σ, 2σ and 3σ regions allowed by $\Delta a_\mu$ in the $M_A$-tan $\beta$ plane taking the limit of $\beta - \alpha = \pi/2$ and $M_{h(H)} = 126 (200)$ GeV in type II (left panel) and type X (right panel) 2HDMs. The regions below the dashed (dotted) lines are allowed at 3σ (1.4σ) by $\Delta a_e$. The vertical dashed line corresponds to $M_A = M_h/2$.

The contribution of the $\tau$ loop is enhanced by a factor $\tan^2 \beta$ both in type II and in X models; it is suppressed by $1/\tan^2 \beta$ in models of type I and Y.
Vacuum stability and perturbativity constraints ($\beta - \alpha = \pi/2$ and $M_h = 126$ GeV). Left: allowed regions for $\Delta M \equiv M_H - M_{H\pm} = \{20, 0, -30\}$ GeV (darker to lighter), $\lambda_{\text{max}} = \sqrt{4\pi}$. Right: $\lambda_{\text{max}} = \{\sqrt{4\pi}, 2\pi, 4\pi\}$, $\Delta M = 0$. $\tan \beta = 50$, but negligible change for $\tan \beta \in [5, 100]$.

$M_A \lesssim 100 \text{ GeV} \rightarrow M_{H\pm} \lesssim 200 \text{ GeV}$

All values of $M_A$ are allowed by EW precision tests if $M_H \sim M_{H\pm}$.

Regions allowed by EW precision constraints: green, yellow, gray for $\Delta \chi^2_{\text{EW}} < 2.3, 6.2, 11.8$, i.e. 68.3, 95.4, 99.7% confidence intervals.
Type I and Y models cannot account for the present value of $\Delta a_\mu$ due to their lack of $\tan^2\beta$ enhancements.

In type II (and Y) models the BR($b \to s\gamma$) sets a strong lower bound on $M_{H^\pm}$ of order 380 GeV → hardly any space left for the light $A$ required to explain $\Delta a_\mu$.

In type X models, no such strong bounds on $M_{H^\pm}$ from BR($b \to s\gamma$), only model-indep. LEP bound $M_{H^\pm} \gtrsim 80$ GeV.

Therefore, out of type I, II, X, Y models, only type X is consistent with all the constraints we considered, provided that $M_A \approx 100$ GeV, $80 \approx M_{H^\pm} \approx 200$ GeV, $M_H \sim M_{H^\pm}$, large $\tan\beta$. 
Testing the SM with the electron g-2


The QED prediction of the electron g-2

\[ a_e^{\text{QED}} = + (1/2)(\alpha/\pi) - 0.32847844400255(33)(\alpha/\pi)^2 \]

Schwinger 1948
Sommerfield; Petermann; Suura&Wichmann ’57; Elend ’66; CODATA Mar ’12

\[ A_1^{(4)} = -0.32847896557919378... \]
\[ A_2^{(4)} (m_e/m_\mu) = 5.19738668 (26) \times 10^{-7} \]
\[ A_2^{(4)} (m_e/m_\tau) = 1.83798 (33) \times 10^{-9} \]

\[ + 1.181234016816(11)(\alpha/\pi)^3 \]

Kinoshita; Barbieri; Laporta, Remiddi; …; Li, Samuel; MP ’06; Giudice, Paradisi, MP 2012

\[ A_1^{(6)} = 1.181241456587... \]
\[ A_2^{(6)} (m_e/m_\mu) = -7.37394162 (27) \times 10^{-6} \]
\[ A_2^{(6)} (m_e/m_\tau) = -6.5830 (11) \times 10^{-8} \]
\[ A_3^{(6)} (m_e/m_\mu, m_e/m_\tau) = 1.90982 (34) \times 10^{-13} \]

\[ - 1.9097 (20)(\alpha/\pi)^4 \]

Kinoshita & Lindquist ’81, …; Kinoshita & Nio ’05; Aoyama, Hayakawa, Kinoshita & Nio 2012;
Kurz, Liu, Marquard & Steinhauser 2014: analytic mass dependent part.

\[ + 9.16 (58) (\alpha/\pi)^5 \]

Complete Result! (12672 mass indep. diagrams!)
Aoyama, Hayakawa, Kinoshita, Nio, PRL 109 (2012) 111807; work in progress to reduce the error.

\[ 0.4 \times 10^{-13} \text{ in } a_e \]
\[ 0.6 \times 10^{-13} \text{ in } a_e \]
\[ NB: (\alpha/\pi)^6 \sim O(10^{-16}) \]
The SM prediction of the electron $g-2$

