# Extracting IR/UV divergences in multiloop diagram by series expansion in ε

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# Outline

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  2. Parameterization  for multiloop diagram
  3. Sector decomposition
- Separating UV/IR divergences
- Example
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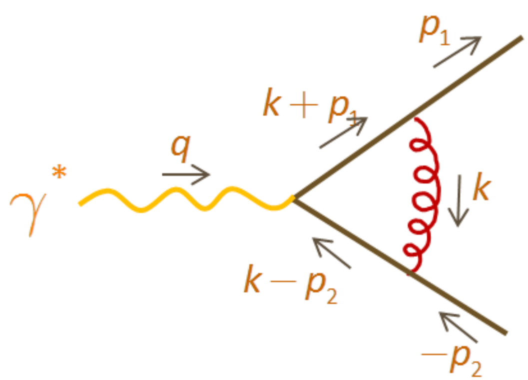
- Conclusion

# UV, IR divergences

Let us consider the process $\gamma^* \longrightarrow q\bar{q}$




*Virtual correction*

$$\sim \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(\not{k}+\not{p}_1)(\not{k}-\not{p}_2)}$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k}+\not{p}_1)(\not{k}-\not{p}_2)}{k^2(k^2+2k\cdot p_1)(k^2-2k\cdot p_2)}$$

UV

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \sim \frac{1}{\epsilon_{\mathrm{UV}}}$$

IR

$k \sim 0$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 k\cdot p_1 k\cdot p_2} \sim \frac{1}{\epsilon_{\mathrm{IR}}}$$

→ Soft divergence

IR

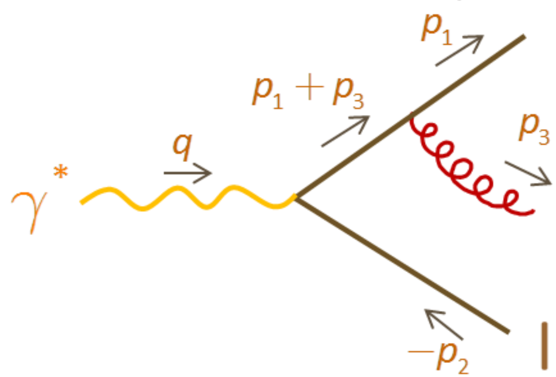
$k \sim p_1$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k^2+2k\cdot p_1)} \sim \frac{1}{\epsilon_{\mathrm{IR}}}$$

→ Collinear divergence

# UV, IR divergences

Let us consider the process $\gamma^* \rightarrow q\bar{q}$




*Real Emission*

$$\sim \int d\Pi_3 \frac{1}{2p_1 \cdot p_3}$$

IR: $p_3 \sim 0$ $\quad \sim \dfrac{1}{\epsilon_{\text{IR}}}$ → Soft divergence

IR: $p_3 \sim p_1$ $\quad \sim \dfrac{1}{\epsilon_{\text{IR}}}$ → Collinear divergence

All the UV divs. are canceled by Renormalization.

All the IR divs. are canceled between Virtual correction and Real emission. → KLN theorem.

# UV, IR divergences

Catani-Seymour Method at NLO Catani, Seymour, NPB (1997)

→ *Systematic method for canceling IR between virtual correction and real emission diagrams. Method for more than one loop is challenging.*

Other works on UV/IR divergences:

Tausk, PLB, (1999) Czakon, CPC, (2005)

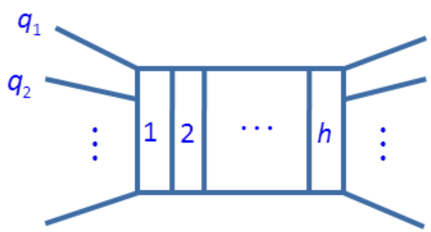
→ *Using MB representation*

Sector Decomposition

Heinrich, 0803.4177

# Parametrization

- Feynman Parameterization



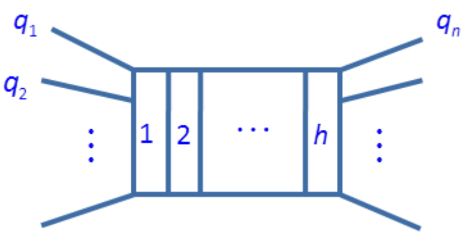

$$= \int [d^d k_1 \cdots d^d k_h] \frac{1}{\mathcal{P}_1^{a_1} \cdots \mathcal{P}_L^{a_L}} \equiv F(q_1, \cdots, q_n; d)$$

$$F_\Gamma(q_1, q_2, \cdots, q_n; d) = \int d^d k_1 \cdots d^d k_h$$

$$\times \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_L \, \delta\left(\sum \xi_l - 1\right) \frac{\prod_l \xi_l^{a_l - 1}}{\left(\sum \mathcal{P}_l \, \xi_l\right)^a}$$

