Topological Solitons & Symmetry Breaking

노트 제목

2014-01-21

- Domain Wall : Toy Example
- Lightening Review of Homotopy
- Vortex Strings, Gauge Field, & Superconductor
- Holonomy of Vortex String
- Non-Abelian Examples : Alice String / Magnetic Monopole

< Domain Wall >

Example : Ising Model (on Lattice)

$$\text{Energy} = J \sum_{\langle ij \rangle}' \sigma_i \sigma_j$$

$J < 0$

$\sigma_i = \pm 1$

$\langle ij \rangle$: Nearest Neighbor Pairs

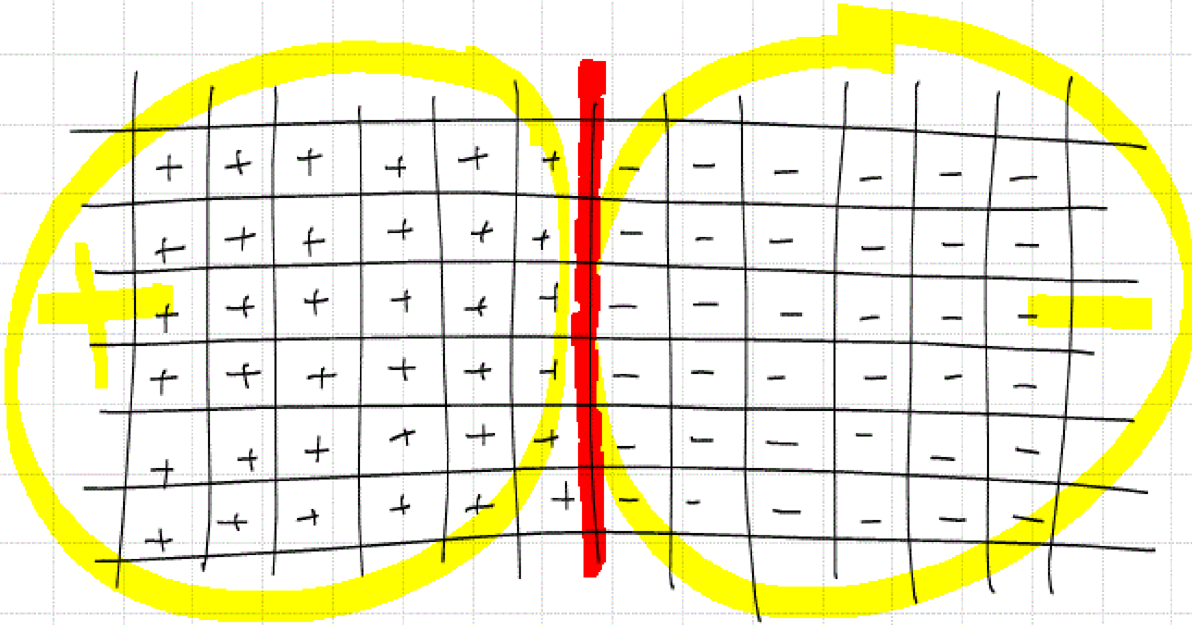
+	−	−	+	−	+	−	−
−	+	+	+	−	−	+	+
+	−	−	+	−	−	+	−
−	+	−	−	+	+	+	+
−	−	+	+	+	+	−	−
−	−	+	−	−	+	−	−

$T > T_{critical}$

random distribution of spin up (+) or down (−)

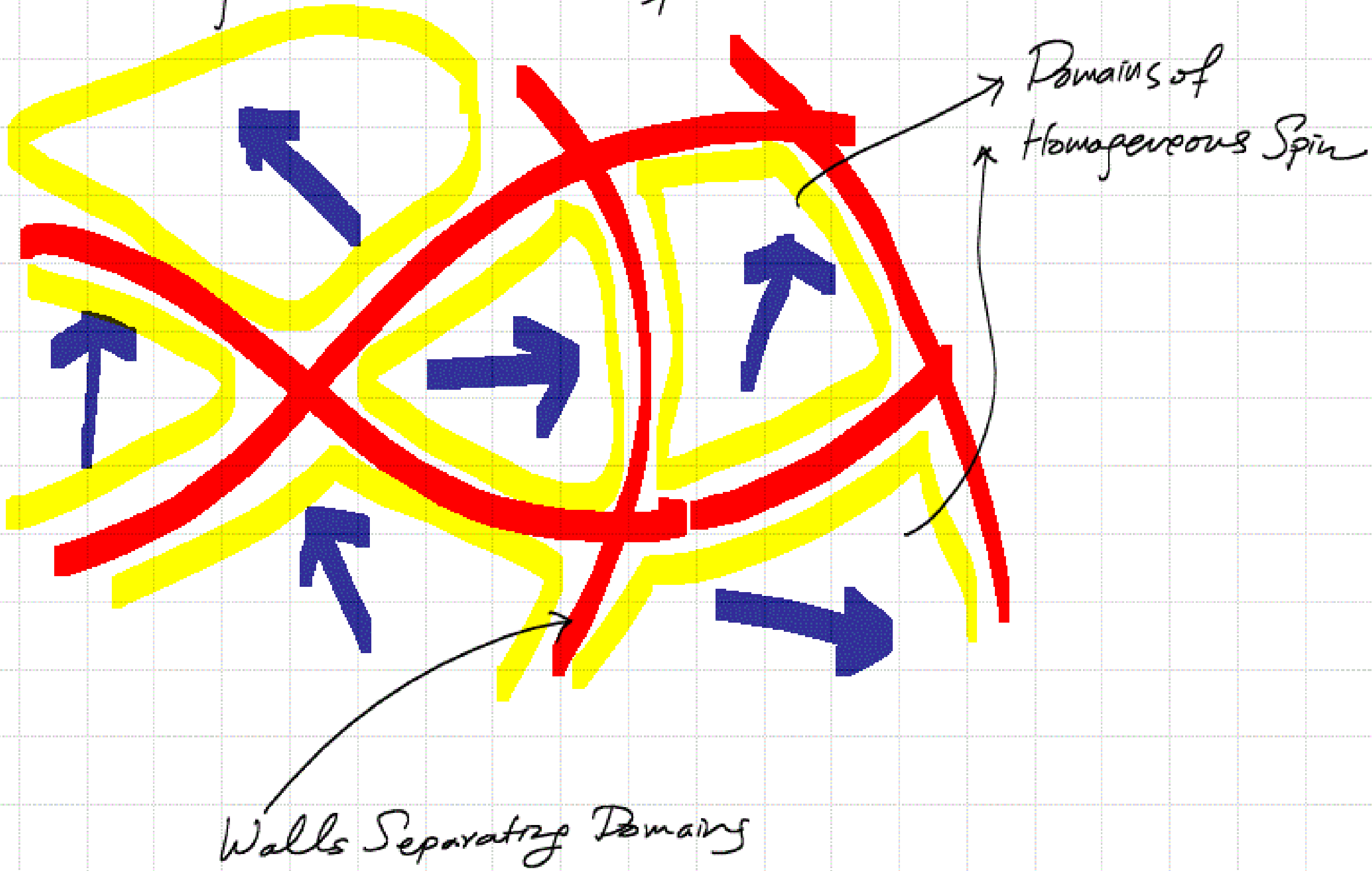
If $T < T_{critical}$,

"lined up spins" are favored locally



(+) Domain (−) Domain

In real magnet with SU(2) spin

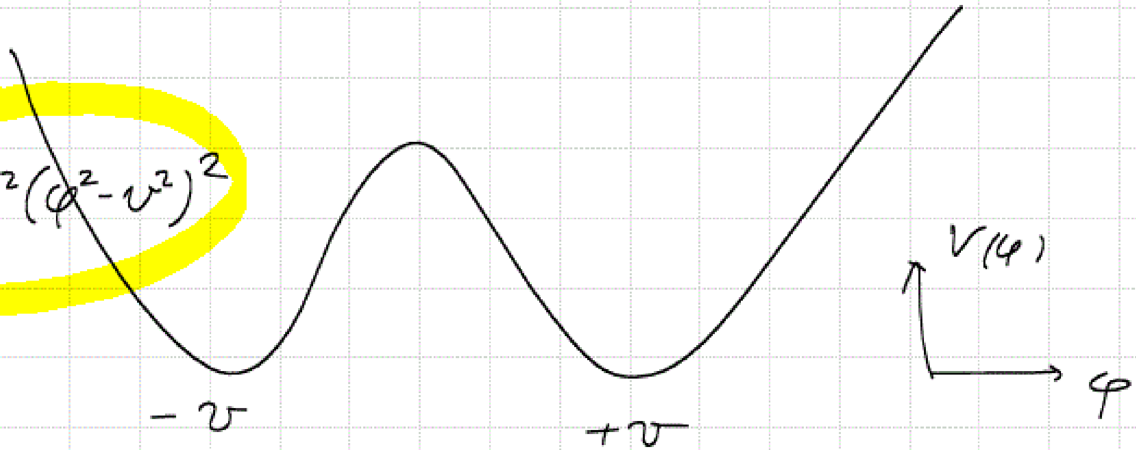


Analog of Ising on Lattice in Continuous Space :

($T < T_{critical}$)

$$\int dt \int d^3x \left(\frac{1}{2} (\partial_\mu \varphi)^2 - V(\varphi) \right) \qquad \left(\begin{array}{l} \partial_{\mu=0} = \frac{\partial}{\partial t} \\ \partial_{\mu=i} = \frac{\partial}{\partial x^i} \end{array} \right)$$

$$V(\varphi) = \frac{1}{2} \lambda^2 (\varphi^2 - v^2)^2$$



Lowest Energy State : $\varphi = +v$ everywhere

or

$\varphi = -v$ everywhere

What if we set $\begin{pmatrix} \varphi(x=+\infty) = +v \\ \varphi(x=-\infty) = -v \end{pmatrix}$ as boundary condition?

What is the static solution?

$$L = \int d^3x \ \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\vec{\partial}\varphi)^2 - V(\varphi)$$

$$\mathcal{H}_{\text{(Hamiltonian)}} = \frac{\delta L}{\delta \dot{\varphi}}\dot{\varphi} - L = \int d^3x \left(\frac{1}{2}\left(\dot{\varphi}^2 + (\vec{\partial}\varphi)^2\right) + V(\varphi) \right)$$

$$\text{EOM}: \quad \underbrace{\ddot{\varphi}}_{0 \text{ if static}} = (\vec{\partial})^2\varphi - V'(\varphi)$$

$$\vec{\partial}^2 \longmapsto \left(\frac{d}{dx}\right)^2$$

$$\left(\frac{d}{dx}\right)^2 \varphi = V'(\varphi)$$

$$\left[\left(\frac{d}{dx}\right)^2 \varphi\right]\left[\frac{d}{dx}\varphi\right] = V'(\varphi)\left[\frac{d\varphi}{dx}\right] = \frac{d}{dx} V(\varphi(x))$$

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - V(\varphi(x))\right] = 0$$

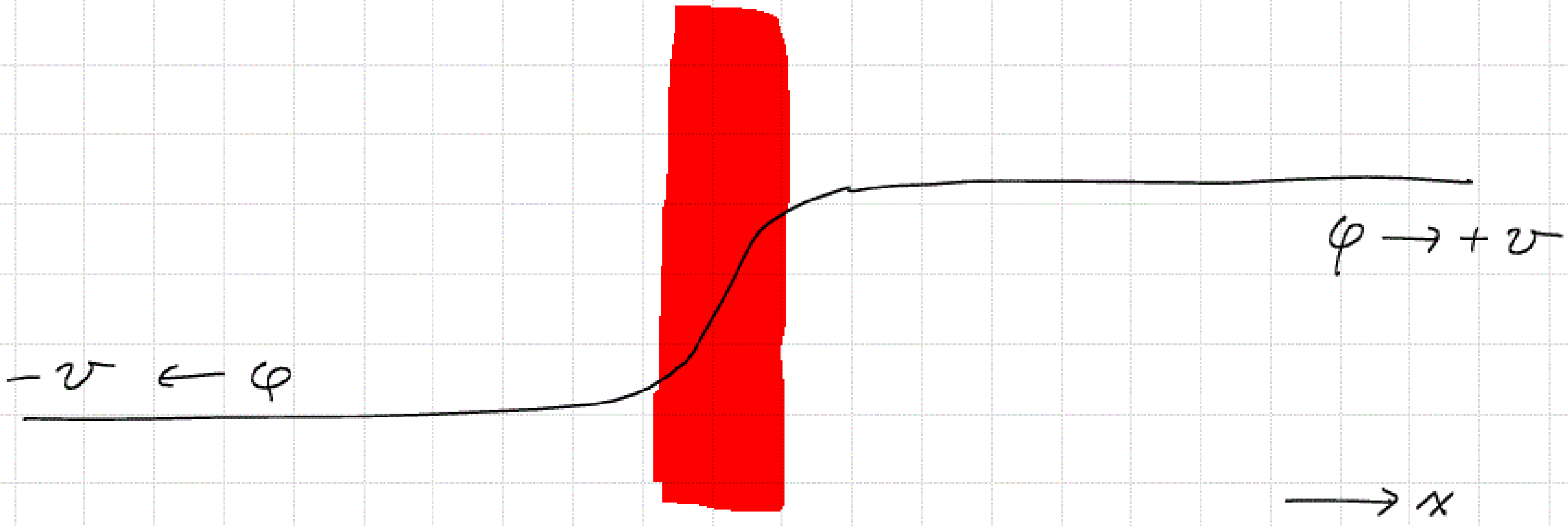
$$\frac{d\varphi}{dx} = \pm\sqrt{2V(\varphi(x)) + \varepsilon}$$

integration constant

Since $V(\varphi = \pm v) = 0$

& $\varphi = \pm v$ far away, $\varepsilon = 0$ is the only option

$$\Rightarrow \boxed{\frac{d\varphi}{dx} = \pm\lambda(\varphi^2 - v^2)}$$



$$dx = \pm \frac{d\varphi}{\lambda(\varphi^2 - v^2)} = \pm \frac{1}{2v\lambda}\left(\frac{1}{\varphi - v} - \frac{1}{\varphi + v}\right) d\varphi$$

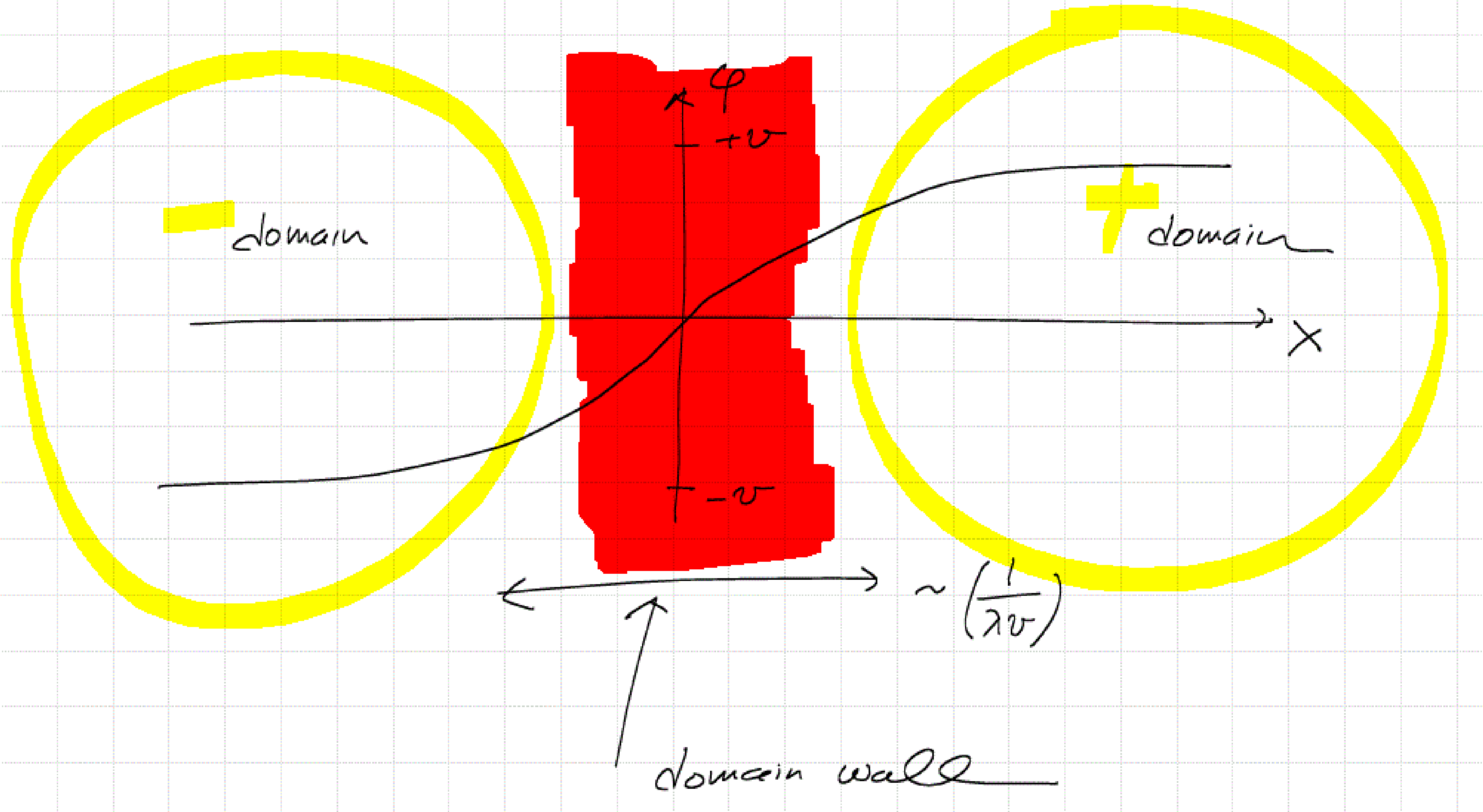
$$= \pm \frac{1}{2v\lambda}\, d \log\left(\frac{\varphi - v}{\varphi + v}\right)$$

$$\frac{\varphi - v}{\varphi + v} = - e^{\pm 2\lambda v x}$$

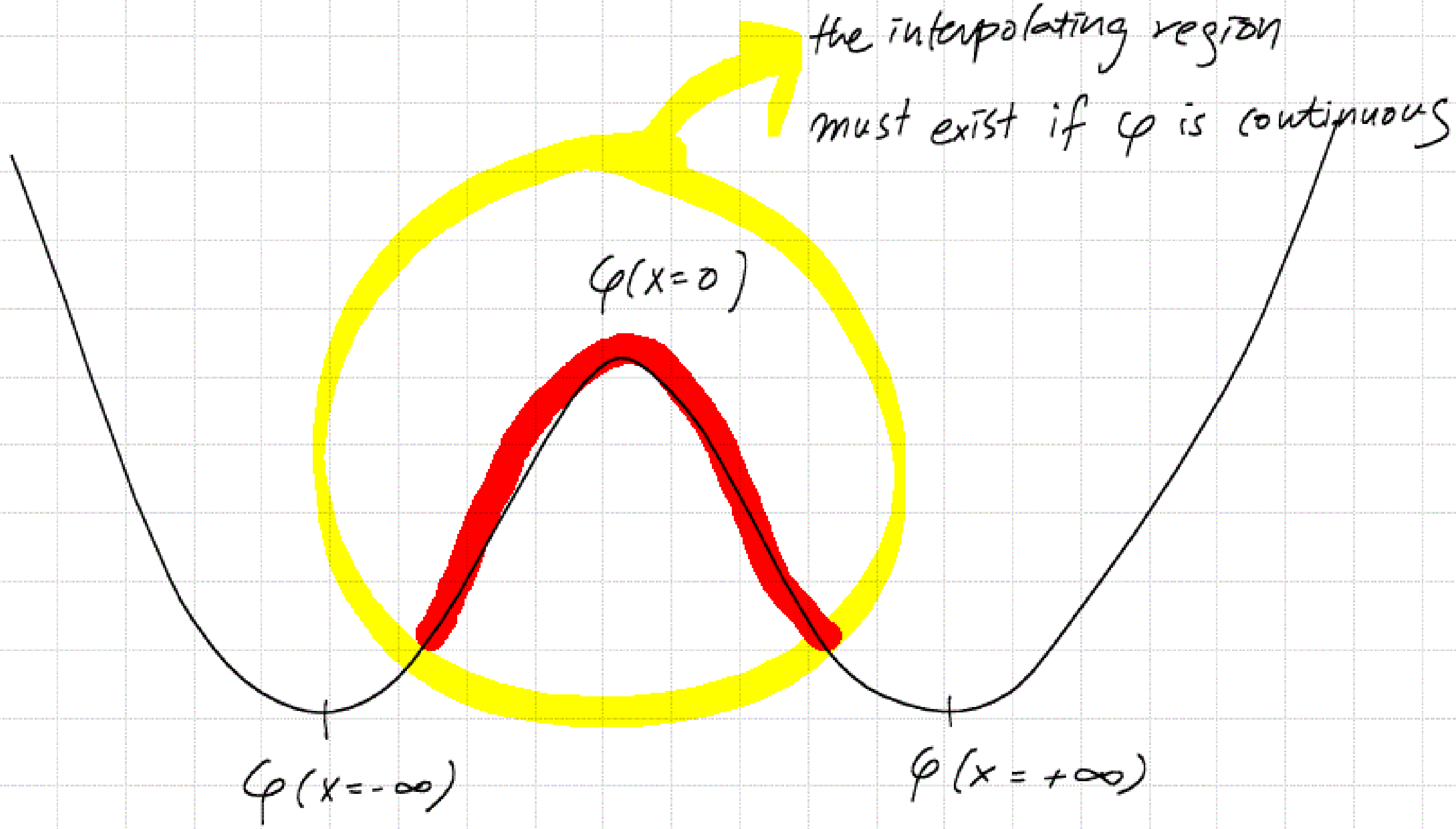
$$(\varphi - v) e^{\mp \lambda v x} = - (v + \varphi) e^{\pm \lambda v x}$$

$$\varphi \cdot (\cosh \lambda v x) = - v \sinh(\pm \lambda v x)$$

$$\varphi(+\infty) = \pm v \implies \boxed{\varphi(x) = \pm v \cdot \tanh(\lambda v x)}$$