The SM prediction is:

$$a_e^{SM} (\alpha) = a_e^{QED} (\alpha) + a_e^{EW} + a_e^{HAD}$$

The EW (1&2 loop) term is: Czarnecki, Krause, Marciano '96 [value from Codata10]

$$a_e^{EW} = 0.2973(52) \times 10^{-13}$$

The Hadronic contribution (LO+NLO+NNLO) is: Nomura & Teubner '12, Jegerlehner & Nyffeler '09; Krause'97; Kurz, Liu, Marquard & Steinhauser 2014

$$a_e^{HAD} = 17.10(17) \times 10^{-13}$$

$$a_e^{HLO} = + 18.66(11) \times 10^{-13}$$

$$a_e^{HNLO} = [-2.234(14)_{\text{vac}} + 0.39(13)_{\text{lbl}}] \times 10^{-13}$$

$$a_e^{HNNLO} = + 0.28(1) \times 10^{-13}$$

Which value of $\alpha$ should we use to compute $a_e^{SM}$?
The electron g-2: SM vs Experiment

Using $\alpha = 1/137.035\,999\,049\ (90)$ [$^{87}\text{Rb, 2011}$], the SM prediction for the electron g-2 is

$$a_e^{\text{SM}} = 115\,965\,218\,18.1\ (0.6)\ (0.4)\ (0.2)\ (7.6) \times 10^{-13}$$

The EXP-SM difference is (note the negative sign):

$$\Delta a_e = a_e^{\text{EXP}} - a_e^{\text{SM}} = -10.8\ (8.1) \times 10^{-13}$$

The SM is in very good agreement with experiment ($1.3\sigma$).

NB: The 4-loop contrib. to $a_e^{\text{QED}}$ is $-556 \times 10^{-13} \sim 70 \delta\Delta a_e$!

(the 5-loop one is $6.2 \times 10^{-13}$)
The electron g-2 sensitivity and NP tests

- The present sensitivity is $\delta \Delta a_e = 8.1 \times 10^{-13}$, ie (10^{-13} units):

\[
\begin{align*}
(0.6)_{\text{QED}4}, & \quad (0.4)_{\text{QED}5}, & \quad (0.2)_{\text{HAD}}, & \quad (7.6)_{\delta \alpha}, & \quad (2.8)_{\delta a_e^{\text{EXP}}} \\
\hline
(0.7)_{\text{TH}} & \xrightarrow{\text{may drop to 0.2 or 0.3}}
\end{align*}
\]

- The $(g-2)_e$ exp. error may soon drop below $10^{-13}$ and work is in progress for a significant reduction of that induced by $\delta \alpha$.

→ sensitivity of $10^{-13}$ may be reached with ongoing exp. work

- In a broad class of BSM theories, contributions to $a_\perp$ scale as

\[
\frac{\Delta a_{\perp i}}{\Delta a_{\perp j}} = \left( \frac{m_{\ell_i}}{m_{\ell_j}} \right)^2
\]

This Naive Scaling leads to:

\[
\begin{align*}
\Delta a_e &= \left( \frac{\Delta a_\mu}{3 \times 10^{-9}} \right) \times 0.7 \times 10^{-13}; & \\
\Delta a_\tau &= \left( \frac{\Delta a_\mu}{3 \times 10^{-9}} \right) \times 0.8 \times 10^{-6}
\end{align*}
\]
The electron g-2 sensitivity and NP tests (2)

- The experimental sensitivity in $\Delta a_e$ is not very far from what is needed to test if the discrepancy in $(g-2)_\mu$ also manifests itself in $(g-2)_e$ under the naive scaling hypothesis.

- NP scenarios exist which violate Naive Scaling: They can lead to larger effects in $\Delta a_e$ and contributions to EDMs, LFV or lepton universality breaking observables. For example:

  - In the MSSM with non-degenerate but aligned sleptons (vanishing flavor mixing angles), $\Delta a_e$ can reach $10^{-12}$ (at the limit of the present exp sensitivity). For these values one typically has breaking effects of lepton universality at few per mil level (within future exp reach).

  - In 2HDMs of type 2 and X with a light pseudoscalar: see earlier slide on 2HDMs solution for $\Delta a_\mu$. 
The leading contribution of positronium to $a_e$ comes from:

Mishima 1311.7109; Fael & MP 1402.1575; Melnikov et al. 1402.5690; Eides 1402.5860; Hayakawa 1403.0416

The $e^+e^-$ bound states appear as poles in the vac. pol. $\Pi(q^2)$ right below the branch-point $q^2 = (2m)^2$. Their contribution is:

$$a_e(\text{vp}) = \frac{\alpha}{\pi^2} \int_0^\infty \frac{ds}{s} \text{Im} \Pi(s + i\epsilon) K(s)$$

$$a_e^P = \frac{\alpha^5}{4\pi} \zeta(3) \left( 8 \ln 2 - \frac{11}{2} \right) = 0.9 \times 10^{-13} = 1.3 \left( \frac{\alpha}{\pi} \right)^5$$