# Parametrization

- alpha Parameterization




$$= \int [d^d k_1 \cdots d^d k_h] \frac{1}{\mathcal{P}_1^{a_1} \cdots \mathcal{P}_L^{a_L}} \equiv F(q_1, \cdots, q_n; d)$$

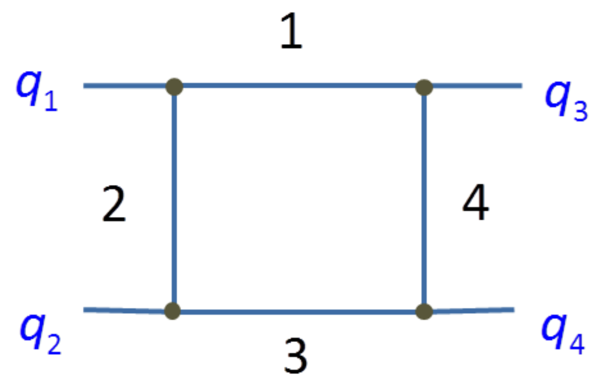
$$F\,(q_1, q_2, \cdots, q_n; d) = \frac{(-1)^a}{\prod_j \Gamma(a_j)} \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_L \delta\left(\sum \alpha_j - 1\right) \frac{\mathcal{U}^{a-(h+1)d/2} \prod_j \alpha_j^{a_j - 1}}{\left(-\mathcal{V} + \mathcal{U} \sum m_j^2 \alpha_j\right)^{a - hd/2}}$$

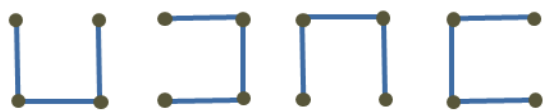
$\mathcal{U} = \sum_{T \in T^1} \prod_{l \notin T} \alpha_l$ → Polynomial of $\alpha$s of order $h$.

$\mathcal{V} = \sum_{T \in T^2} \prod_{l \notin T} \alpha_l (q^T)^2$ → Polynomial of $\alpha$s of order $h+1$.

# Parametrization

For example




$T_1$ : 

$T_2$ : 

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\mathcal{V} = s\,\alpha_1\alpha_3 + t\,\alpha_2\alpha_4$$

$$F(s,t;\varepsilon) = \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_4 \delta\left(\sum \alpha_j - 1\right) \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 1)^{2\varepsilon}}{\left(-t\,\alpha_1\alpha_3 - s\alpha_2\alpha_4\right)^{2+\varepsilon}}$$

# Parametrization

Where are UV/IR divergences?

→ Looking for denominator to be zero.

$$F(q_1, q_2, \cdots, q_n; d) =$$

$$\frac{(-1)^a}{\prod_j \Gamma(a_j)} \int_0^\infty d\alpha_1 \cdots d\alpha_L \frac{\delta(\sum \alpha_j - 1) \prod_j \alpha_j^{a_j - 1}}{\underbrace{\mathcal{U}^{-a+(h+1)d/2}}_{\text{UV source}} \underbrace{(-\mathcal{V} + \mathcal{U} \sum m_j^2 \alpha_j)^{a-hd/2}}_{\text{IR source}}}$$

Divergences arise when denominator becomes zero as same number of alphas with integer power of denominator have boundary values.