φ
+v
−v
x
− domain
+ domain
~ (1/λv)
domain wall


the interpolating region
must exist if φ is continuous
φ(x=0)
φ(x=-∞)
φ(x=+∞)

Topology of Vacuum & Domain Wall

Topology → Property of an object that cannot be continuously deformed away

- $\oplus$ vacuum : $\varphi(-\infty) = v = \varphi(+\infty)$
- $\ominus$ vacuum : $\varphi(-\infty) = -v = \varphi(-\infty)$
- domain wall : $\varphi(-\infty) = -v$ & $\varphi(\infty) = +v$

Topology

Set of All Vacua

$\sim$ Distinct Boundary Conditions

$\wr$

$\sim$ Possible Choices for $\varphi(\pm\infty)$

Distinct Topology $=$ (Discrete Choices of Boundary Condition)

Boundary : $\{ x = +\infty \,,\ x = -\infty \}$

$\{\text{Vacua}\} = \{ +v \,,\ -v \} \longrightarrow$ "Vacuum Manifold"

(Distinct B.C.) :

+Vacuum	−vacuum	domain wall	anti-domain wall
$+\infty \to +v,\ -\infty \to +v$	$+\infty \to -v,\ -\infty \to -v$	$+\infty \to +v,\ -\infty \to -v$	$+\infty \to -v,\ -\infty \to +v$

We understand such situation by saying

$$\underbrace{\pi_0}_{\substack{\text{count \# of}\\ \text{disconnected objects}}} \left(\begin{matrix} \text{Vacuum} \\ \text{Manifold} \end{matrix} \right) = \{+, -\}$$

$\boxed{0^{th} \text{ homotopy group}}$

Homotopy Group

Let M be a manifold. We define n^{th} homotopy group $\boxed{\pi_n(M)}$ as collection of maps from S^n to M, up to the following equivalence

$$f_0 : S^n \longrightarrow M \quad , \; f_1 : S^n \longrightarrow M$$

are equivalent if $\exists\, F : \underset{t}{[0,1]} \times S^n \longrightarrow M$

$\ni$ $F_{t=0} = f_0$, $F_{t=1} = f_1$; F is continuous

N.B. What is S^0?

$$S^n = \{(x_1, \dots, x_{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}$$

$$\Rightarrow S^0 = \{x_1 \mid x_1^2 = 1\} = \{+1, -1\}$$

Example : $M = S^1$

(i) $\pi_0(M) =$ "0" : Group with only 1 element

$S^0 = \{pt\}$ $\quad f_\theta : pt \longmapsto \theta \in [0, 2\pi)$

$f_\theta \sim f_{\theta'}$ via $F(t;pt) = t\theta' + (1-t)\theta \mod 2\pi$

$$\begin{pmatrix} F_0 = f_\theta \\ F_1 = f_{\theta'} \end{pmatrix}$$

All such maps are equivalent homotopically

In fact, π_0 simply counts # of connected components of M

(ii) $\pi_1(S^1) = \mathbb{Z}$: Integer group

$$f_k : \underset{S^1}{\varphi_{\in[0,2\pi)}} \longmapsto \underset{M = S^1}{\underbrace{k}_{\text{winding } \#}\varphi}$$

Any function from S^1 to S^1 has well-defined winding #, which can be thought to represent the element of $\pi_1(S^1)$.

That is, if $g : S^1 \longrightarrow M = S^1$ has winding # k

$$g \sim f_k$$

(iii) $\pi_n(S^1) = 0 \qquad n > 1$

Obvious ?

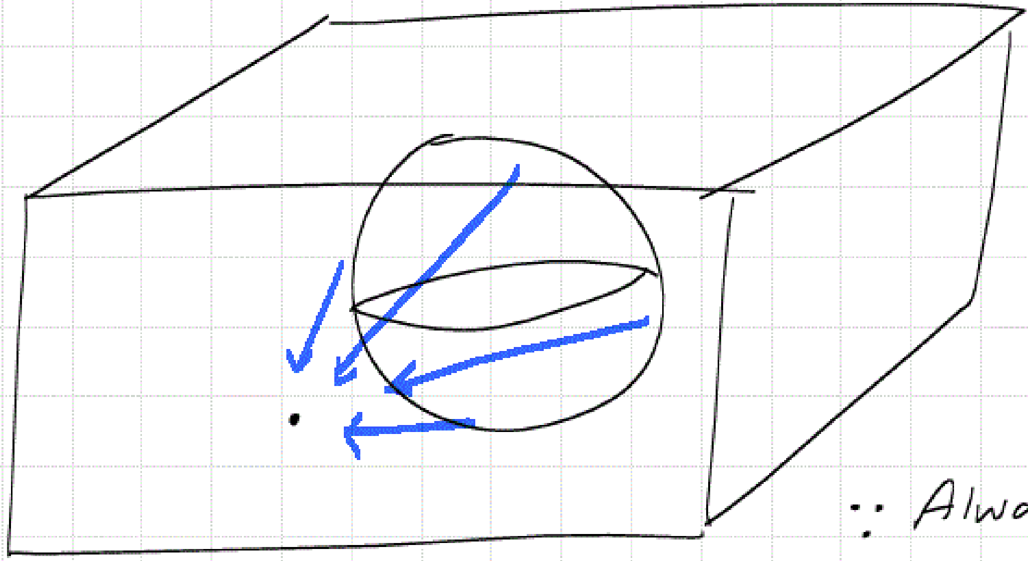
We will come back to the question of

$$\pi_n(S^k)$$

for more general n & $k \geqslant 1$

(iv) Even simpler $\to$ take R^n for any n

$$\pi_k(\mathbb{R}^n) = 0 \quad \forall\, k, n \geq 0$$



$\because$ Always contractible to a point

Let us go back to definition of π_n

Let M be a manifold. We define n^{th} homotopy group $\pi_n(M)$ as collection of maps from S^n to M, up to the following equivalence

$$f_0 : S^n \longrightarrow M \quad , \; f_1 : S^n \longrightarrow M$$

are equivalent if $\exists \; F : \underset{t}{[0,1]} \times S^n \longrightarrow M$

$\ni$ $F_{t=0} = f_0$, $F_{t=1} = f_1$; F is continuous

NOT QUITE CORRECT!

π_n is a group :

① Identity Element ?

$i : S^n \longrightarrow$ Any point of M

$\Rightarrow$ Thus for $n \geq 1$, π_n is meaningful for each connected component of M

② Multiplication Rule

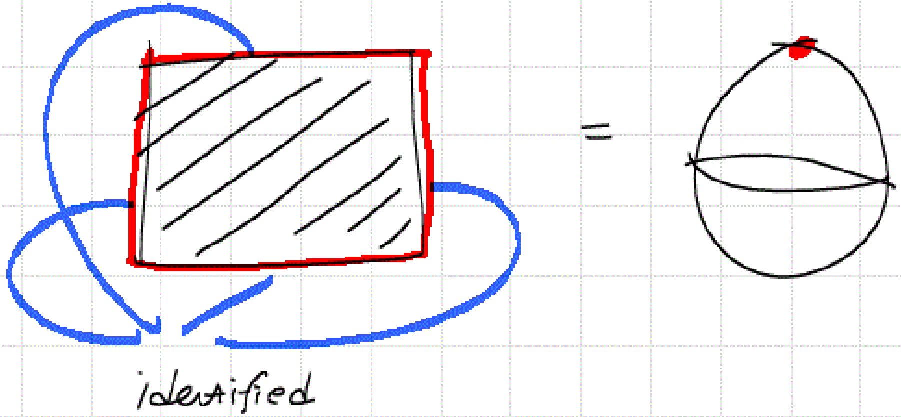
$$f : S^n \longrightarrow M$$

$$g : S^n \longrightarrow M$$

Useful Parametrization of S^n is to think it as a Solid Cube

$$I^n = [0,1] \times [0,1] \times \cdots \times [0,1]$$

with its boundary ∂I^n identified as a single pt.



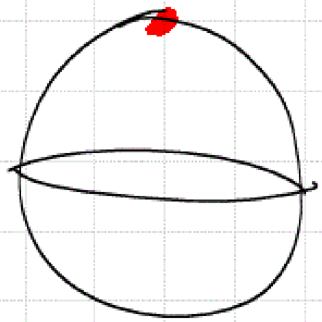
$$f' : I^n \xrightarrow{\eta} S^n \xrightarrow{f} M$$

$$g' : I^n \xrightarrow{\eta} S^n \xrightarrow{g} M$$

$$g' \circ f' : (x_1, \cdots) \longmapsto \begin{cases} f'(2x_1, \cdots) & \text{if } x \le 1/2 \\ g'(2x-1, \cdots) & \text{if } x \geq 1/2 \end{cases}$$

This defines the multiplied map

$$g \circ f : S^n \longrightarrow M$$

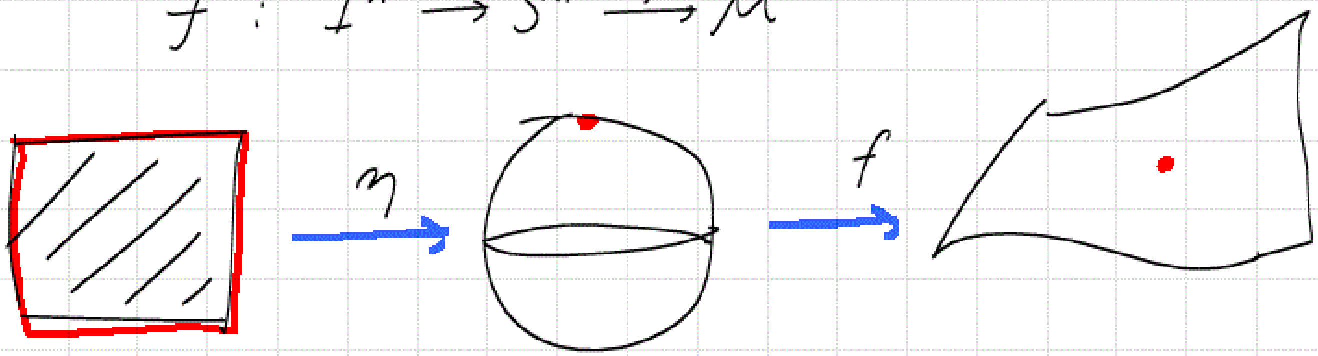


if and only if $g(\bullet) = f(\bullet)$

Thus, for $\pi_n(M)$ to form a group, the definition of π_n requires a basepoint on M. However, properties of $\pi_n(M)$ do not depend on this choice.

③ Inverse

$$f' : I^n \xrightarrow{\eta} S^n \xrightarrow{f} M$$



$$(f^{-1})' : (x, \cdots) \longmapsto f'(1-x, \cdots)$$

$$(f^{-1})' \circ f'$$

$$\updownarrow$$

Interpolated by F_t': $(x, \cdots) \mapsto \begin{cases} f'(2tx, \cdots) & x \le \frac{1}{2} \\ f'(2t(1-x), \cdots) & x \geq \frac{1}{2} \end{cases}$

$$i_\bullet'$$

$$\Longrightarrow \quad i_\bullet \sim f^{-1} \circ f$$

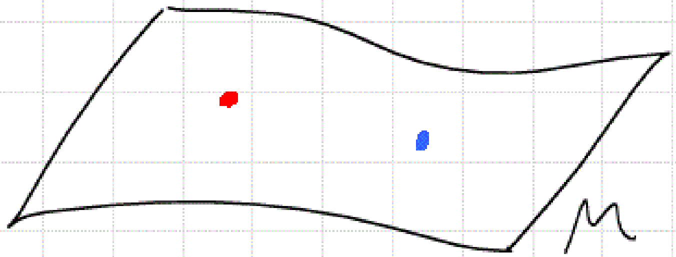
Thus $\pi_n(M, \bullet)$ is a Group:

$$\text{elements} \left(\begin{array}{llll} i & : S^n \longmapsto \bullet & \Rightarrow & \text{Identity} \\ f_\bullet & : S^n \longmapsto M_\bullet & \Rightarrow & [f_\bullet] \end{array} \right)$$

$$\text{group operation} \left(f_\bullet \circ g_\bullet \qquad \Rightarrow [f_\bullet][g_\bullet] = [f_\bullet \circ g_\bullet] \right)$$

More properties :

- $\pi_n(M, \bullet)$ is commutative for $n \geq 2$
- $\pi_n(M, \bullet) = \pi_n(M, \bullet)$ for connected M



Back to $\pi_n(S^k)$

<Fact 1> $\pi_n(S^k) = \begin{cases} 0 & n < k \\ \mathbb{Z} & n = k \end{cases}$

$\pi_1(S^1) = \mathbb{Z}$

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^3) = \mathbb{Z}$

$\vdots$

Some additional facts for $k > n$ cases

$$\pi_n(S^2) = \pi_n(S^3) \quad n \geq 3$$

$$\left(\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z} \right)$$

$$\pi_{n+1}(S^n) = \mathbb{Z}_2 \quad n \geq 2$$

$$\pi_{n+2}(S^n) = \mathbb{Z}_2 \quad n \geq 2$$

$$\pi_{n+3}(S^n) = \mathbb{Z}_{24} \quad n \geq 5$$

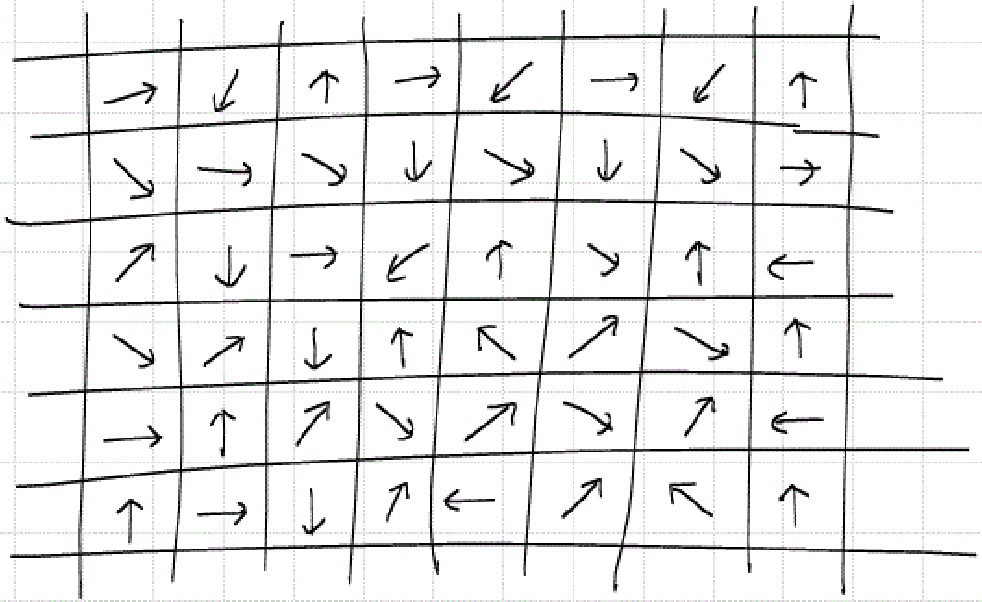
Vortex String

π_0 (Vacuum Mfld) $\longrightarrow$ Domain Wall

π_1 (Vacuum Mfld) $\longrightarrow$ Vortex String

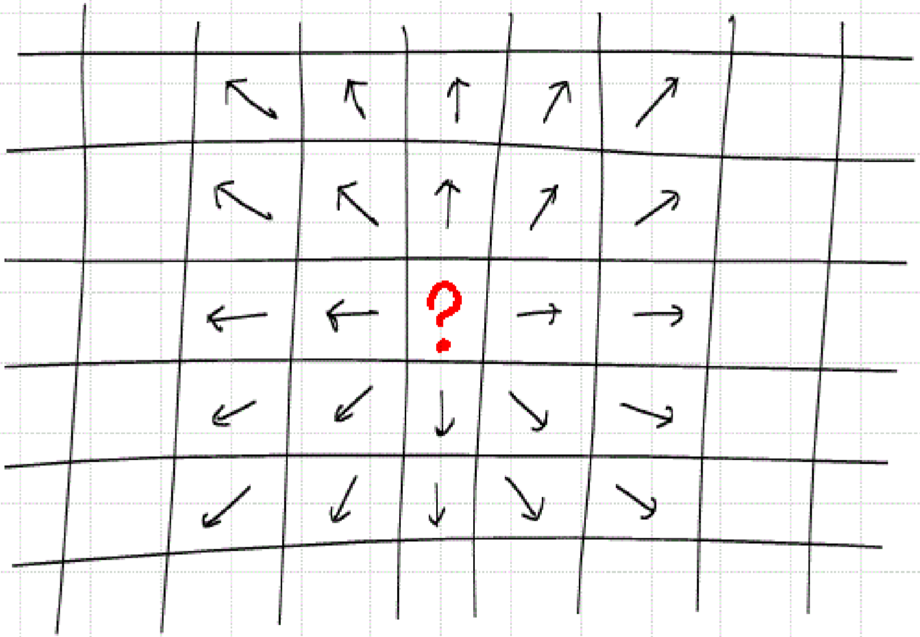
When do we have vacuum mfld $= S^1$?

