Project: A flavor model in extra dimensions

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I. INTRODUCTION

A model of flavor hierarchy can be built based on a simple 5D spacetime.

- First, you can build a model where each fermion field has different wave function profile along extra dimension. Here, you only need to keep the zero mode if the size of extra dimension is small enough.
- Second, taking a localized Higgs field on a brane, you can obtain effective Yukawa couplings by taking the wave function overlaps.

The gamma matrices in 5D are given by $\Gamma^M = (\gamma^\mu, i\gamma_5)$ which satisfy the extended Clifford algebra

$$\left\{\Gamma^M, \Gamma^N\right\} \quad = 2\eta^{MN} = 2\text{diag}(1, -1, -1, -1, -1). \tag{1}$$

In Weyl basis, gamma matrices are given by Pauli matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where $\sigma^\mu = (1, \vec{\sigma}), \overline{\sigma}^\mu = (1, -\vec{\sigma})$.

II. PROJECT

We start from the action involves a single real scalar field $\Phi(x^\mu, y)$ and Dirac spinors $\Psi_i(x^\mu, y)$ in a flat 5D:

$$S_5 = \int d^4x \int_{-L}^{L} dy \, \sum_i \bar{\Psi}_i(i\Gamma^M\partial_M - f_i\Phi)\Psi_i + \tfrac{1}{2}\eta^{MN}\partial_M\Phi\partial_N\Phi - \tfrac{\lambda}{4}(\Phi^2 - v^2)^2. \tag{3}$$

The flavor index runs $i = 1, 2, \cdots N_f$. The couplings f_i and λ are real. Here gauge interactions are irrelevant so that we will neglect them.

1. Check the mass-dimensions of f, λ and v parameters.
2. Find the Hamiltonian of the scalar field,

$$\mathcal{H}_\Phi = \int dy \; (\pi\partial_t\Phi - \mathcal{L}_\Phi) \tag{4}$$

   where the canonical momentum is $\pi(x^\mu, y) = \frac{\partial\mathcal{L}}{\partial\partial_t\Phi}$.

3. Show that the static vacuum configuration of the scalar field $\Phi(x^\mu, y) = h(y)$, which minimize the energy (Hamilotonian) satisfies the following condition

$$\partial_5^2 h(y) = \lambda(h(y)^2 - v^2)h(y). \tag{5}$$

   (Note: As it is in vacuum state, it is independent of x^μ.)

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4. When $\sqrt{\lambda} vL \gg 1$, show that the solution to the Eq.(5) subject to the following boundary conditions $h(-L) = -h(L)$ is of the form of

$$h(y) = v \tanh(\mu y). \tag{6}$$

Discuss the physical condition allowing that we may regard the mass of Ψ_i as a kink

$$\begin{aligned} m_5^i(y) &= f_i h(y) & (7) \\ &\xrightarrow{??} \mu_i \mathrm{sgn}(y) = \begin{cases} -\mu_i & \text{if } y < 0 \\ +\mu_i & \text{if } y > 0 \end{cases}. & (8) \end{aligned}$$

5. (Here, for simplicity, the flavor indices are suppressed) Derive the Euler-Lagrange equations for $\Psi_L(x^\mu, y) = P_L \Psi$ and $\Psi_R(x^\mu, y) = P_R \Psi$ by respectively taking variations $\delta\overline{\Psi_L}$ and $\delta\overline{\Psi_R}$. In general, when the fermion belongs to a complex representation of the symmetry group, the KK modes can only acquire Dirac masses. Take the Kaluza-Klein decomposition

$$\Psi_{L/R} = \sum_n \psi^n_{L/R}(x) f^n_{L/R}(y)\,, \tag{9}$$

where $\psi^n_{L/R}$ are 4D spinors which satisfy the Dirac equations:

$$i\gamma^\mu \partial_\mu \psi^n_{L/R} = m_n \psi^n_{R/L}\,. \tag{10}$$

By plugging the expansion into the bulk equations obtained above, obtain the following set of coupled first order differential equations for the zero mode, $n = 0$, the massless solution ($m_0 = 0$):

$$(\mp \partial_5 - m_5)\, f^0_{R/L} = 0\,. \tag{11}$$

Assuming $\mu_\Psi > 0$, find the normalized solutions f^0_L and f^0_R satisfying $\int_{-L}^{L} dy |f^0_{R/L}|^2 = 1$. Draw the wave function profile of zero modes (which can be regarded as the Standard model fermions).

6. Assume that the Higgs field H is localized on a brane at the middle of the extra dimension ($y = 0$) thus the Yukawa interaction is of the form

$$S \ni \int d^4x \int dy \quad \delta(y) y_{ij} H(\overline{\Psi^0_{i\,L}}(x^\mu) \Psi^0_{j\,R}(x^\mu) f^i_L f^j_R + c.c.). \tag{12}$$

From the above expression, derive the 4D effective Yukawa coupling:

$$y^{\text{eff}}_{ij} = y_{ij} \int_{-L}^{L} dy f^i_L(y) f^j_R(y) \tag{13}$$

where i and j could have different couplings f_i so as μ_i. Discuss how the hierarchal Yukawa structure can be obtained. Can you make a realistic Yukawa matrix?