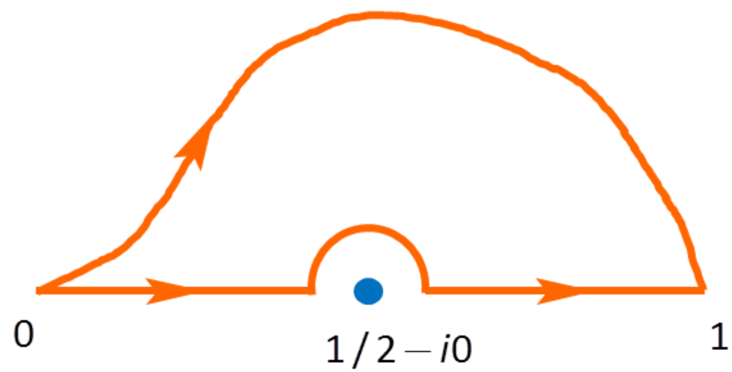
# Parametrization

For example-1

$$\int_0^1 d\alpha \frac{1}{(\alpha + i0)^{1-\varepsilon}} = \frac{1}{\varepsilon}$$

$$\int_0^1 d\alpha \frac{1}{(\alpha - 1/2 + i0)^{1-\varepsilon}} = \log\left(\frac{1}{2}\right) - \log\left(-\frac{1}{2} + i0\right) = -\pi i$$

→ *zero denominator with non-boundary values of alphas generate imaginary values.*



# Parametrization

For example-2

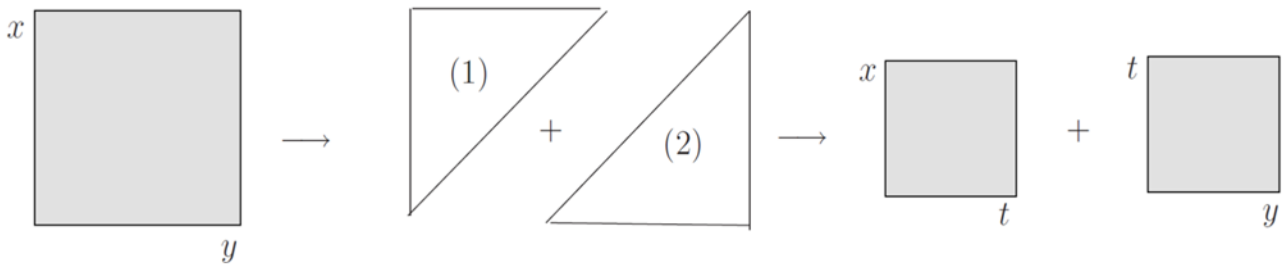
$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{1-\varepsilon}} = \int_0^1 d\alpha_1 \ln\left(\frac{1+\alpha_1}{\alpha_1}\right) + \mathcal{O}(\varepsilon) = 2\ln 2 + \mathcal{O}(\varepsilon)$$

→ No div. where the number of alphas with boundary values that make denominator zero is greater than integer power of denominator.

$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}} = \int_0^1 d\alpha_1 \frac{\alpha_1^{\varepsilon-1} - (\alpha_1+1)^{\varepsilon-1}}{1-\varepsilon} = \frac{1}{\varepsilon} + 1 - \ln 2 + \mathcal{O}(\varepsilon)$$

# Sector Decomposition

Heinrich, 0803.4177




$$\int_0^1 d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}} = \left(\int_{0\le\alpha_1\le\alpha_2}^1 d\alpha_1 d\alpha_2 + \int_{0\le\alpha_2\le\alpha_1}^1 d\alpha_1 d\alpha_2\right)\frac{1}{(\alpha_1+\alpha_2)^{2-\varepsilon}}$$

$$= \int_0^1 d\alpha_1 d\alpha_2 \frac{1}{\alpha_2^{1-\varepsilon}(\alpha_1+1)^{2-\varepsilon}} + \int_0^1 d\alpha_1 d\alpha_2 \frac{1}{\alpha_1^{1-\varepsilon}(\alpha_2+1)^{2-\varepsilon}}$$

$$= \frac{1}{\varepsilon} + 1 - \ln 2 + \mathcal{O}(\varepsilon)$$

→ *Perform this decomposition for the multi-dimensional hyper-surface of alphas.*

→ *Algorithmic.* ***Suitable for numeric calculation.***

# Separation of IR/UV

We first apply 'Cheng-Wu' theorem for all alphas except 1.