Back to Lattice



Insted of $\{+, -\}$, an arrow on a plane $\approx$ one angle

Vortex Configuration

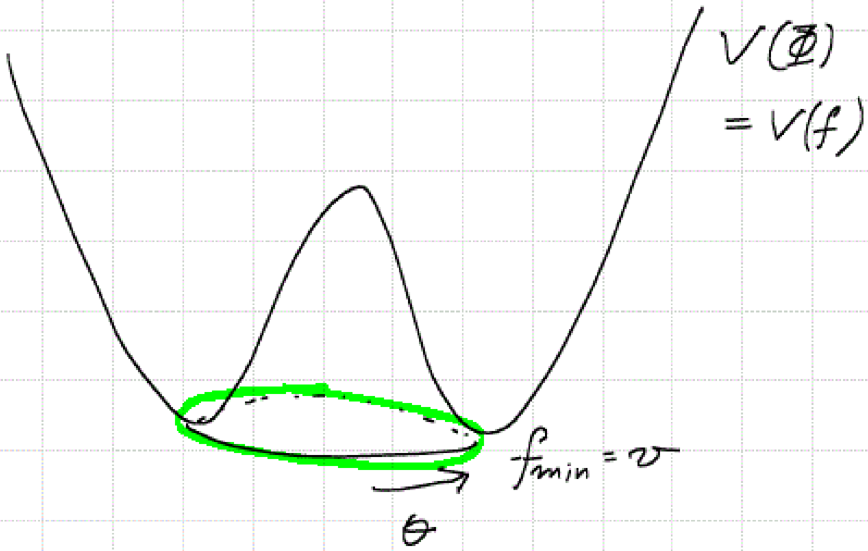


Again, continuum model of this is useful:

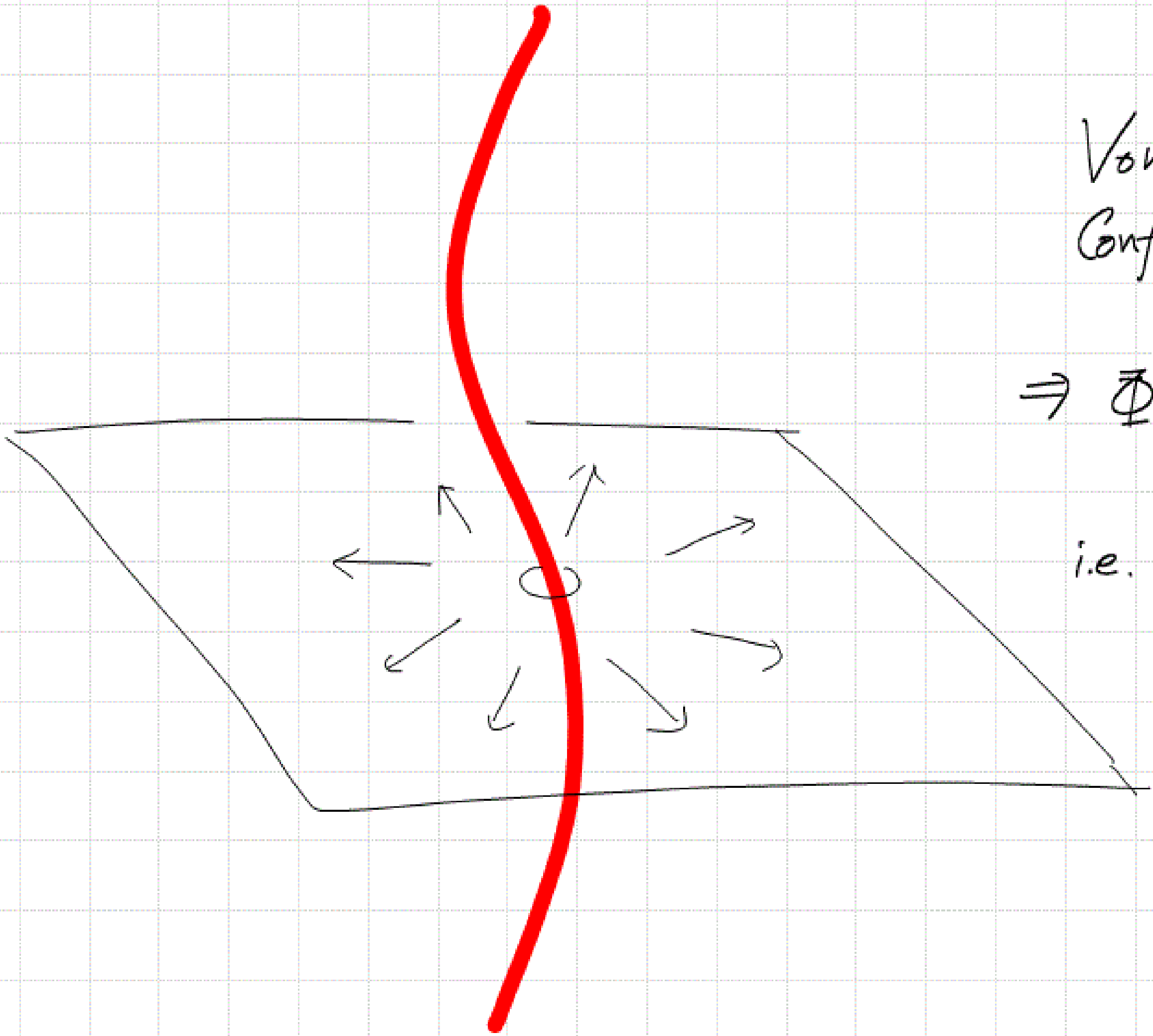
$$\int dt \int dx^3 \left(\partial \Phi \, \partial \Phi^* - V(|\Phi|) \right)$$

$$\Phi = f e^{i\theta}$$

angle



$$\pi_1(\bigcirc) = \mathbb{Z}$$



Vortex Configuration

$$\Rightarrow \Phi = f_{min} \cdot e^{i n \phi}$$

i.e. $|\Phi| = |\Phi|_{min}$

$\theta = n\phi$

(angle in X-Y plane)

Unlike the case of Domain Wall, No Such Solution Exists

$$\Phi = f_{min} e^{i\varphi}$$ (Take $n=1$ from now on)

$$\rightarrow \begin{cases} V(\Phi) = 0 \\ \partial\Phi \partial\Phi^* = -\vec{\partial}\Phi \cdot \vec{\partial}\Phi^* \end{cases}$$

$$= -(f_{min})^2 (\vec{\partial}\varphi)^2$$

$$= -(f_{min})^2 \left(\frac{1}{r}\right)^2$$

$$\int d^3x \left(-\partial\Phi\partial\Phi^*\right) = 2\pi \int_{-\infty}^{\infty} dz \int_0^{\infty} r\,dr\,(f_{min})^2 \cdot \left(\frac{1}{r}\right)^2 = \int dz \cdot (\infty)$$

$\Rightarrow$ Vortex Configuration is infinitely heavy (per unit length)

$\Rightarrow$ No Isolated Vortex Exists

Cure : Introduce a gauge field

$$\partial_\mu \Phi \longmapsto (\overbrace{\partial_\mu - iA_\mu}^{D_\mu})\Phi$$

$$\partial_\mu \Phi^* \longmapsto (\partial_\mu + iA_\mu)\Phi^*$$

$$\int dt \int d^3x \left((D_\mu \Phi)(D^\mu \Phi)^* - V(|\Phi|) - \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Vortex Configuration

$$\left(\begin{array}{l} \Phi \cong f_{min} e^{i\phi} \\ A_i \cong \partial_i \phi \end{array} \right) \quad \text{at large } r$$

$$\Rightarrow \quad D_i \Phi \cong i f_{min} e^{i\phi} \left(\partial_i \phi - A_i \right) = 0 \quad \text{at large } r$$

$\Rightarrow$ $\boxed{\text{Asymptotic Winding Allowed}}$

Note : $A_i = \partial_i \phi \implies F_{ij} = [\partial_i, \partial_j]\phi = 0$

This is related to "gauge invariance" where

$$\Phi \longmapsto \Phi e^{i\tilde{\theta}(x)}$$

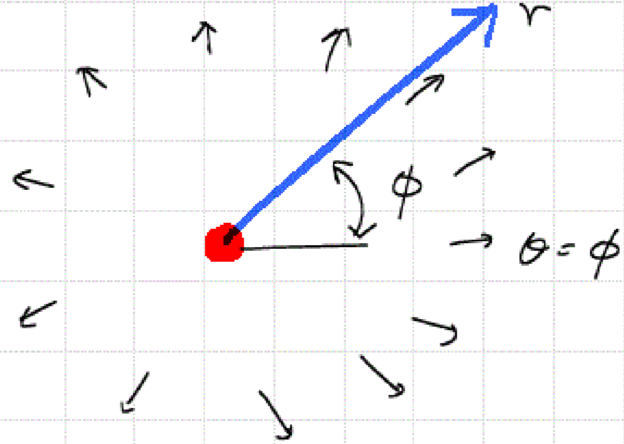
become irrelevant provided that

$$A_\mu \longmapsto A_\mu - \partial_\mu \tilde{\theta}$$

$$\Rightarrow \quad D_\mu \Phi \longmapsto (D_\mu \Phi) e^{i\tilde{\theta}(x)}$$

$$|D_\mu \Phi|^2 \longmapsto |D_\mu \Phi|^2$$

$$\begin{pmatrix} \Phi = f(r)\, e^{i\phi} \\ A_i = u(r)\, \partial_i \phi \end{pmatrix}$$

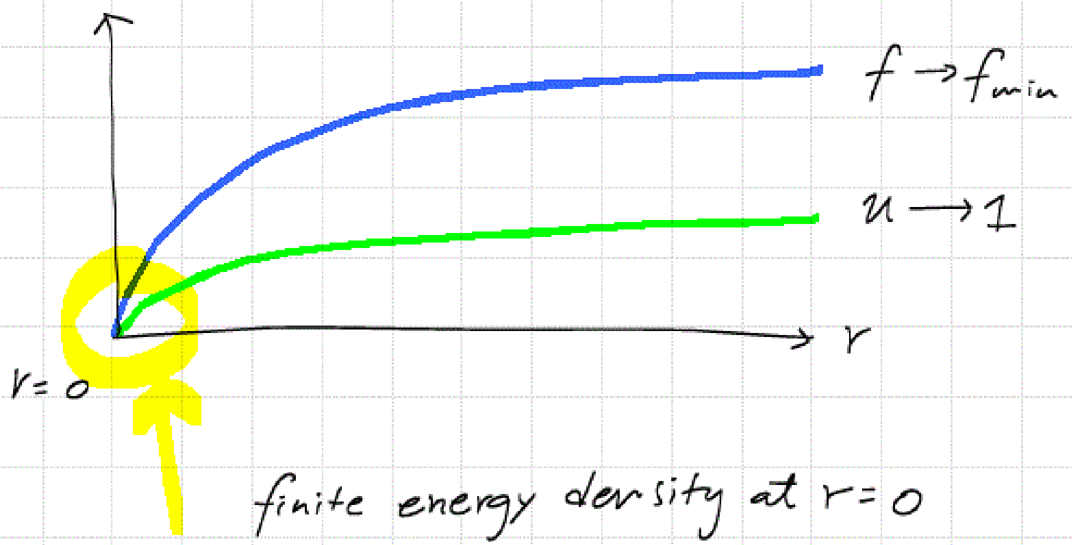


$$0 = \delta \int d^3x \left(D_i \Phi D_i \Phi + V(\Phi) + F_{ij}^2 / 4g^2 \right)$$

$$0 = \delta \int_0^\infty r\,dr \left((f')^2 + \frac{f^2}{r^2}(1-u)^2 + V(f) + \frac{1}{2g^2} \frac{(u')^2}{r^2} \right)$$

Far away ($r \to \infty$), $f \to f_{min}$, $u \to 1$ for any finite energy configuration.

Nonsingular origin ($r=0$) requires $f(0)=0$, $u(0)=0$

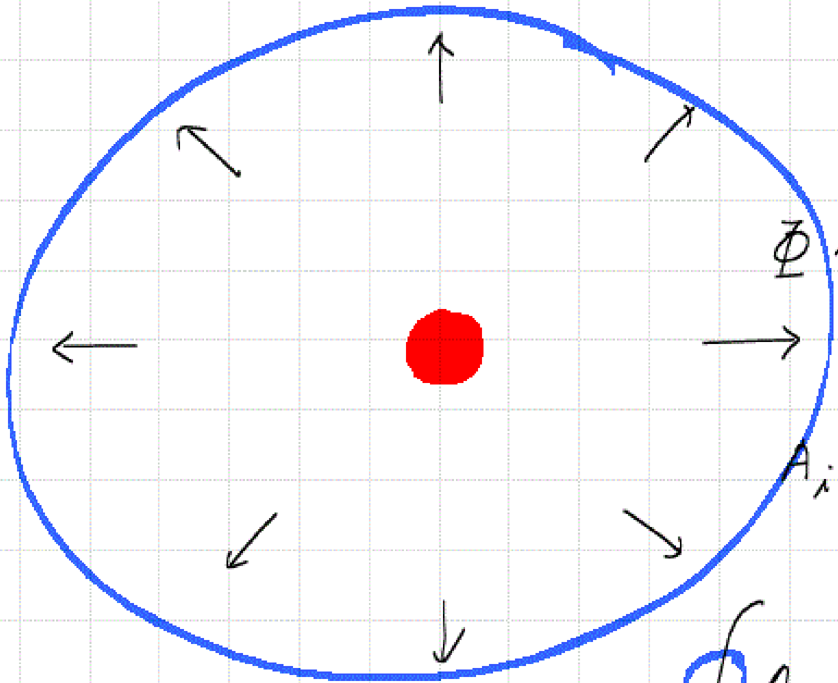


finite energy density at $r=0$

Expanding for small $\frac{1}{r}$, $\delta f = f - f_{min}$, $\delta u = u - 1$

$$\left\{ \begin{aligned} \delta f'' &= \left(\frac{d^2V}{df^2}\right)\Bigg|_{f=f_{min}} \cdot \delta f \\ \delta u'' &= \left(2g^2 f_{min}^2\right) \delta u \end{aligned} \right\}$$

$$\left\{ \begin{aligned} f &= f_{min} - O\left(e^{-\left(\sqrt{V''_{min}}\, r\right)}\right) \\ u &= 1 - O\left(e^{-\left(\sqrt{2g^2 f_{min}^2}\, r\right)}\right) \end{aligned} \right\}$$



$$\Phi \sim f_{min} e^{i\phi}$$

$$A_i = u(r)\,\partial_i \phi \sim \partial_i \phi$$

$$\oint A_i\, dx^i = \int \partial_i \phi\, dx^i = \int_0^{2\pi} d\phi$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\|$$

$$\int (\nabla \times A)_z\, dx\,dy = 2\pi$$

Introduction of Gauge Field Forces

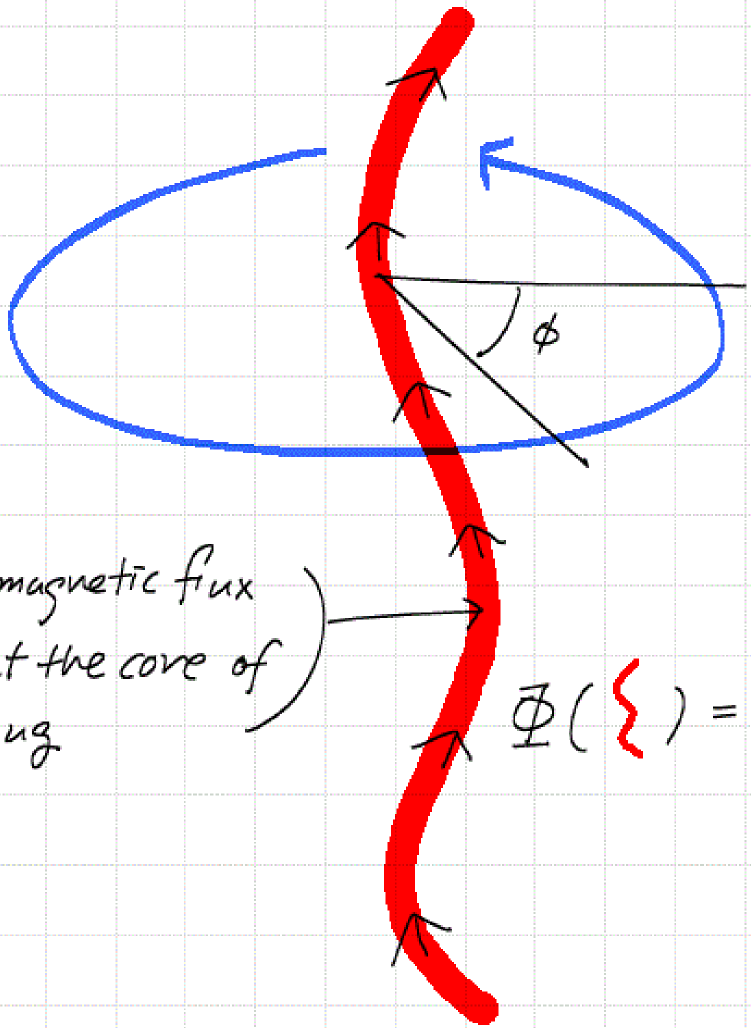
Winding # of $\Phi|_\infty$ $\leftrightarrow$ "Winding" of A_i

$\downarrow$ $\qquad\qquad$ $\downarrow$

Vorticity (Integer Quantized) $\leftrightarrow$ Magnetic Flux (Integer Quantized)

N.B. If we used $D_i\Phi = (\partial_i - iqA_i)\Phi$ instead

Flux would have been $\oint A = (2\pi/q)$ per unit winding of Φ



$$\Phi = \underbrace{|\Phi|}_{\hookrightarrow\ f_{min}\ \text{as}\ r\to\infty} e^{i\phi}$$

Unit Winding

(quantized magnetic flux localized at the core of vortex string)

$$\Phi(\xi) = 0$$

$$L = \int d^3x \left((D_\mu \Phi)(D^\mu \Phi)^* - V(|\Phi|) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$D_\mu \Phi = \partial_\mu \Phi - i A_\mu \Phi$$

This gives a decent low energy theory of superconducting material

with $|\langle \Phi \rangle| = |f_{min}|$ ~ Cooper pair condensate.

Earliest Description of Superconducting Phase

$$\left(\vec{\nabla} \times \vec{J} = -m_*^2 \vec{B} \right)$$

London's Equation

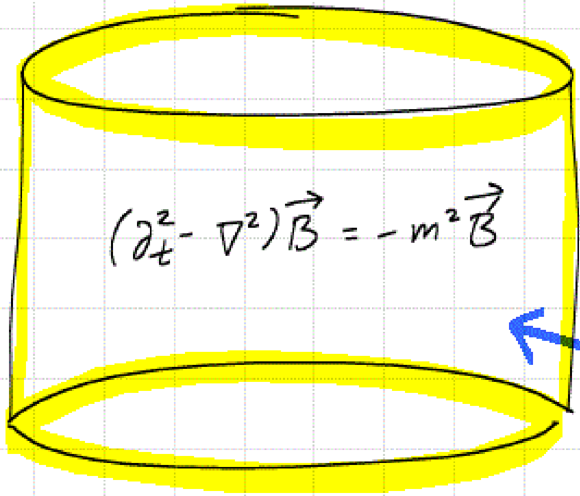
in addition to usual Maxwell Eqs

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B} = -\frac{\partial}{\partial t}\vec{E} + \vec{J} \right) \Longrightarrow -\nabla^2 \vec{B} = -\partial_t \underbrace{(\nabla \times \vec{E})}_{\frac{\partial}{\partial t}\vec{B}} + \underbrace{\vec{\nabla} \times \vec{J}}_{-m^2 \vec{B}}$$

$\nabla \cdot B = 0$

$$\boxed{\Rightarrow \left[\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2 \right] \vec{B} = -m_*^2 \vec{B}}$$

$\Rightarrow$ (Electro) Magnetic Field Become Massive !!!