Of the same magnitude of the exp. unc. of $a_e$ & the “naively rescaled” muon $\Delta a_\mu$. Of the same order of $\alpha$ as the 5-loop term!
Melnikov, Vainshtein & Voloshin (MVV) \(^{1402.5690}\) determined a nonpert. contrib. of the \(e^+e^-\) continuum right above threshold that cancels one-half of \(a_e^p\):

\[
a_e(vp)^{\text{cont, np}} = -\frac{|\alpha|^5}{8\pi} \zeta(3) \left(8 \ln 2 - \frac{11}{2}\right)
\]

In fact the total positronium poles + continuum nonperturbative contribution to \(a_e\) arising from the threshold region at LO in \(\alpha\) is:

\[
a_e^{\text{thr}}(vp) = -\frac{\alpha}{\pi} K(4m^2) \Re A(1)
\]

with

\[
A(\beta) = -\frac{\alpha^2}{2} \left[ \gamma + \psi \left(1 - \frac{i\alpha}{2\beta}\right) \right] = \frac{\alpha^2}{2} \sum_{k=1}^{\infty} \zeta(k + 1) \left(\frac{i\alpha}{2\beta}\right)^k
\]

so that

\[
a_e^{\text{thr}}(vp) = \frac{\alpha^5}{8\pi} \zeta(3) K(4m^2) = \frac{a_e^p}{2}
\]
So, should we add this total threshold contribution $a_e^P/2$ to the perturbative QED 5-loop result of Kinoshita and collaborators?

Using the Coulomb Green’s function, MVV 1402.5690 argued that it is already contained in the contribution of $O(\alpha^5)$.

Hayakawa 1403.0416 claimed that positronium contributes to $a_e$ only through a specific class of diagrams of $O(\alpha^7)$.

To address this question: study the 5-loop QED contribution to $a_e$ arising from the insertion of the 4-loop VP in the photon line. This has been computed via:

\[ e^-(\gamma) e^+ (\Pi^{(8)}) e^- \]

Aoyama, Hayakawa, Kinoshita & Nio 2012
Using explicit expressions for $\Pi^{(8)}(q^2)$ (Baikov, Maier, Marquard ’13) in

$$a_{e}^{(10)}(\text{vp}) = -\frac{\alpha}{\pi} \int_{0}^{1} dx \ (1 - x) \Pi^{(8)}\left(\frac{-m^2 x^2}{1 - x}\right)$$

we obtain:

$$a_{e}^{(10)}(\text{vp}) = n_{e} \frac{\alpha^5}{8\pi} \zeta(3) K(4m^2) + \cdots = \frac{a_{e}^{P}}{2} + \cdots$$

$\alpha^P/2$ is already included in the 5-loop contrib. of class I(i).

There is no additional contrib of QED bound states beyond PT!

M.A. Braun 1968; Barbieri, Christillin, Remiddi 1973
Conclusions

μ \[ \Delta a_\mu \sim 3\div3.5 \sigma. \] New upcoming experiment: QED & EW ready; progress in the hadronic sector, but not yet ready!

μ Can \[ \Delta a_\mu \] be solved by 2HDMs? Not by type I, II, and Y! Type X is still allowed by all the constraints we considered if \[ M_A \approx 100\text{GeV}, \ 80 \lesssim M_{H\pm} \lesssim 200\text{GeV}, \ M_H \sim M_{H\pm}, \text{& large } \tan\beta. \]

e It may soon be possible to test NP with \( (g-2)_e \). In particular, whether the \( a_\mu \) discrepancy shows up also in \( a_e \).

e There is no additional contribution of QED bound states to \( (g-2)_e \) beyond perturbation theory!
The End
The electron g-2 gives the best determination of alpha

- The 2008 measurement of the electron g-2 is:
  \[ a_e^{\text{EXP}} = 11596521807.3 \times 10^{-13} \]  
  Hanneke et al, PRL100 (2008) 120801

- vs. old (factor of 15 improvement, 1.8\( \sigma \) difference):
  \[ a_e^{\text{EXP}} = 11596521883 \times 10^{-13} \]  
  Van Dyck et al, PRL59 (1987) 26

- Equate \( a_e^{\text{SM}}(\alpha) = a_e^{\text{EXP}} \) → best determination of alpha:
  \[ \alpha^{-1} = 137.035\,999\,177 (34) \]  
  [0.25 ppb]

- Compare it with other determinations (independent of \( a_e \)):
  \[ \alpha^{-1} = 137.036\,000\,0 (11) \]  
  [7.7 ppb]  
  PRA73 (2006) 032504 (Cs)
  \[ \alpha^{-1} = 137.035\,999\,049 (90) \]  
  [0.66 ppb]  
  PRL106 (2011) 080801 (Rb)

Excellent agreement → beautiful test of QED at 4-loop level!