$$F(q_1,q_2,\cdots,q_n;d)=\frac{(-1)^a}{\prod_j \Gamma(a_j)}\int_0^\infty d\alpha_1\cdots\int_0^\infty d\alpha_L\,\delta(\sum\alpha_j-1)\frac{\mathcal{U}^{a-(h+1)d/2}\prod_j \alpha_j^{a_j-1}}{(-\mathcal{V}+\mathcal{U}\sum m_j^2\alpha_j)^{a-hd/2}}$$

*We can replace* $\delta(\sum\alpha_j-1)$ *with* $\delta(\sum_{\nu\in S}\alpha_\nu-1)$ *for any subset* $S$ *of alpha variables set.*

*Our choice is* $\delta(\sum\alpha_j-1)\rightarrow\delta(\alpha_L-1)$

Namely, we choose one alpha and make it 1, then integrate from zero to infinity for all the other alphas.

# Separation of IR/UV

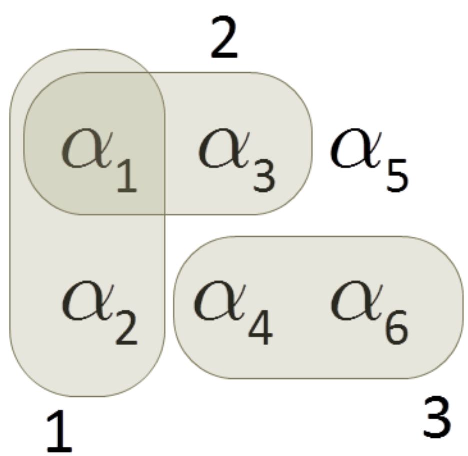
Investigate denominator to find divergences

$$\frac{\square}{\mathcal{U}^{2-\varepsilon}\left(-\mathcal{V}+\mathcal{U}\sum m_j^2\alpha_j\right)^{3-\varepsilon}}$$

Construct a set $S$ that consists of set of variables that cause divergences

1. For example

$$S_1=\{\alpha_1,\alpha_2\}$$
$$S_2=\{\alpha_1,\alpha_3\}$$
$$S_3=\{\alpha_4,\alpha_6\}$$




3. Do the variable change

$$\alpha_1 \to \eta_1\eta_2\alpha_1,$$
$$\alpha_2 \to \eta_1\alpha_2,$$
$$\alpha_3 \to \eta_2\alpha_3,$$
$$\alpha_4 \to \eta_3\alpha_4,$$
$$\alpha_5 \to \eta_3\alpha_5$$

2. Multiply by $1=\int_0^\infty d\eta_1 d\eta_2 d\eta_3 \delta(\eta_1-\alpha_2)\delta(\eta_2-\alpha_3)\delta(\eta_3-\alpha_4-\alpha_6)$

# Separation of IR/UV

The IR/UV div. are separated as

$$\int_0^\infty d\eta_1\, d\eta_2\, d\eta_3 \underbrace{\eta_1^{-1+a_1\varepsilon}\eta_2^{-1+a_2\varepsilon}\eta_3^{-1+a_3\varepsilon}}_{\text{Div. part}} \underbrace{\delta(\alpha_2-1)\delta(\alpha_3-1)\delta(\alpha_4+\alpha_6-1)F(\eta_i,\alpha_i,\varepsilon)}_{\text{Div. free, safely expanded in }\varepsilon}$$

After variable change : $\eta_j \to \frac{(1-\xi_j)}{\xi_j}, \quad \int_0^\infty d\eta_j \to \int_0^1 \frac{d\xi_j}{\xi_j^2}$