$$(\partial_t^2 - \nabla^2)\vec{B} = -m^2\vec{B}$$

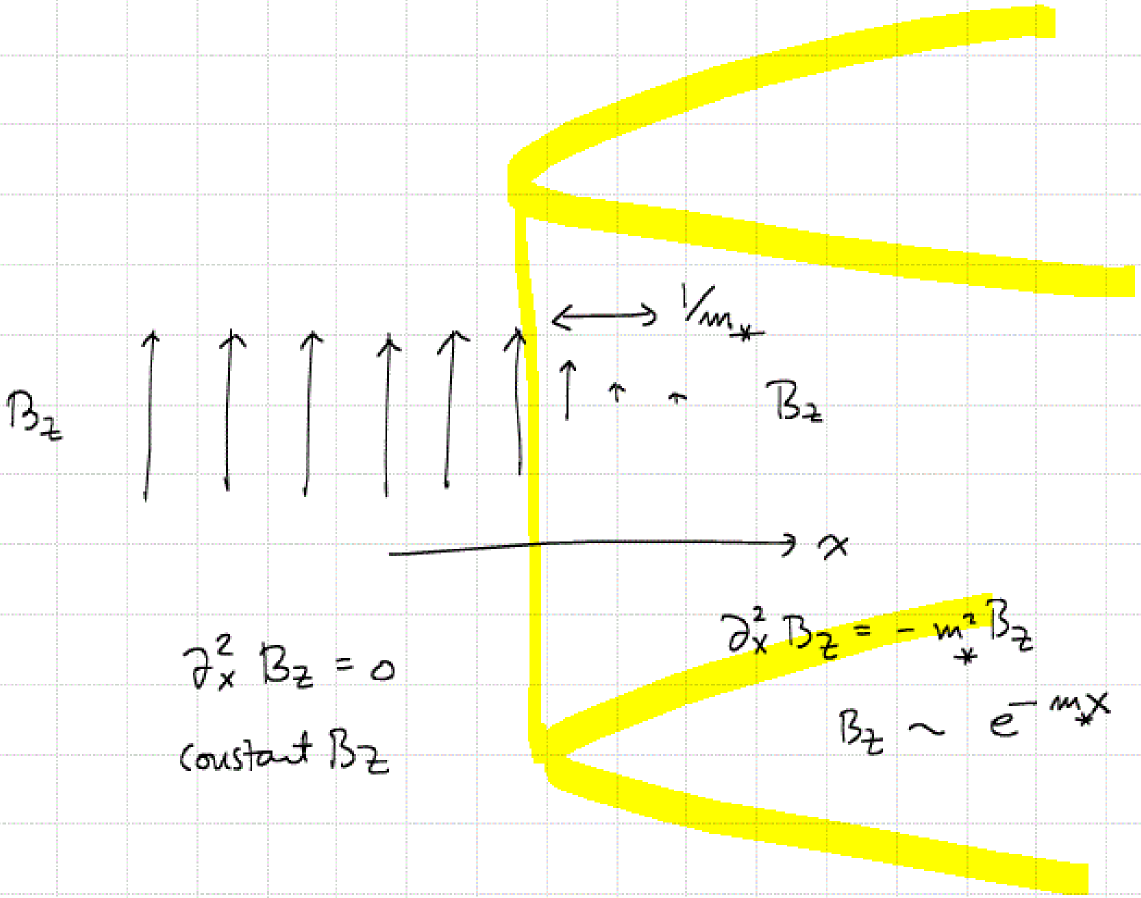
outside

$$(\partial_t^2 - \vec{\nabla}^2)\vec{B} = 0$$

inside superconductor

Since $\vec{B} = \vec{\nabla}\times\vec{A}$, this modification also implies

$$\left[\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2\right]\vec{A} = -m_*^2\vec{A} \quad \underline{\underline{\text{inside}}}$$



$\partial_x^2 B_z = 0$

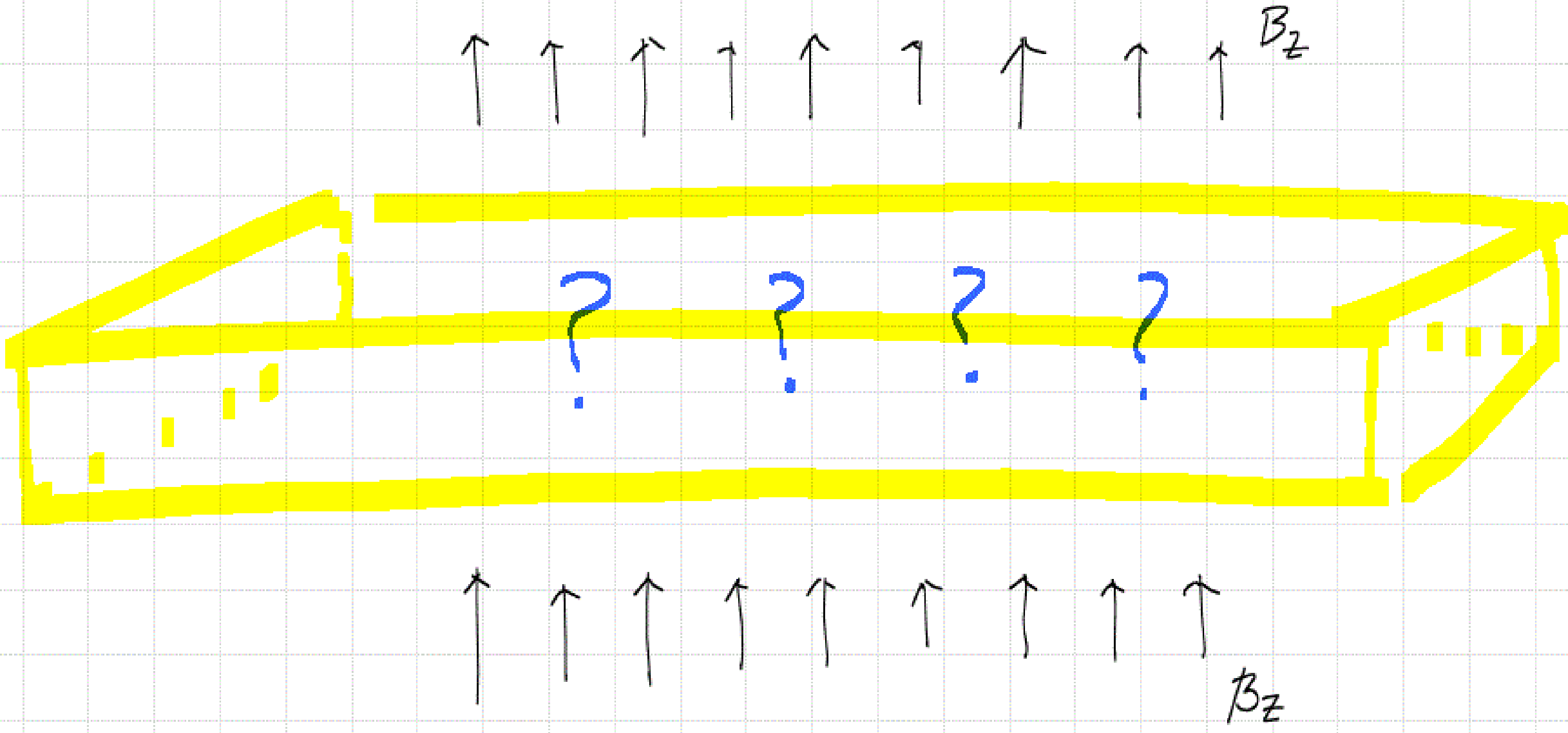
constant B_z

$\partial_x^2 B_z = -m_*^2 B_z$

$B_z \sim e^{-m_* x}$

$\Rightarrow$ Magnetic Flux Tries to Avoid Superconductor

(Meissner's Effect)



B_z
?
?
?
?
B_z

On the other hand, E.O.M for (Φ, A_μ) include

$$\frac{1}{g^2}\partial_\mu F^{\mu\alpha} = \frac{\delta}{\delta A_\alpha}\left(D\Phi (D\Phi)^*\right)$$

Writing $A_\mu = (A_0, A_i)$, A_0, A_i are the scalar & vector potentials of Electromagnetic Field.

$$(F_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \partial_0 A_i - \partial_i A_0 = E_i \\ \hline -E_i & \epsilon_{ijk} B_k \end{array}\right)$$

If we remove Φ, $\partial_\mu F^{\mu\alpha} = 0$ is the same as Maxwell's Eq without source terms

With Φ with $|\langle \Phi \rangle| = f_{min} \neq 0$

$$|\Phi| = f_{min}, \quad |D\Phi|^2 \rightarrow \underbrace{(f_{min})^2 \cdot (-A_i A_i}_{\text{mass term for } A_i} + A_0 A_0)$$

$$\partial_\mu F^{\mu k} = -g^2 \cdot \frac{\delta}{\delta A_k} f_{min}^2 (\vec{A})^2 = -2g^2 f_{min}^2 A^k$$

(gauge fix $\partial_\mu A^\mu = 0$)

$$(\partial_t^2 - \vec{\partial}^2) A^k = - m_*^2 A^k$$

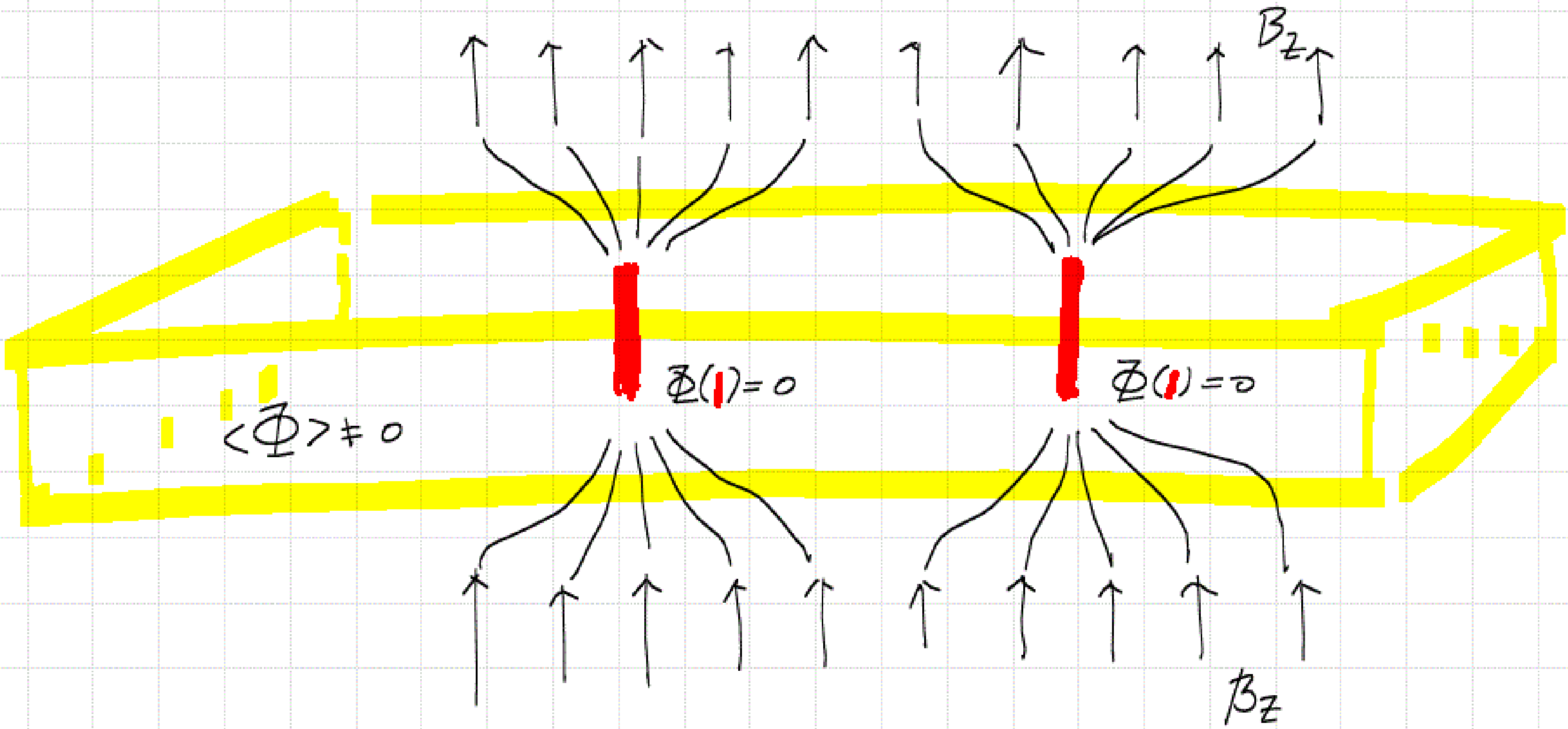
Superconducting Phase

$\Leftrightarrow$ $\left(\begin{array}{l} \langle \Phi \rangle \neq 0 \text{ for some charged field} \\ \text{Photon acquire mass} = \sqrt{2} \cdot g \, |\langle \Phi \rangle| \end{array} \right)$

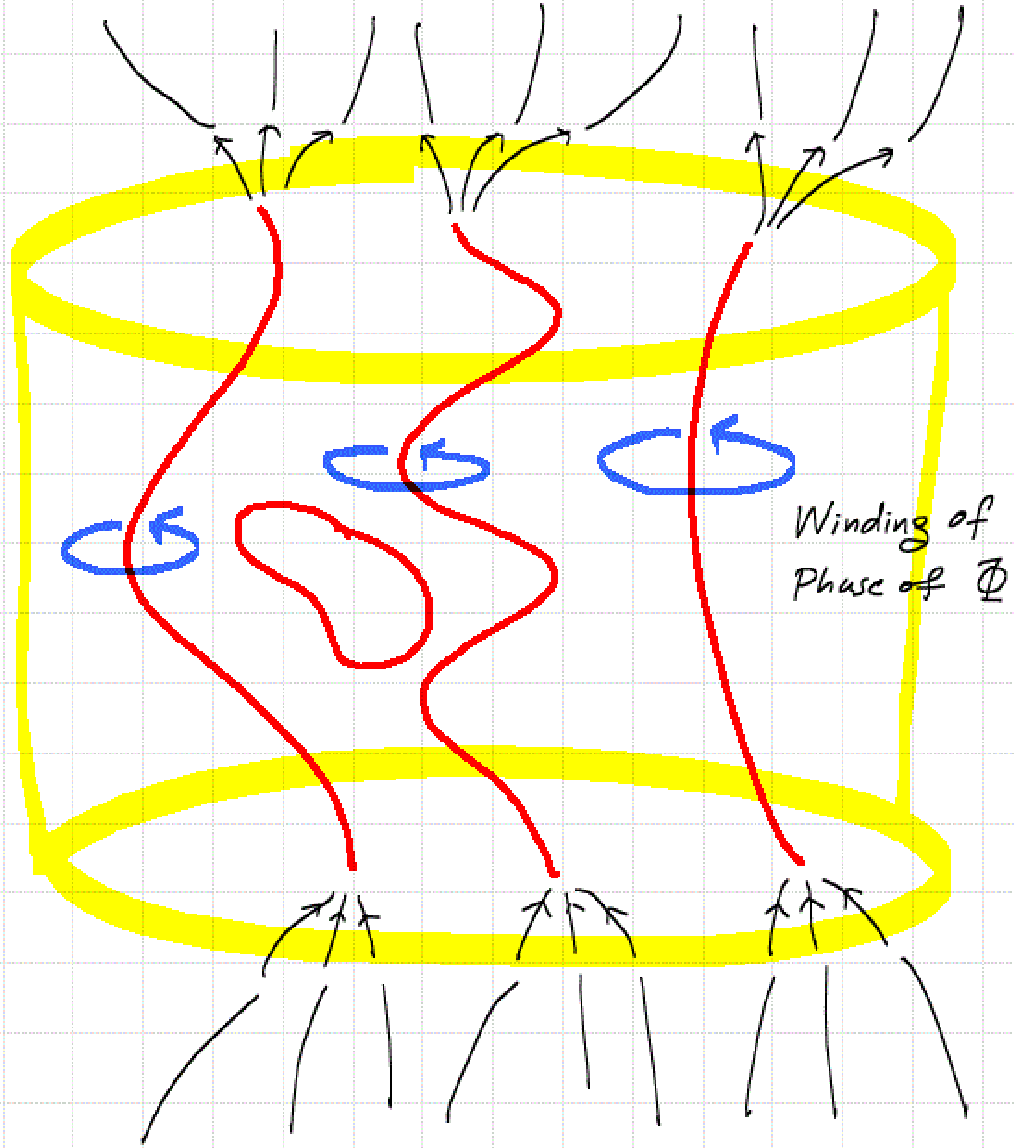
$\downarrow$

Magnetic Field Lines Get Pushed Away

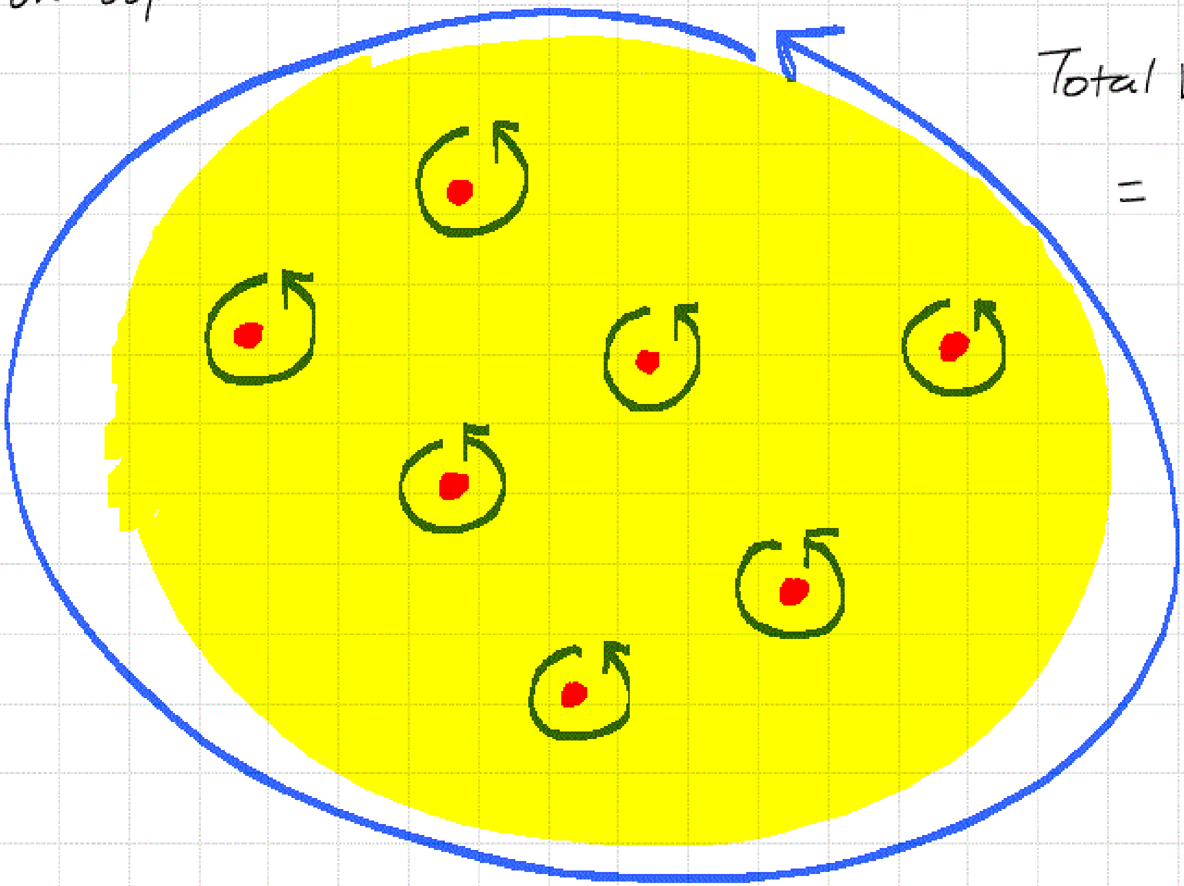
But this still allows localized & quantized magnetic flux where $\Phi \to 0$: Magnetic Vortex



B_z
$\Phi(|)=0$
$\Phi(|)=0$
$\langle\Phi\rangle\neq 0$
B_z


Winding of
Phase of Φ

View on top



Total Winding #

= #(Flux Vortex)

= n

represents an element of $\pi_1(\;)$

$\boxed{\text{Holonomy}}$

Suppose there are two charged fields $\Phi_{1,2}$

$$D_\mu \Phi_1 = (\partial_\mu - i A_\mu)\Phi_1$$

$$D_\mu \Phi_2 = (\partial_\mu - i q A_\mu)\Phi_2$$

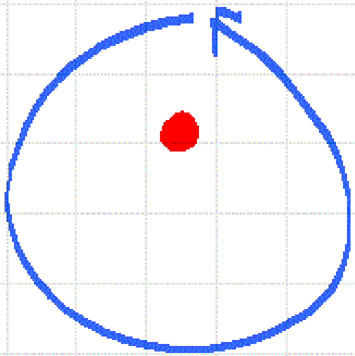
and $\langle \Phi_2 \rangle \neq 0$

Unit winding of $\langle \Phi_2 \rangle = f^{(2)}_{min}\, e^{i\phi}$

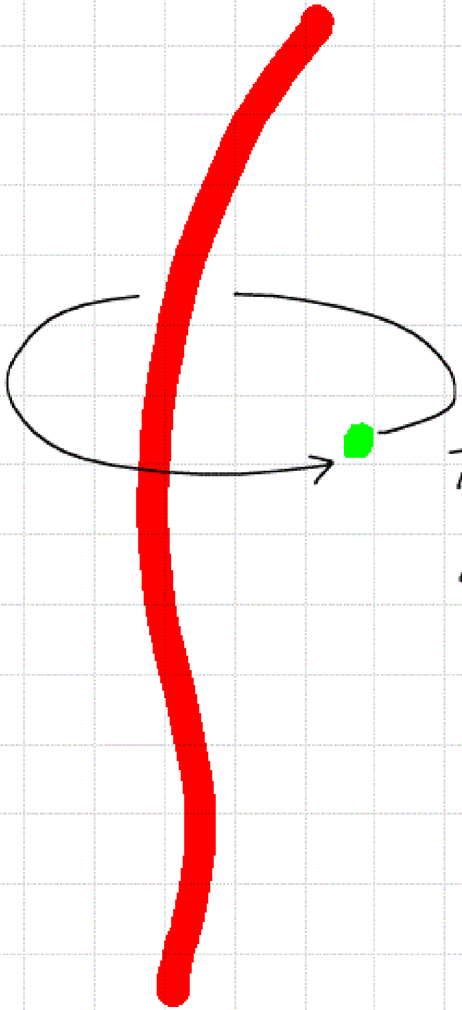
$$\Rightarrow \quad A_i(r\to\infty) = \frac{1}{q}\,\partial_i \phi$$

$\Rightarrow$ Magnetic Vortex of Flux $\frac{2\pi}{q}$

$$\left(\oint B = \oint A = \frac{1}{q}\int d\phi = \frac{1}{q}\,2\pi \right)$$



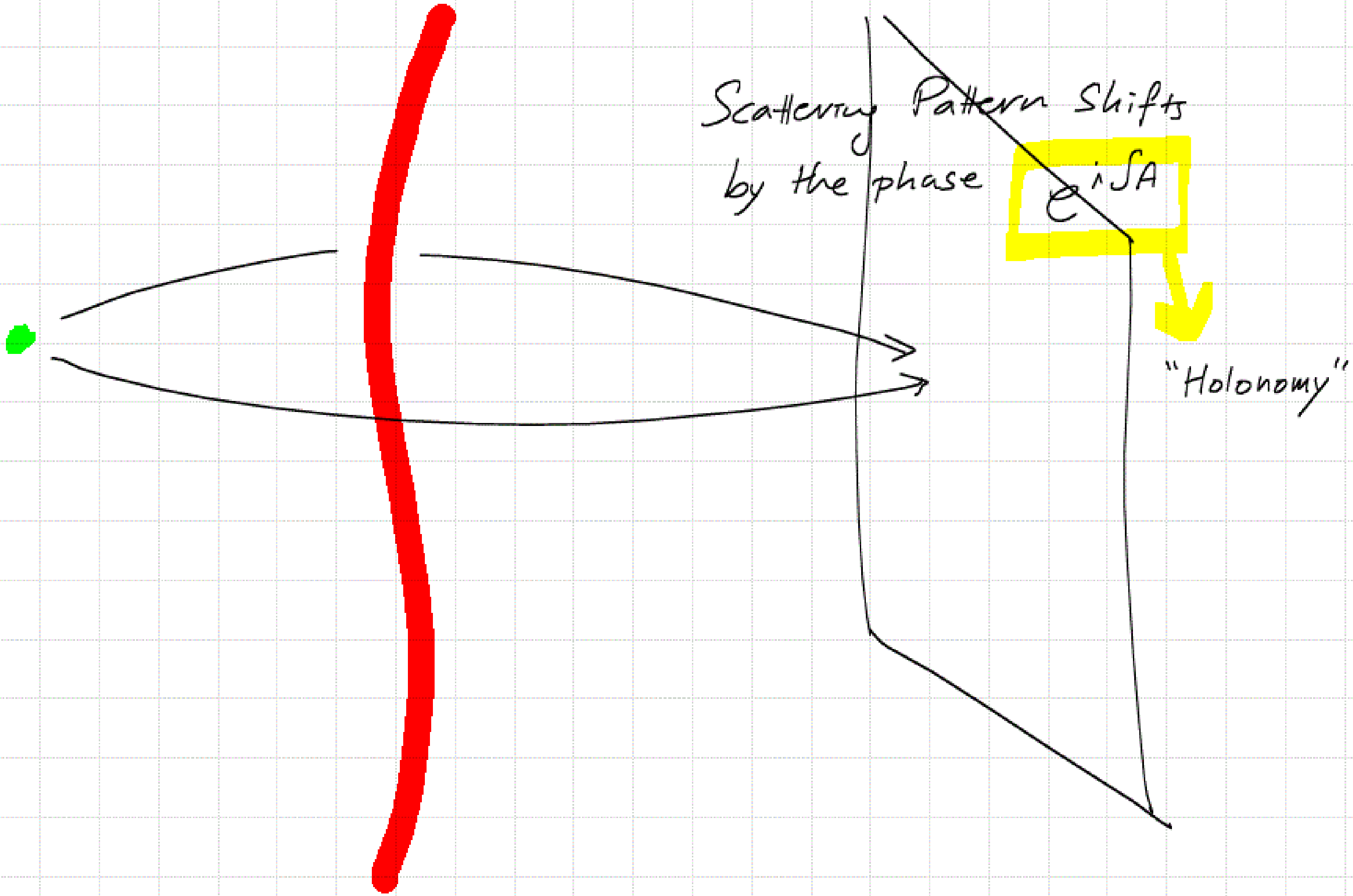
$$\oint A = \frac{2\pi}{q}$$



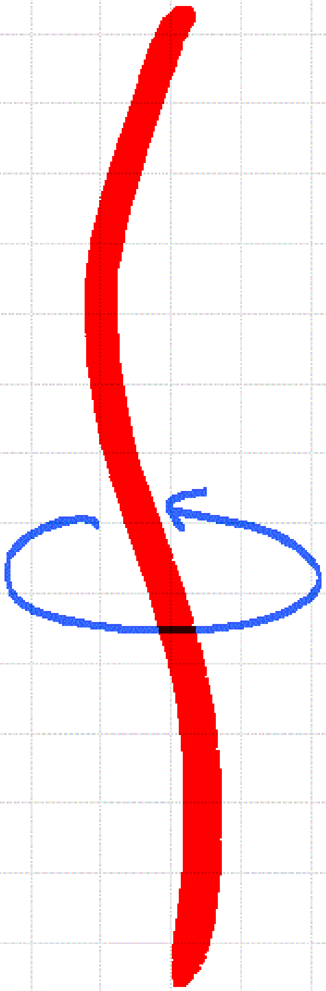
Take a Φ_1 particle a round trip around the vortex string

$\Rightarrow$ Its wavefunction acquire a phase $e^{i2\pi/q}$

Aharonov-Bohm phase


Scattering Pattern Shifts
by the phase $e^{i\int A}$
"Holonomy"

Vortex Strings

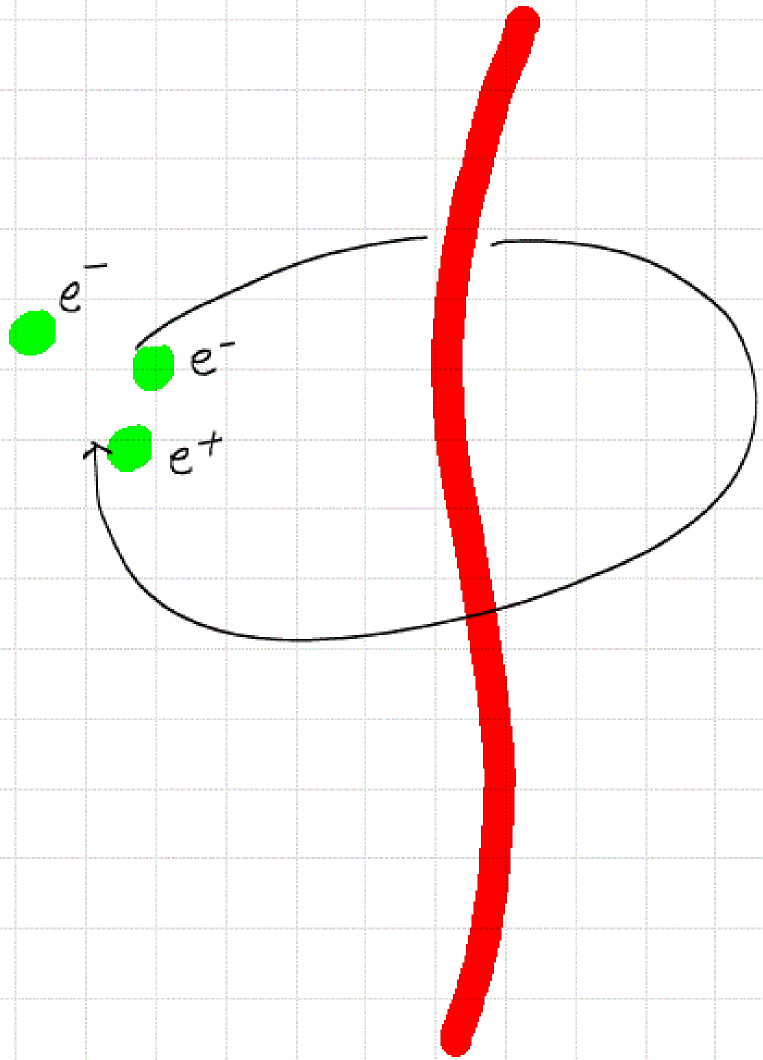


π_1

Solitonic String

Gauge Field

Holonomy
(A-B phase)



"Alice String" will generate more exotic type of holonomy (= Aharonov-Bohm "phase")

Taking a charged particle (e^-) around an Alice string, holonomy transform it to (e^+)

Back to $\pi_n(S^k)$

<Fact 1> $\pi_n(S^k) = \begin{cases} 0 & n < k \\ \mathbb{Z} & n = k \end{cases}$

$\pi_1(S^1) = \mathbb{Z}$

$\pi_2(S^2) = \mathbb{Z}$

$\pi_3(S^3) = \mathbb{Z}$

$\vdots$

Some additional facts for $k > n$ cases

$$\pi_n(S^2) = \pi_n(S^3) \qquad n \geq 3$$

$$\left(\pi_3(S^2) = \pi_3(S^3) = \mathbb{Z} \right)$$

$$\pi_{n+1}(S^n) = \mathbb{Z}_2 \qquad n \geq 2$$

$$\pi_{n+2}(S^n) = \mathbb{Z}_2 \qquad n \geq 2$$

$$\pi_{n+3}(S^n) = \mathbb{Z}_{24} \qquad n \geq 5$$

Basic Tool of Homotopy Computing : Exact Sequence

If $X = (B/Y)$, there is an exact sequence

$$\longrightarrow \pi_{n+1}(X) \xrightarrow[\eta_{n+1}]{} \pi_n(Y) \xrightarrow[i_n]{} \pi_n(B) \xrightarrow[h_n]{} \pi_n(X) \xrightarrow[\eta_n]{} \pi_{n-1}(Y) \longrightarrow \cdots$$

i.e., $\mathrm{Im}(\eta_{n+1}) = \mathrm{Ker}(i_n)$ ...... etc for each step.

Image

Subspace of $\pi_n(Y)$
Killed by (i_n)

Use of Exact Sequences

① $0 \xrightarrow{f} A \xrightarrow{g} 0$

$0 = \mathrm{Im}\, f = \mathrm{Ker}\, A = 0 \Rightarrow g$ is 1-1 map $\Rightarrow \boxed{A = 0}$

② $0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} 0$

$0 = \mathrm{Im}(f) = \mathrm{Ker}(g) \Rightarrow g$ is 1-1 map

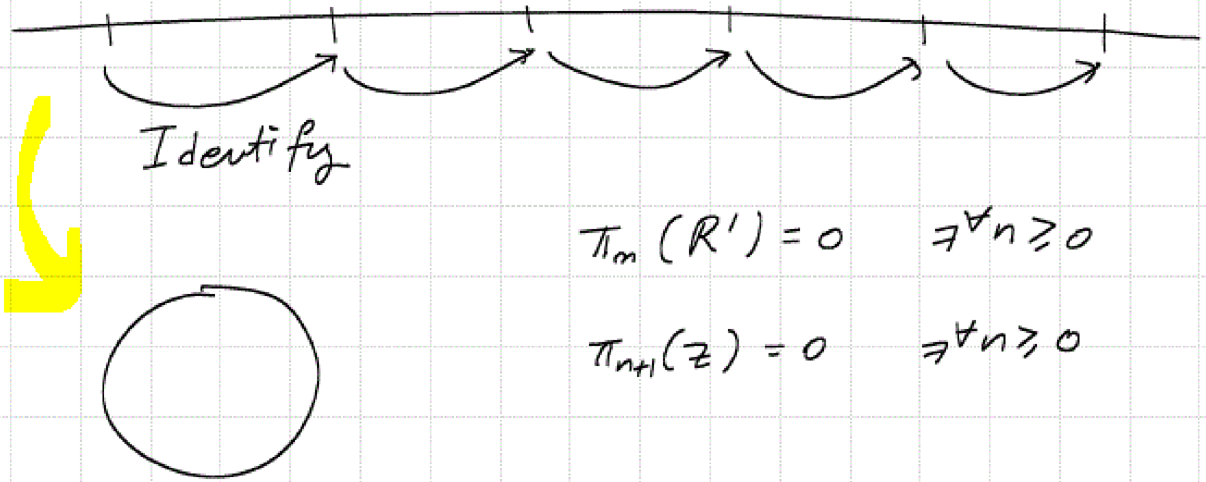
$h(B) = 0 \Rightarrow \mathrm{Ker}(h) = B = \mathrm{Im}(g) \Rightarrow g$ is onto-map

$\therefore g$ is an isomorphism $\Rightarrow \boxed{A = B}$

③ $0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \xrightarrow{k} 0$

$$\Rightarrow \left(\begin{array}{l} g \text{ is } 1\text{-}1 \text{ map} \\ \\ h \text{ is onto map} \end{array} \right) \Rightarrow A = \left(B/C \right)$$

For example, $S^1 = \mathbb{R}/\mathbb{Z}$



$\pi_n(\mathbb{R}^1) = 0 \quad \forall n \geq 0$

$\pi_{n+1}(\mathbb{Z}) = 0 \quad \forall n \geq 0$

$$0 = \pi_{2+n}(\mathbb{R}^1) \longrightarrow \underbrace{\pi_{2+n}(S^1)}_{\substack{\| \\ 0}} \longrightarrow \pi_{1+n}(\mathbb{Z}) = 0$$

$n \geq 0$

Also $S^2 = (S^3/S^1)$

$$\to \underset{\substack{\| \\ 0 \text{ if } n \geq 2}}{\pi_n(S^1)} \to \pi_n(S^2) \to \pi_n(S^3) \to \underset{\substack{\| \\ 0 \text{ if } n \geq 3}}{\pi_{n-1}(S^1)} \to$$

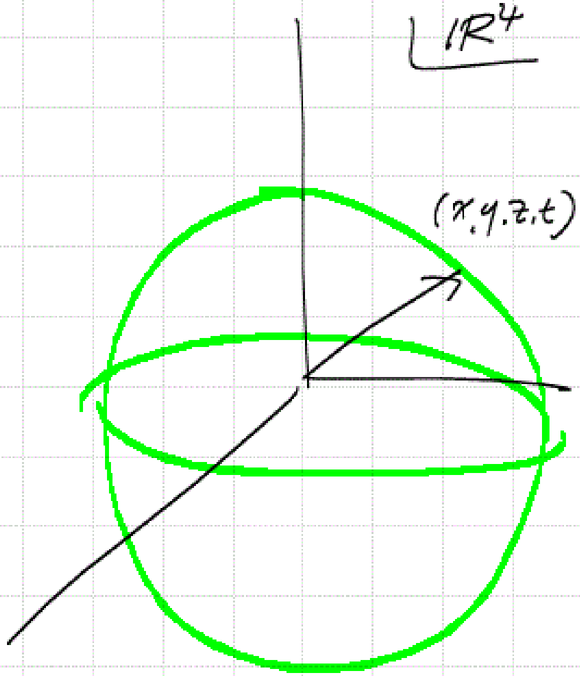
$$\Rightarrow \qquad 0 \to \pi_n(S^2) \to \pi_n(S^3) \to 0$$

if $n \geq 3$

equal

$$\boxed{S^3/S^1 = S^2}$$ Why?

$$S^3 = \{ (x, y, z, t) \mid x^2 + y^2 + z^2 + t^2 = 1 \}$$



Write :

$$\left(\begin{aligned} x + iy &= (\sin\theta)\, e^{i(\psi+\varphi)/2} \\ z + it &= (\cos\theta)\, e^{i(\psi-\varphi)/2} \end{aligned} \right)$$

$$\left\{ \begin{aligned} 0 \le \theta &\le \pi \\ 0 \le \varphi &\le 2\pi \\ 0 \le \psi &\le 4\pi \end{aligned} \right\}$$

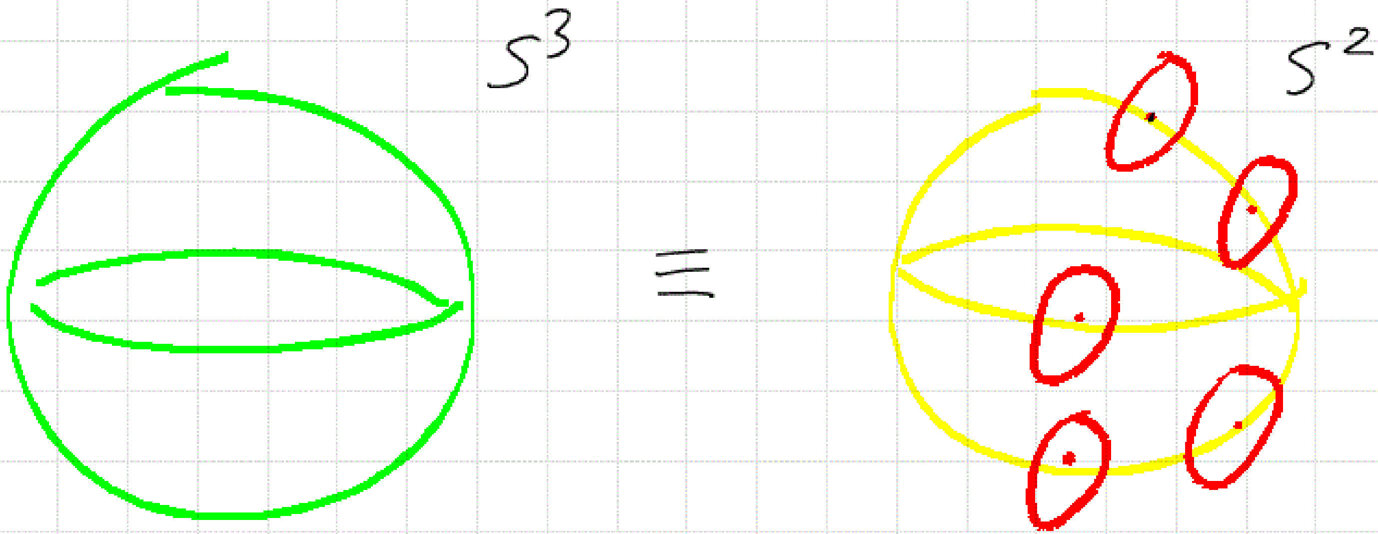
S^1 act as $\left\{ \begin{array}{l} x+iy \longmapsto (x+iy)e^{i\delta} \\ z+it \longmapsto (z+it)e^{i\delta} \end{array} \right\}$

which shifts $\psi \longrightarrow \psi + 2\delta$ with $\delta \in [0, 2\pi)$.