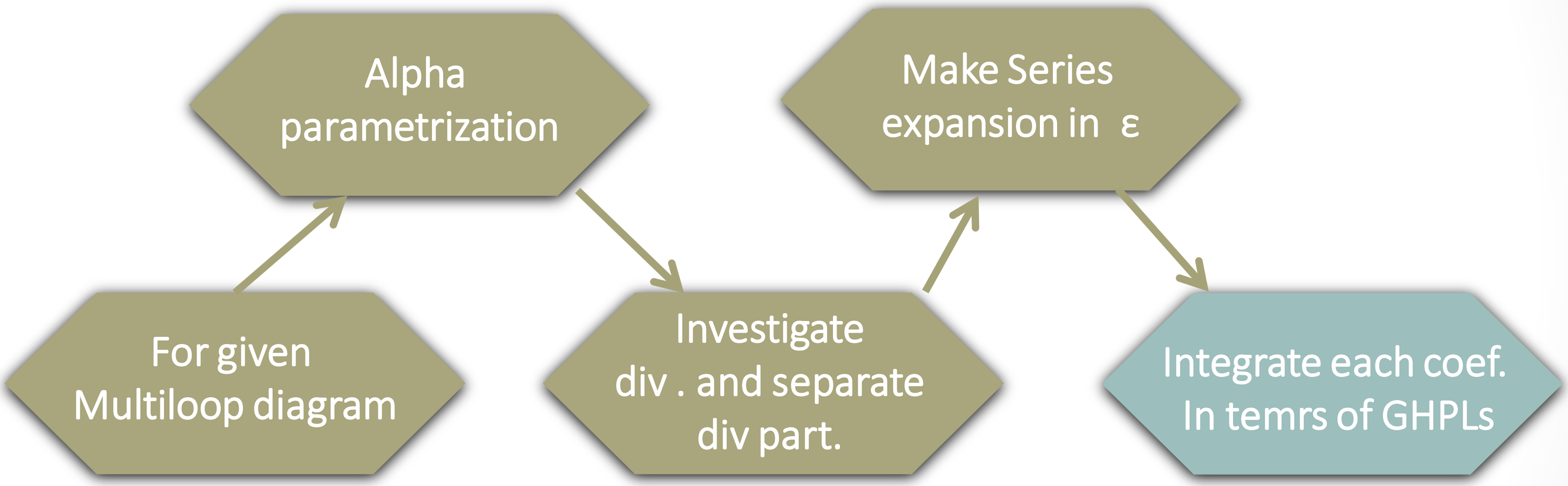
we use following expansion formula for div. part.

$$\xi^{-1+a\varepsilon} = \frac{\delta(\xi)}{a\varepsilon} + \sum_k \frac{(a\varepsilon)^k}{k!}\left[\frac{\ln^k \xi}{\xi}\right]_+$$

$$\left(\xi(1-\xi)\right)^{-1+a\varepsilon} = \frac{\Gamma(a\varepsilon)^2}{2\Gamma(2a\varepsilon)}\left(\delta(\xi)+\delta(1-\xi)\right) + \sum_k \frac{(a\varepsilon)^k}{k!}\left[\frac{\ln^k \xi(1-\xi)}{\xi(1-\xi)}\right]_+$$

# Separation of IR/UV

## Summary



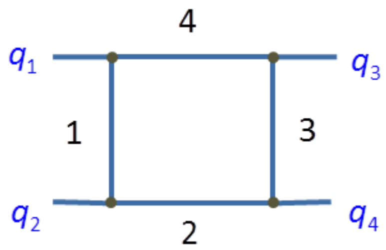

*working in progress*

GHPL : Generalized Harmonic Poly-Logarithm func.

$$G(p_1,\cdots,p_m;x) \equiv \int_0^x \frac{dy_1}{y_1 - p_1} \int_0^{y_1} \frac{dy_2}{y_2 - p_2} \cdots \int_0^{y_{m-1}} \frac{dy_m}{y_m - p_m}$$

# Example

Massless one loop box diagram




$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$\mathcal{V} = t\,\alpha_1\alpha_3 + s\,\alpha_2\alpha_4$$

$$F(s,t;\varepsilon) = \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_4 \delta\left(\sum \alpha_j - 1\right) \frac{(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)^{2\varepsilon}}{\left(-t\,\alpha_1\alpha_3 - s\alpha_2\alpha_4\right)^{2+\varepsilon}}$$

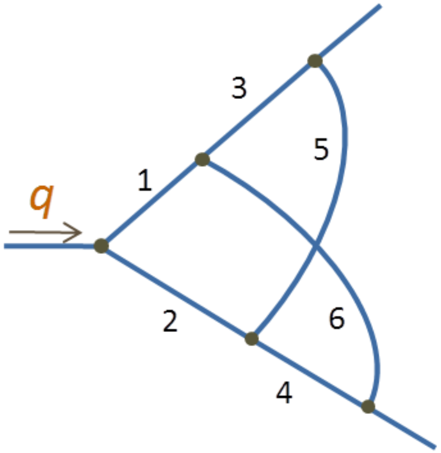
After IR separation

$$F(s,t;\varepsilon) = \frac{(-t)^{-\varepsilon}}{st}\Gamma(2+\varepsilon)\frac{\Gamma(-\varepsilon)^2}{\Gamma(-2\varepsilon)}\int_0^1 \xi_1\xi_2\ (\xi_2\bar{\xi}_2)^{-1-\varepsilon}\left(x\xi_1\xi_2 + \bar{\xi}_2\bar{\xi}_2\right)^{\varepsilon}$$