Then we choose $\psi = \phi$ slice

$$\Rightarrow \left(\begin{array}{l} x+iy = (\sin\theta)e^{i\phi} \\ z+it = (\cos\theta) \end{array} \right\} \Rightarrow (\theta, \phi) : \text{angular coordinates in } \mathbb{R}^3$$

$$\Rightarrow S^3/S^1 = \{ (x, y, z, 0) \mid x^2+y^2+z^2 = 1 \} = S^2$$


S^3
$\equiv$
S^2

Homotopy Groups of some Lie Group Manifold

① $SO(2) = U(1)$

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad 0 \le \theta < 2\pi \right\} \rightarrow \text{Circle } S^1$$

$$\pi_0 = 0$$

$$\pi_1 = \mathbb{Z}$$

$$\pi_2 = \pi_3 = \cdots = 0$$

(2) $SU(2) \sim S^3$

$\Downarrow$

Space of all unitary 2×2 matrices of determinant 1

$$U = \begin{pmatrix} x & z \\ y & w \end{pmatrix}, \quad \mathrm{Det}\, U = xw - zy = 1$$

$$UU^{+} = \begin{pmatrix} x & z \\ y & w \end{pmatrix}\begin{pmatrix} x^* & y^* \\ z^* & w^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \left\{ \begin{aligned} &|x|^2 + |z|^2 = 1 = |y|^2 + |w|^2 \\ &xy^* = -zw^* \\ &xw - yz = 1 \end{aligned} \right\}$$

Solution : $z = -y^*$, $w = x^*$, $|x|^2 + |y|^2 = 1$

$$\therefore \quad U = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \quad \text{with } \underbrace{|x|^2 + |y|^2}_{(\mathrm{Re}\,x)^2 + (\mathrm{Im}\,x)^2 + (\mathrm{Re}\,y)^2 + (\mathrm{Im}\,y)^2} = 1$$

$$\Rightarrow \quad SU(2) = \text{Unit } S^3 \text{ in } \mathbb{R}^4 = \mathbb{C}^2$$

Therefore $SU(2) = S^3$

$$\pi_0(SU(2)) = 0$$

$$\pi_1(SU(2)) = 0$$

$$\pi_2(SU(2)) = 0$$

$$\pi_3(SU(2)) = \mathbb{Z}$$

$$\pi_4(SU(2)) = \mathbb{Z}_2$$

$$\pi_5(SU(2)) = \mathbb{Z}_2$$

(3) $SO(3) = SU(2)/\mathbb{Z}_2 = S^3/\mathbb{Z}_2$

$\pi_0 = 0$

$\pi_1 = \mathbb{Z}_2$

$\pi_2 = 0$

$\pi_3 = \mathbb{Z}$

$\pi_4 = \mathbb{Z}_2$

$\pi_5 = \mathbb{Z}_2$

$$\begin{array}{ll}
\downarrow & \\
\pi_n(\mathbb{Z}_2) & = \mathbb{Z}_2 \text{ only for } n=0 \\
\downarrow & \\
\pi_n(S^3) & \\
\downarrow & \\
\pi_n(SO(3)) & \\
\downarrow & \\
\pi_{n-1}(\mathbb{Z}_2) & = \mathbb{Z}_2 \text{ only for } n=1 \\
\downarrow & \\
\pi_{n-1}(S^3) & \\
\downarrow &
\end{array}$$

equal except for $n=1$

Non-Abelian Generalization

Consider $\Phi = |\Phi| e^{i\theta}$. We may represent Φ as a traceless symmetric 2x2 matrix $\begin{pmatrix} \mathrm{Re}\Phi & \mathrm{Im}\Phi \\ \mathrm{Im}\Phi & -\mathrm{Re}\Phi \end{pmatrix}$

Then shift of $\theta \to \theta + 2\alpha$ is achieved by $SO(2)/\mathbb{Z}_2$

$$\underbrace{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}}_{O(\alpha)} \begin{pmatrix} \mathrm{Re}\Phi & \mathrm{Im}\Phi \\ \mathrm{Im}\Phi & -\mathrm{Re}\Phi \end{pmatrix} \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ +\sin\alpha & \cos\alpha \end{pmatrix}}_{O(\alpha)^T = O(\alpha)^{-1}} \qquad \alpha \in [0, \pi)$$

As a simplest generalization of this, take 3x3 matrix

$$\Phi = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{pmatrix} \qquad \left(\begin{array}{c} \Phi^T = \Phi \\ \mathrm{Tr}\,\Phi = 0 \end{array} \right)$$

- analog of $\Phi = |\Phi| e^{i\theta}$

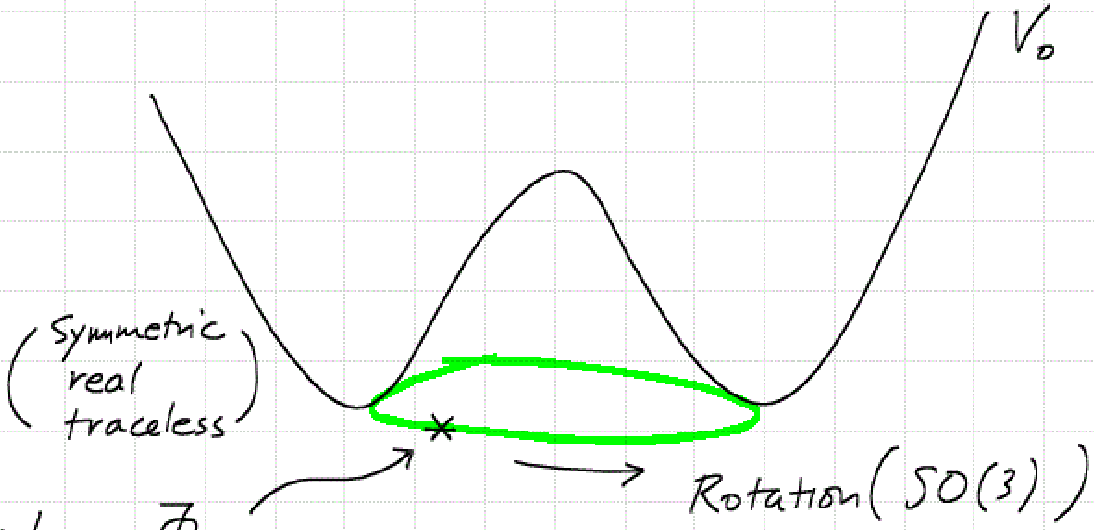
$$\Phi = O \Phi_0 O^T \quad \text{with} \quad O \in SO(3)$$

- analog of $V(|\Phi|)$ is an $SO(3)$-invariant potential

$$V_0(\Phi) = V_0(O \Phi O^T)$$

Without A_μ

$$\Rightarrow \quad L = \int d^3x \left(\frac{1}{2} \mathrm{Tr}\left(\partial_\mu \Phi \, \partial^\mu \Phi^T\right) - V_0(\Phi) \right)$$



Pick any value Φ_*

Diagonalize it as $\Phi_* = O_* \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -(\lambda_1+\lambda_2) \end{pmatrix} O_*^T$

Then, the vacuum manifold $\mathcal{M}_*$ is

$$\mathcal{M}_* = \left\{ O \Phi_* O^T = O O_* \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -(\lambda_1+\lambda_2) \end{pmatrix} (O O_*)^T \;\middle|\; O \in SO(3) \right\}$$

$$= \left\{ O \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & -(\lambda_1+\lambda_2) \end{pmatrix} O^T \;\middle|\; O \in SO(3) \right\}$$

Take $\lambda_1 = \lambda_2 = \lambda$ for simplicity $\quad (\Leftarrow V_0 = -a \operatorname{Tr} \Phi^2 + b \operatorname{Tr} \Phi^4, \; a>0, \; b>0)$

$$\Rightarrow \boxed{\mathcal{M}_* = \left\{ O \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix} O^T \;\middle|\; O \in SO(3) \right\}}$$

Among O's in $SO(3)$,

$$H = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\equiv R} \underbrace{\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boxed{e^{i\theta Q},\ Q = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}} \ \middle|\ \theta \in [0, 2\pi) \right\}$$

Leave $\lambda \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$ invariant

$$\Rightarrow \quad \mathcal{M}_* = SO(3)/H \quad \text{topologically}$$

$$= \left(S^3/\tilde{\mathbb{Z}}_2\right)/H \qquad \tilde{\mathbb{Z}}_2 : \text{rotation by } 2\pi \in H$$

$$= S^3/H = \left(S^3/S^1\right)/\mathbb{Z}_2 = S^2/\mathbb{Z}_2$$

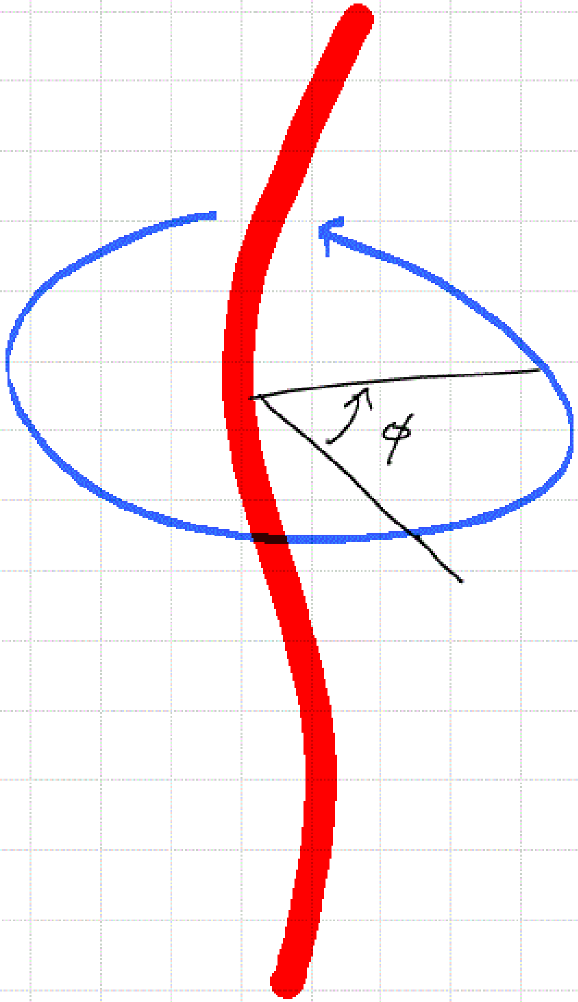
$$\longrightarrow \underset{\substack{\| \\ 0}}{\pi_1(S^2)} \longrightarrow \pi_1\left(S^2/\mathbb{Z}_2\right) \longrightarrow \pi_0(\mathbb{Z}_2) \longrightarrow \underset{\substack{\| \\ 0}}{\pi_0(S^2)} \longrightarrow$$

identical

$$\therefore \quad \pi_1(\mathcal{M}_*) = \mathbb{Z}_2$$

cf) $\pi_1(S^1) = \mathbb{Z}$ for magnetic vortex case

$\Longrightarrow$ We expect vortex string with "$\mathbb{Z}_2$" winding #



$$\Phi = O(\phi) \begin{pmatrix} x & & \\ & x & \\ & & -2x \end{pmatrix} O(\phi)^T$$

with

$$O(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi/2 & \sin\phi/2 \\ 0 & -\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$O(0) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad O(2\pi) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = R$$

($\sim$ Analog of $\Phi = |\Phi| e^{i\phi}$)

But $\partial_\mu \Phi = (\partial_\mu O)\begin{pmatrix}\lambda & & \\ & \lambda & \\ & & -2\lambda\end{pmatrix} O^T + O\begin{pmatrix}\lambda & & \\ & \lambda & \\ & & -2\lambda\end{pmatrix}\partial_\mu O^T$

$$= \left[(\partial_\mu O)O^T\right]\Phi + \Phi\left[O\partial_\mu O^T\right]$$

$$= -(O\partial_\mu O^T)\Phi + \Phi(O\partial_\mu O^T)$$

$$\left(\because\ 0 = \partial_\mu(\underbrace{OO^T}_{\mathbb{1}_{3\times 3}}) = (\partial_\mu O)O^T + O(\partial_\mu O^T)\right)$$

$$\int (\partial_\mu \Phi)(\partial^\mu \Phi)\, d^3x \longrightarrow \infty$$

Thus, as before, configuration with winding of Φ is infinitely massive. One must introduce gauge field to proceed!

Analog of $\boxed{D_\mu \Phi = (\partial_\mu - iA_\mu)\Phi}$ is

$$\boxed{D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi - \Phi A_\mu \quad \text{with } \underbrace{A_\mu = -A_\mu^T}_{\text{real antisymmetric}}}$$

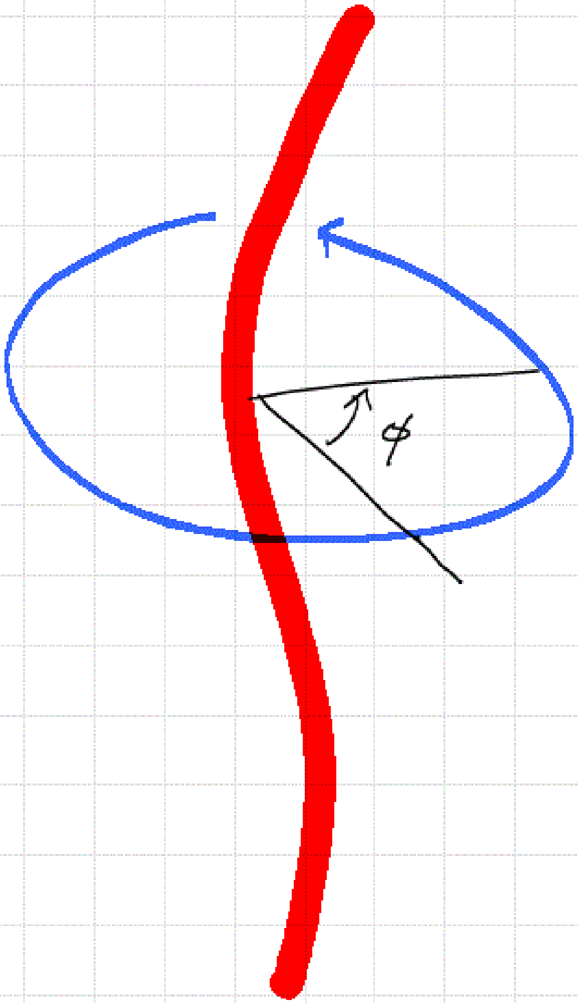
$\Rightarrow$ We may then compensate by $A_\mu \sim O \partial_\mu O^T$ asymptotically

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$$

$$-\cancel{(O\partial_\mu O^T)}\Phi + \Phi\cancel{(O\partial_\mu O^T)} \qquad \cancel{O\partial_\mu O^T}\Phi - \Phi \cancel{O\partial_\mu O^T}$$

$$D_\mu \Phi \longrightarrow 0 \quad \text{as} \quad r \to \infty$$

Effect of Asymptotic Winding Does Not Contribute ∞ Energy



$$\boxed{\begin{aligned} \Phi(r\to\infty) &= O(\phi)\begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix} O(\phi)^T \\ A_i(r\to\infty) &= O(\phi)\,\partial_i\, O(\phi)^T \end{aligned}}$$

with

$$O(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi/2 & \sin\phi/2 \\ 0 & -\sin\phi/2 & \cos\phi/2 \end{pmatrix}$$

$$O(0) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad O(2\pi) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = R$$

Lagrangian including the gauge field A_μ (3×3 antisymmetric) is

$$L = \int d^3x \left(Tr\left(\frac{1}{2} D_\mu \Phi D_\mu \Phi^T\right) - V_0(\Phi) - \frac{1}{4g^2} Tr F_{\mu\nu} F^{\mu\nu} \right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$$

For $A_\mu = O\partial_\mu O^T$

$$F_{\mu\nu} = \underline{\partial_\mu (O\partial_\nu O^T)} - \underline{\partial_\nu (O\partial_\mu O^T)}$$

$$+ \underline{(O\partial_\mu O^T)(O\partial_\nu O^T)} - \underline{(O\partial_\nu O^T)(O\partial_\mu O^T)}$$

$$= \underline{\partial_\mu O \cancel{\partial_\nu O^T}} - \underline{\partial_\nu O \cancel{\partial_\mu O^T}}$$

$$- \cancel{\underline{\partial_\mu O \partial_\nu O^T}} + \cancel{\underline{\partial_\nu O \partial_\mu O^T}} = 0$$

$\Rightarrow$ No Energy Cost in $F_{\mu\nu}$ from Winding Either

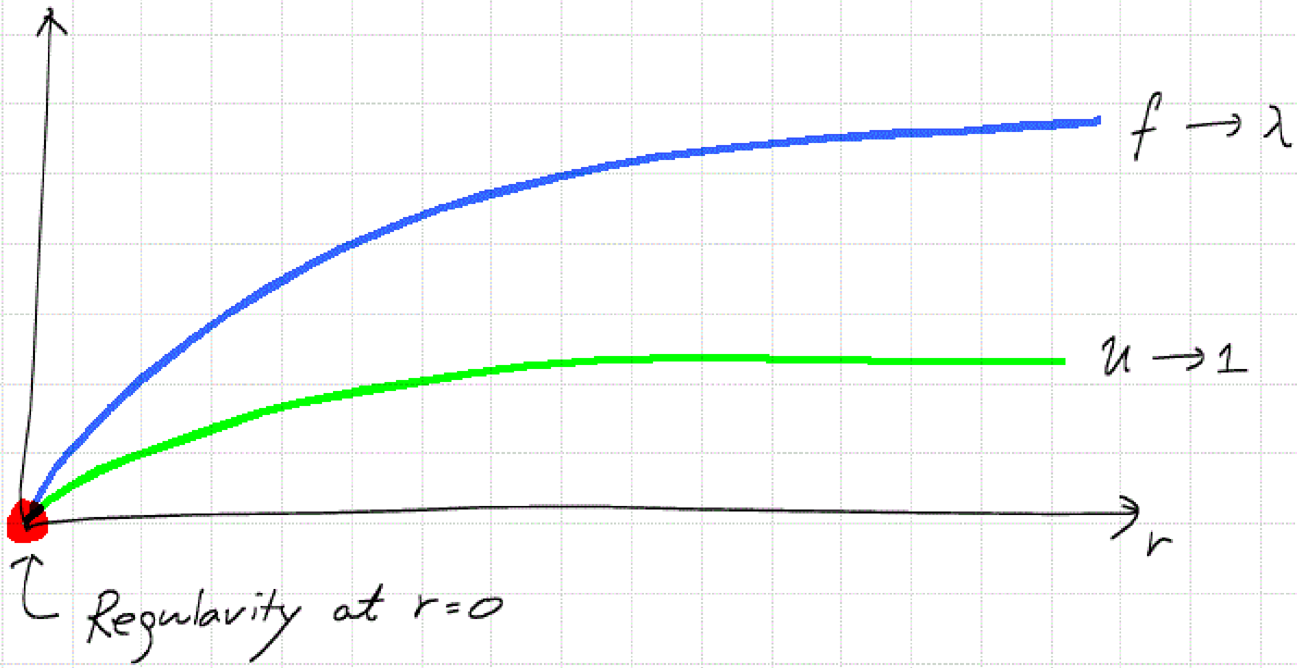
More Generally

$$\left\{\begin{aligned}
\Phi &\longmapsto O(x)\,\Phi\, O(x)^T \\
A_\mu &\longmapsto O(x) A_\mu O(x)^T + O(x)\partial_\mu O^T \\
&\quad = O(x)\left(A_\mu - O(x)^T\partial_\mu O(x)\right) O(x)^T \\
\Big(F_{\mu\nu} &\longmapsto O(x) F_{\mu\nu} O(x)^T \Big)
\end{aligned}\right\}$$

Leaves L invariant

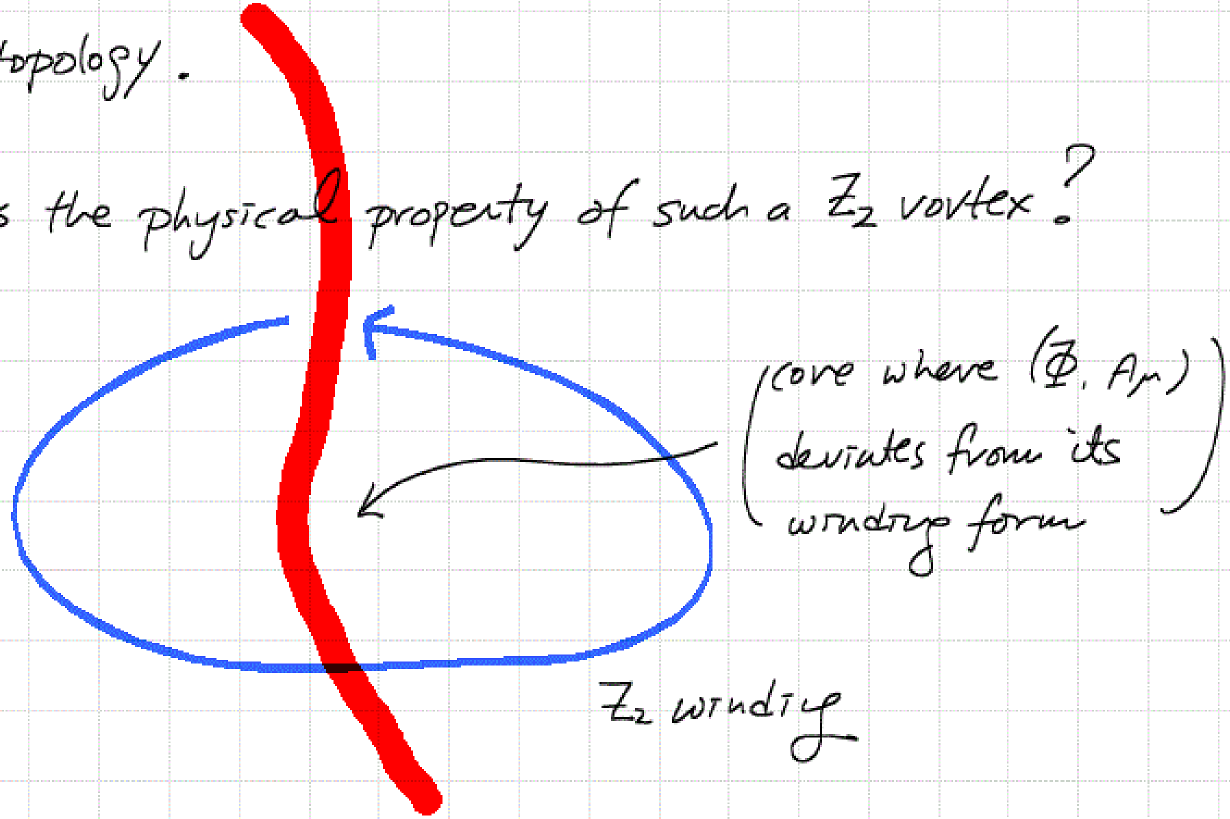
Ansatz $A_i = u(r)\, O(\phi)\, \partial_i\, O(\phi)^T$

$$\Phi = f(r)\, O(\phi) \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} O(\phi)$$



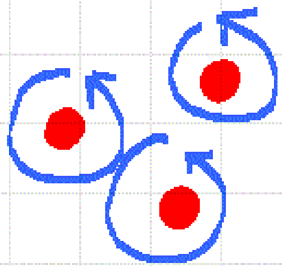
We will leave it to mathematician to solve for actual solution, whose "existence" is guaranteed by the topology.

What is the physical property of such a Z_2 vortex?

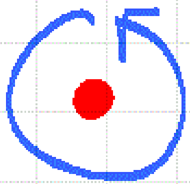


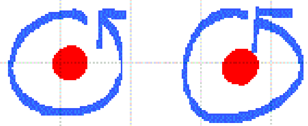
① Since it belongs to $\pi_1(\mathcal{M}_*) = Z_2$,

having a pair of them negates topology



$\approx$ topologically

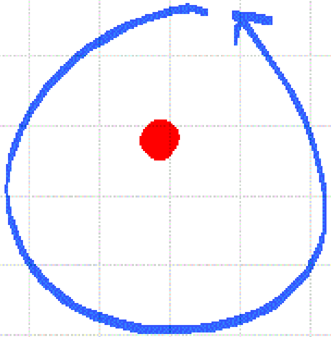




$\approx$ No Winding

$\longrightarrow$ Property of "Z_2"

② Specific to the example here



Aharonov-Bohm "phase"

$$\sim \underbrace{\text{"}e^{\oint A}\text{"}}_{\text{must be ordered correctly}} = O(2\pi)\,O(0)^{-1} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

Rotation by π in y-z plane

$$R \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{=\; iQ} R^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{=\; -iQ}$$

What is special about Q?

Consider $A_\mu = \cdots + i a_\mu O(\phi) Q\, O(\phi)^T + \cdots$

Since $Q \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix} = \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix} Q$

$$\mathrm{Tr}\,(D_\mu \Phi)^2 = \cdots + a_\mu a^\mu \underbrace{\mathrm{Tr}\left([Q, \Phi]\right)^2}_{=\,0} + \cdots$$

With $A_\mu = O(\phi)\begin{pmatrix} 0 & a_\mu & -b_\mu \\ -a_\mu & 0 & c_\mu \\ b_\mu & -c_\mu & 0 \end{pmatrix} O(\phi)^T$

$$\mathrm{Tr}\,(D_\mu \Phi)^2 = b_\mu b^\mu \,\mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix}\right]^2$$

$$+ c_\mu c^\mu \,\mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & -2\lambda \end{pmatrix}\right]^2$$

$$= (18\lambda^2) \times (b_\mu b^\mu + c_\mu c^\mu)$$

Of 3 vector fields, only a_μ remains massless and behave like ordinary Maxwell field

We say $SO(3)$ is spontaneously broken to $H = \{ e^{i\theta Q} , R e^{i\theta Q} \}$

$$= SO(2) \oplus R \cdot SO(2)$$

Also Consider $\Phi = \cdots + O(\phi) \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ \xi & \eta & 0 \end{pmatrix} O(\phi)^T + \cdots$

$$e^{i\theta Q} \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ \xi & \eta & 0 \end{pmatrix} (e^{i\theta Q})^T = \begin{pmatrix} 0 & 0 & \xi' \\ 0 & 0 & \eta' \\ \xi' & \eta' & 0 \end{pmatrix}$$

with $\xi' + i\eta' = (\xi + i\eta) e^{i\theta}$

$\Rightarrow$ $(\xi + i\eta)$ part of Φ is like

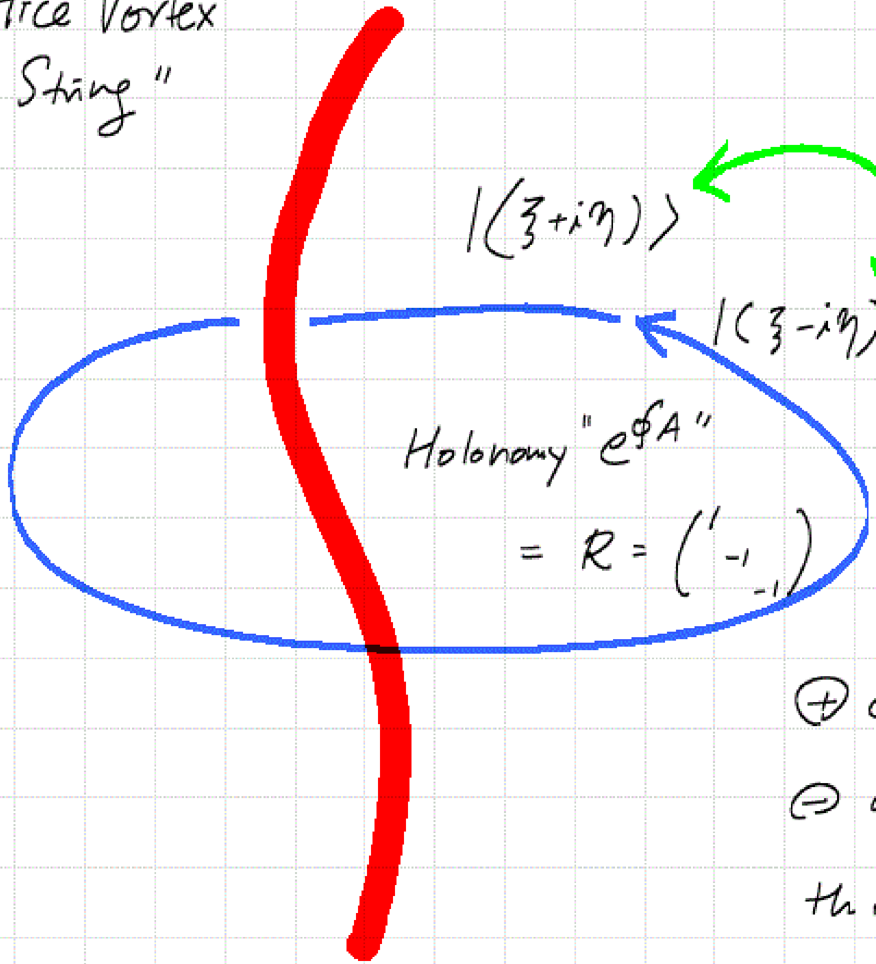
electrically charged particle (field) under Q

On the other hand, going around a $\mathbb{Z}_2$ vortex

$$\begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ \xi & \eta & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & \eta \\ \xi & \eta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & -\eta \\ \xi & -\eta & 0 \end{pmatrix}$$

"Alice Vortex String"



$|(\xi+i\eta)\rangle$

$|(\xi-i\eta)\rangle$

Holonomy "$e^{\oint A}$"

$= R = \begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix}$

complex conjugate

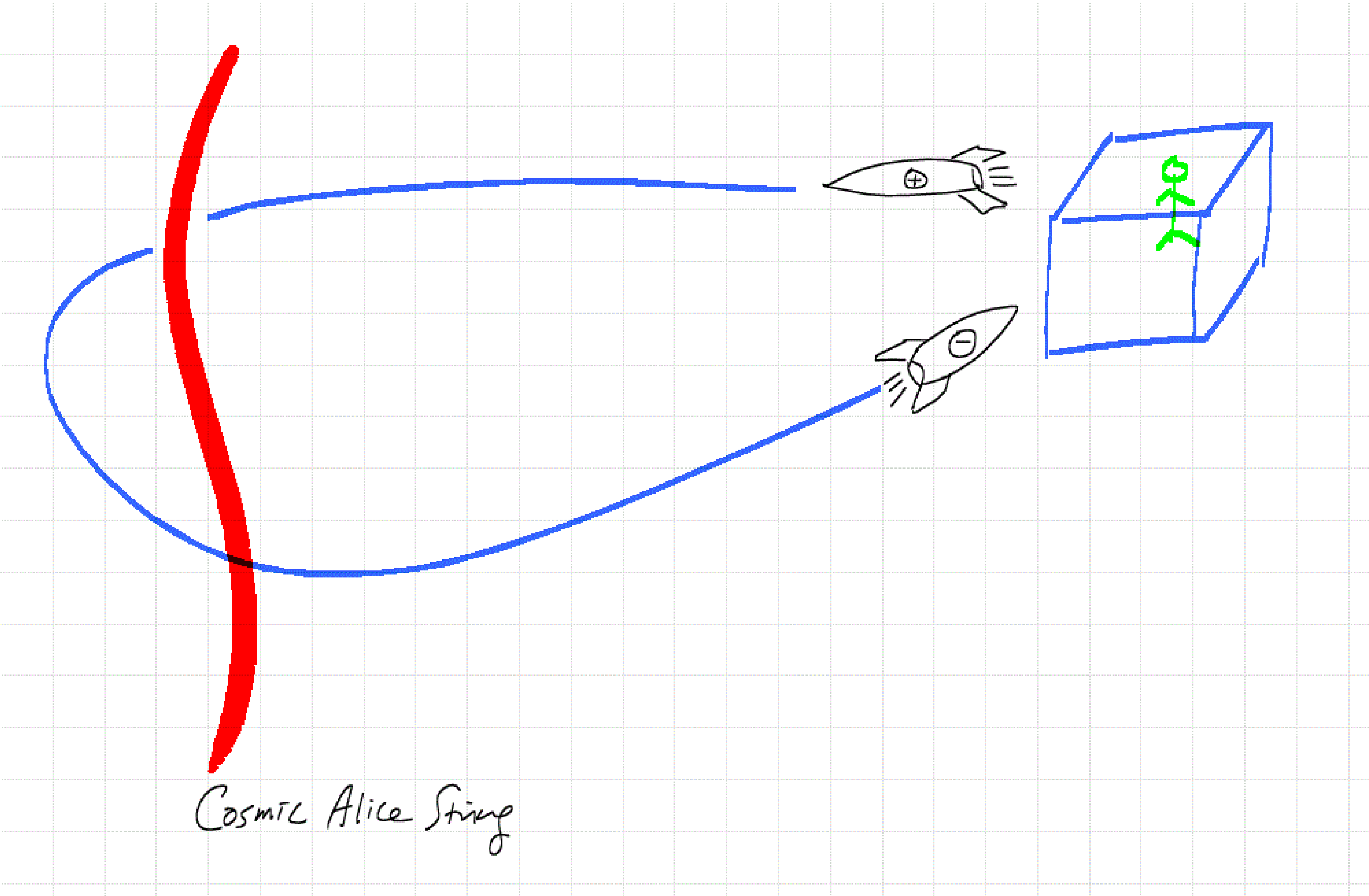
~ charge conjugate

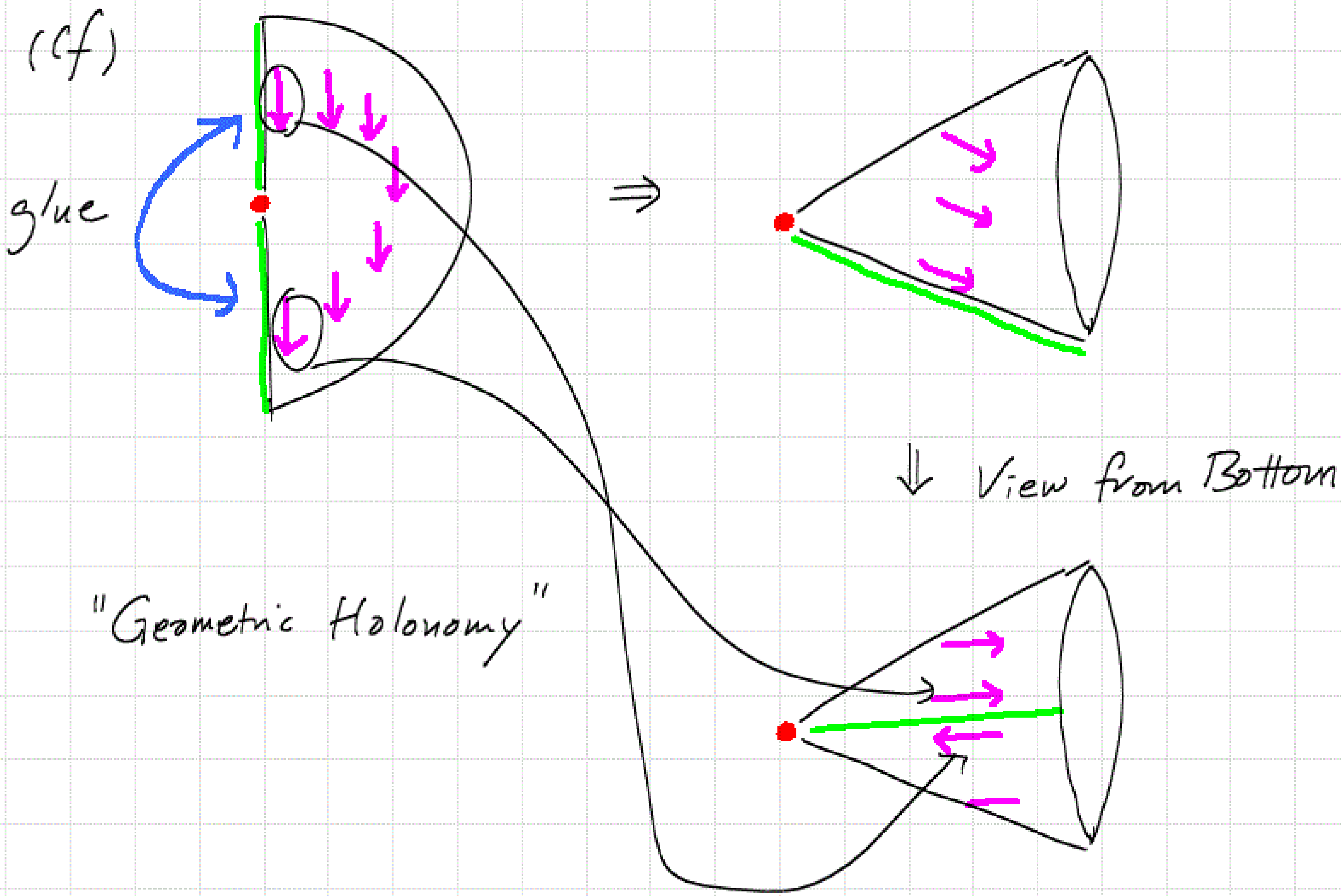
Because $RQR^T = -Q$

⊕ charge transforms to

⊖ charge by moving around

this $\mathbb{Z}_2$ vortex


Cosmic Alice String


(cf)
glue
⇒
⇓ View from Bottom
"Geometric Holonomy"

Magnetic Monopole

$$\longrightarrow \underset{\substack{\| \\ 0}}{\pi_2(\mathbb{Z}_2)} \longrightarrow \overset{\substack{\mathbb{Z} \\ \|}}{\pi_2(S^2)} \longrightarrow \pi_2(M_* = S^2/\mathbb{Z}_2) \longrightarrow \underset{\substack{\| \\ 0}}{\pi_1(\mathbb{Z}_2)} \longrightarrow$$

Identical

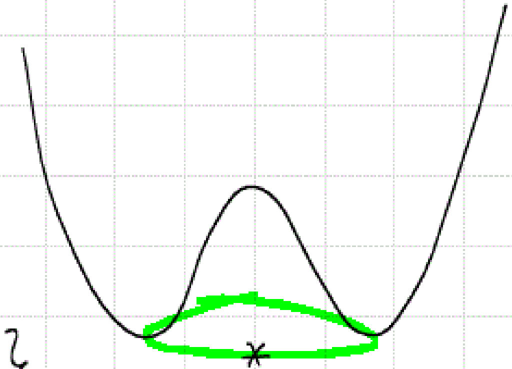
$\pi_2(M_*) = \mathbb{Z} \Rightarrow$ Possibility of New Winding #

$\Rightarrow \exists$ Magnetic Monopole

A simpler magnetic monopole arises if we replace symmetric traceless Φ by antisymmetric Φ

$$\Phi_* = O_* \underbrace{\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} O_*^T$$

$V(\Phi) = V(O\Phi O^T)$



Left invariant under

$$H = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \theta \in [0, 2\pi] \right\}$$

$$\therefore\ M_* = \left\{ O\,\Phi_*\,O^T \;\middle|\; O \in SO(3) \right\}$$

$$= SO(3)/SO(2) = \left(S^3/\underbrace{\mathbb{Z}_2}_{\text{part of } SO(2)}\right)/SO(2)$$

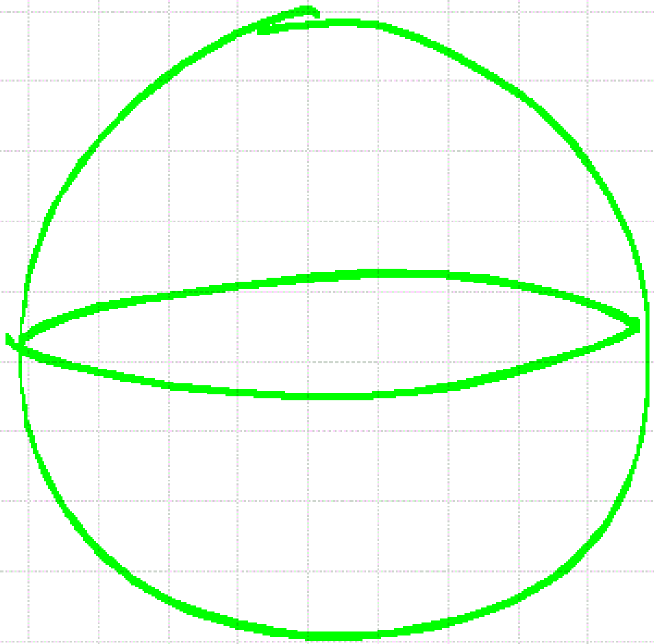
$$= S^3/SO(2) = S^3/S^1 = S^2 \quad \Rightarrow \quad \pi_2(M_*) = \mathbb{Z}$$

Alternatively

$$\longrightarrow \underset{\substack{\| \\ 0}}{\pi_2(S^3)} \longrightarrow \pi_2\left(M_* = S^3/S^1\right) \longrightarrow \underset{\substack{\| \\ \mathbb{Z}}}{\pi_1(S^1)} \longrightarrow \underset{\substack{\| \\ 0}}{\pi_1(S^3)} \longrightarrow$$

identical

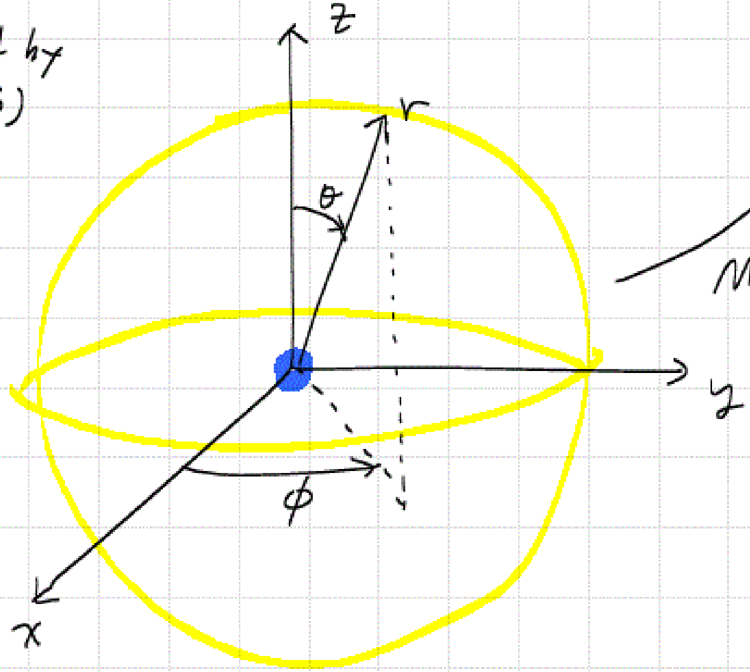
$M_* = S^2$

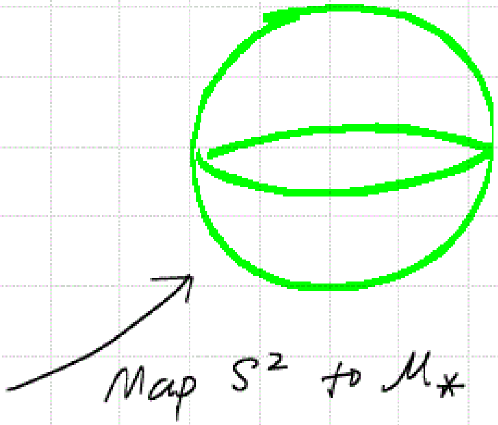


N.B. (Simpler than symmetric traceless $\otimes$
because M_* is simpler)

How does $\pi_2(M_*) = \pi_2(S^3/S^1) = \pi_1(SO(2))$ winding look like?