The result is

$$F(s,t;\varepsilon) = \frac{1}{st}\left(\frac{4}{\varepsilon^2} - \frac{2}{\varepsilon}\left(\ln(-s)+\ln(-t)\right) + 2\ln(-s)\ln(-t) - \frac{4\pi^2}{3}\right)$$

# Example




$$\mathcal{U} = (\alpha_3+\alpha_5)(\alpha_4+\alpha_6)+(\alpha_1+\alpha_2)(\alpha_3+\alpha_5+\alpha_4+\alpha_6)$$

$$\mathcal{V} = q^2(\alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5+\alpha_6)+\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_6)$$

After IR separation

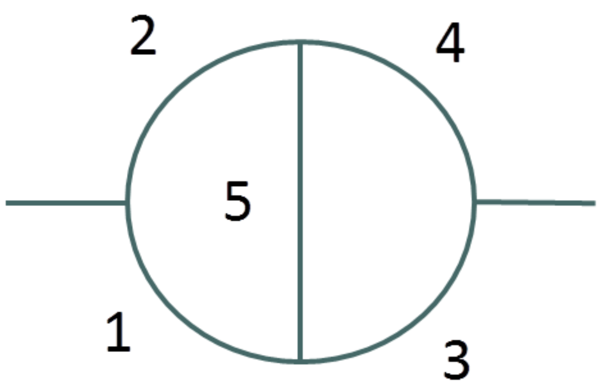
$$F(q^2;\varepsilon) = \frac{\Gamma(2+2\varepsilon)}{(-q^2)^{2+2\varepsilon}} \int_0^1 \xi_1\xi_2\xi_3\xi_4\xi_5 \; (\xi_1\bar{\xi}_1\xi_2\bar{\xi}_2)^{-1-2\varepsilon} (\xi_3\bar{\xi}_3)^{-1-\varepsilon} \, \xi_4^{-1-2\varepsilon} \bar{\xi}_4^{1+\varepsilon} \xi_5^{-\varepsilon}$$

$$\times \left(\xi_4\bar{\xi}_5+\bar{\xi}_4\right)^{-2-2\varepsilon} (\xi_5\bar{\xi}_4+\xi_1\xi_2\xi_4\xi_5+\bar{\xi}_1\bar{\xi}_2\xi_4\bar{\xi}_5)^{3\varepsilon}$$

$$= \frac{1}{(-q^2)^{2+2\epsilon}} \left( \frac{1}{\epsilon^4} - \frac{\pi^2}{\epsilon^2} - \frac{83\zeta(3)}{3\epsilon} - \frac{59\pi^4}{120} \right)$$

# Example

Massless two loop diagram




$$\mathcal{U} = \alpha_{1234}\alpha_5 + \alpha_{12}\alpha_{34}$$

$$\mathcal{V} = q^2\left(\alpha_5\alpha_{13}\alpha_{24} + \alpha_1\alpha_3\alpha_{24} + \alpha_2\alpha_4\alpha_{13}\right)$$

There is no div. term.

$$F(s,t;\varepsilon) = \frac{1}{q^2}\int_0^1 d\xi_1 \cdots \int_0^1 d\xi_4 \frac{1}{(\bar{\xi}_2\xi_2\xi_3 + \xi_1(1-\bar{\xi}_2\xi_3(1+\xi_2-2\xi_4)-\xi_4-\bar{\xi}_2{}^2\xi_3^2\xi_4))}$$

$$F(s,t;\varepsilon) = \frac{6}{q^2}\zeta(3)$$

# Conclusion

- We propose a new method of extracting IR/UV divergences by series expansion in ε.
- This method is algorithmic and quite improves the access to analytic calculation.
- We are working on generalized procedure for analytic calculation for the remaining integration in terms of GHPLs and making Mathematica code for that.
- This method will be useful for higher order correction in the future LHC era.