Must map $\underbrace{\boxed{S^2}}_{\text{Spanned by } (\theta,\phi)}$ to $M_* = \boxed{S^2}$





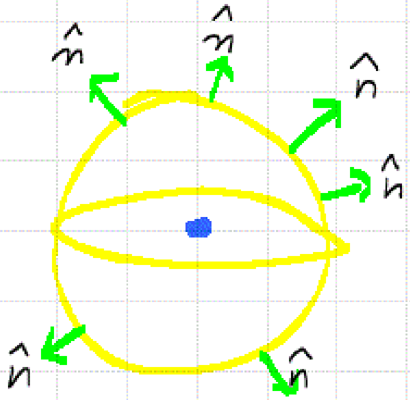
Unit Winding Configuration: "Hedge Hog"

$$\Phi = \tilde{O}(\theta,\phi) \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tilde{O}(\theta,\phi)^T$$

(rotation of $\hat{z}$ to $\hat{n}$)

$$= \lambda \begin{pmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{pmatrix}$$

$$n_z = \cos\theta$$
$$n_x = \sin\theta \cos\phi$$
$$n_y = \sin\theta \sin\phi$$

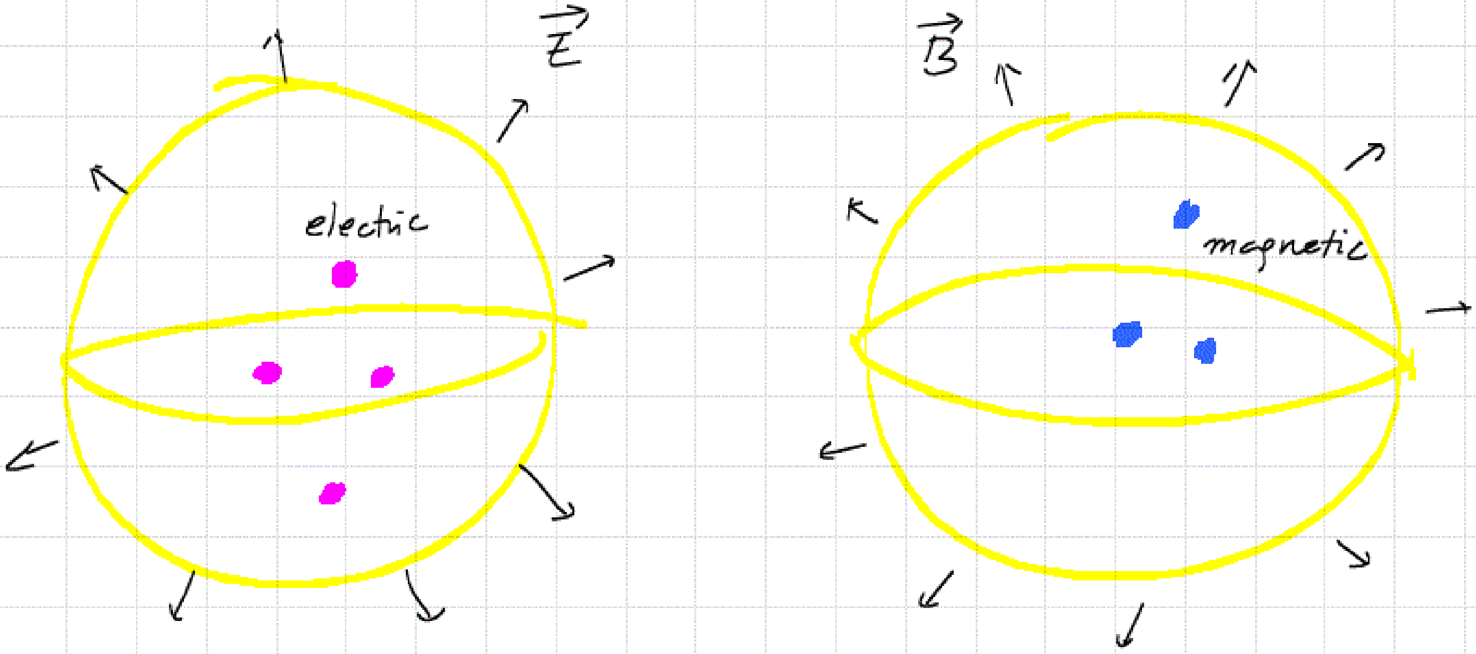


$\hat{n} = (n_x, n_y, n_z)$: unit vector along radial direction

Why is a magnetic monopole?

What is a magnetic monopole?

$$\begin{aligned}
\nabla \cdot E &= \rho_e \\
\nabla \cdot B &= 0 \\
\vec{\nabla} \times \vec{E} &= \frac{\partial}{\partial t} \vec{B} \\
\vec{\nabla} \times \vec{B} &= -\frac{\partial}{\partial t} \vec{E} + \vec{J}_e
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
\nabla \cdot E &= \rho_e \\
\nabla \cdot B &= \underline{\rho_m} \\
\vec{\nabla} \times \vec{E} &= \frac{\partial}{\partial t} \vec{B} + \underline{\vec{J}_m} \\
\vec{\nabla} \times \vec{B} &= -\frac{\partial}{\partial t} \vec{E} + \vec{J}_e
\end{aligned}$$



$$\oint \hat{n}\cdot\vec{E} = Q_e \qquad\qquad \int \hat{n}\cdot\vec{B} = Q_m$$

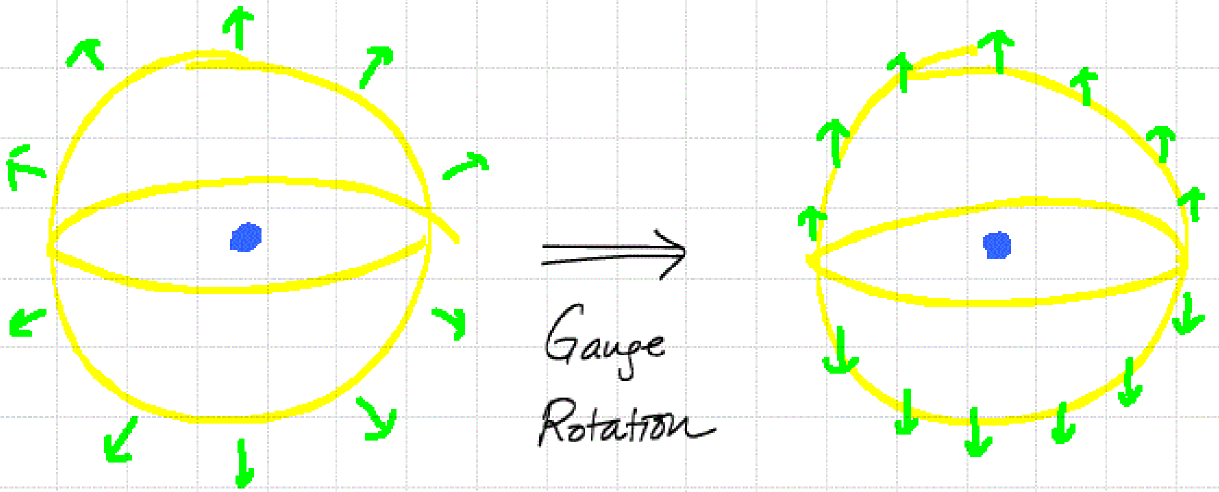
$\pi_2(\mathcal{M}_*)$ winding # $\longrightarrow$ Magnetic Charge?

(Φ, A_μ)

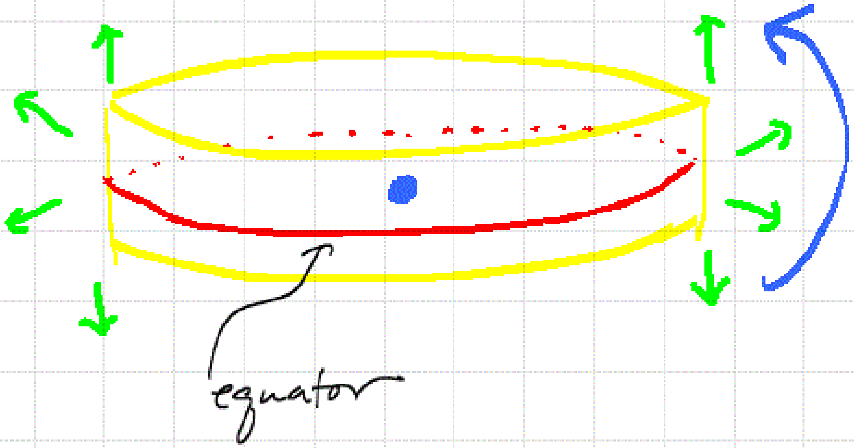
① Use gauge-invariance to "unwind" Φ

$$\Phi = \tilde{O}(\theta,\varphi)\begin{pmatrix}0&\lambda&0\\-\lambda&0&0\\0&0&0\end{pmatrix}\tilde{O}(\theta,\varphi)^T$$

$$\Phi = \begin{cases}\begin{pmatrix}0&\lambda&0\\-\lambda&0&0\\0&0&0\end{pmatrix} & \text{for } 0\leq\theta<\pi/2\\[2ex] \begin{pmatrix}0&-\lambda&0\\\lambda&0&0\\0&0&0\end{pmatrix} & \text{for } \pi/2<\theta\leq\pi\end{cases}$$



π_2 winding
is compressed at equator



$$\mathcal{O}_\epsilon(\theta,\phi) \equiv \tilde{\mathcal{O}}\left(\frac{\theta - \pi/2}{\epsilon} + \pi/2 \,,\, \phi \right)$$

$$\pi/2(1-\epsilon) \le \theta \le \pi/2(1+\epsilon)$$

Φ changes from $\begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

to $\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

via rotation of $180°$

along an axis $\perp \hat{z}$

Except at Equator, Unbroken $SO(2)$ is associated with

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Upper Hemisphere

$$\Rightarrow \text{Analog of Maxwell Field is } A_\mu = \begin{cases} \cdots + a_\mu^U \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots \\ \text{or} \\ \cdots + a_\mu^L \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \cdots \end{cases}$$

Lower Hemisphere

Gauge Transformation Between Upper & Lower Hemisphere

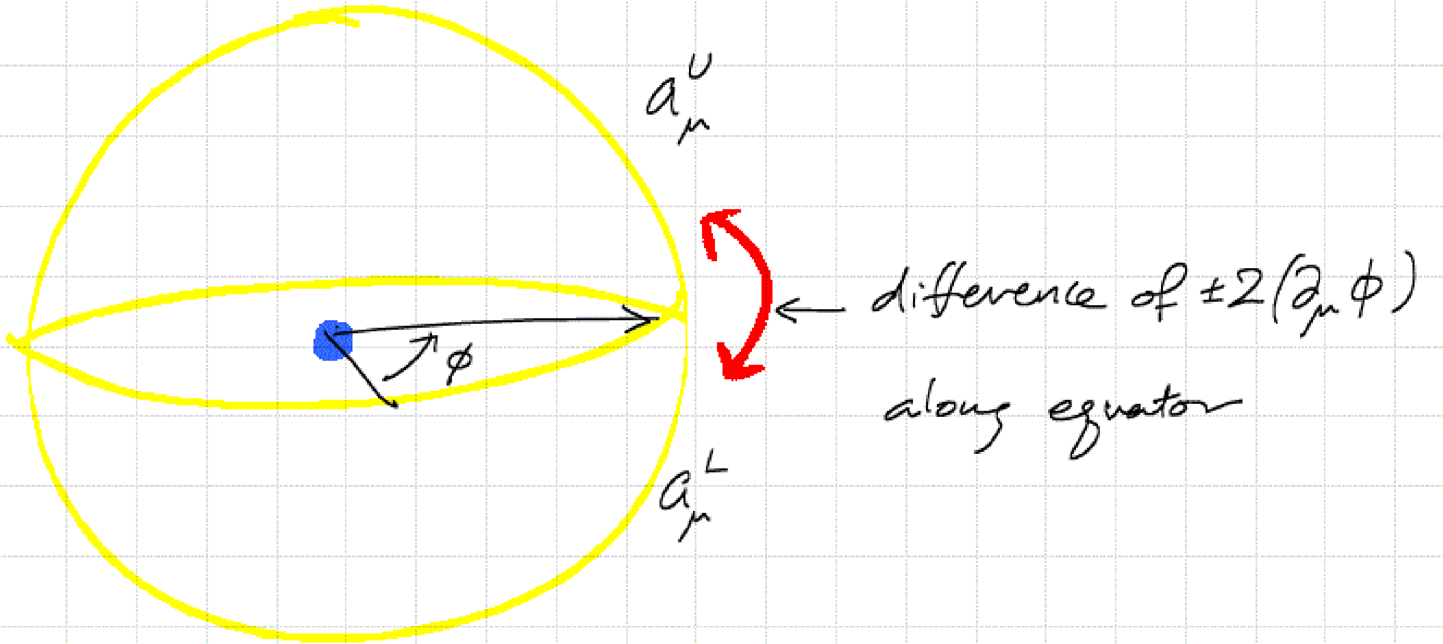
$$A_\mu^{\text{Upper}} = O_\epsilon A_\mu^{\text{lower}} O_\epsilon^T + O_\epsilon \partial_\mu O_\epsilon^T$$

Extracting "massless" pieces

$$a_\mu^U = a_\mu^L \pm 2\, O_z(\phi)\, \partial_\mu O_z(\phi)^T$$

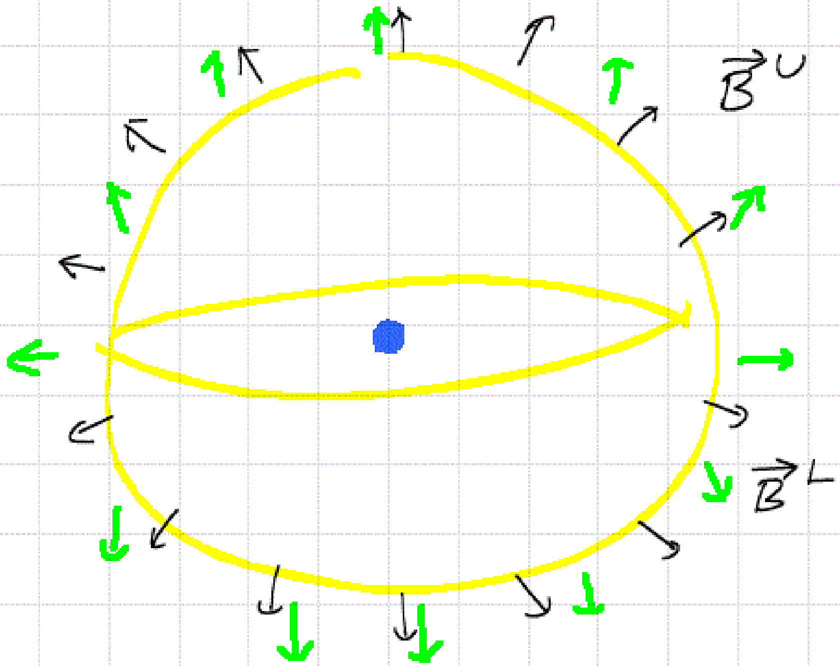
with $O_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Ordinary Maxwell Field
in Upper (Lower) Hemisphere



$$\oint_{\text{Upper}} \vec{B}_U \cdot \hat{n} + \oint_{\text{Lower}} \vec{B}_L \cdot \hat{n} = \oint_{\text{Equator}} \vec{a}^U \cdot d\vec{\ell} - \oint_{\text{Equator}} \vec{a}^L \cdot d\vec{\ell} = \pm 2 \oint d\phi$$

$$= 4\pi$$

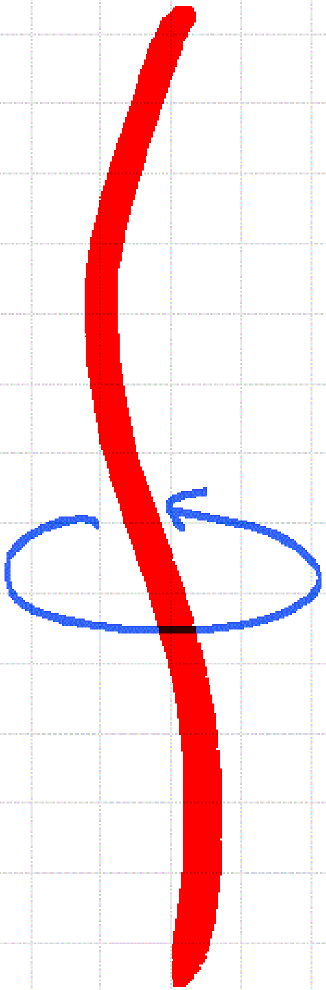


$$\oint_{S^2} \vec{B}\cdot\vec{n} = 4\pi$$

Unit Winding in $\pi_2(\mathcal{M}_* = SO(3)/SO(2))$

$\Longrightarrow$ Magnetically Charged Object !!!

Vortex Strings



π_1

Solitonic String

Gauge Field

Holonomy
(A-B phase)

Homotopy	Lump	Diff'l Topology (~ Cohomology)
π_0	Domain Wall	?
$\pi_1(G/H)$	Vortex String	Quantized Holonomy
$\pi_2(G/H)$	Monopole	$\oint \vec{B}\cdot\hat{n}$
$\pi_3(G)$	Instanton	$\int \epsilon_{\mu\nu\alpha\beta} F^{\mu\nu} F^{\alpha\beta}$
$\vdots$	$\vdots$	$\vdots$

More for Ambitions : Differential Topology