Goldstone Bosons in Condensed Matter System

Outline of lectures:

1. Examples (Spin waves, Phonon, Superconductor)
2. **Meissner effect, Gauge invariance and Goldstone boson in SC**
3. Higgs mode (boson) in Superconductors

“**Goldstone bosons:** continuous symmetry breaking always induce massless mode (boson) ”

Goldstone bosons in Magnets (Spin-wave) are easy to Understand and visualize it.

Goldstone boson in SC is subtle, not even detectable, but plays a crucial role to make SC in reality.

Meissner effect

$$J = \sigma E$$

$$J = K A$$

Gauge invariance problem of Meissner effect

* $\mathbf{J} = \mathbf{K}\,\mathbf{A}$ is "not gauge invariant" → "not physical"

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{A}' = \mathbf{A} + \nabla\chi \quad \rightarrow \quad \mathbf{B}' = \mathbf{B}$$

$$\Phi' = \Phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \quad \rightarrow \quad \mathbf{E}' = \mathbf{E}$$

$\mathbf{J} = \mathbf{K}\,\mathbf{A}$ → $\mathbf{J}' = \mathbf{K}\,\mathbf{A} + \mathbf{K}\,\nabla\chi$ This is Non-sense !!

Cure: $\mathbf{J}' = \mathbf{K}_1\,\mathbf{A} + \mathbf{K}_2\nabla\chi$ and $\mathbf{K}_2 = 0$

57' BCS , 58' Anderson, 60' Nambu, 61' Higgs, …

Meissner effect

J = K A "**London Eq.** ('35)" automatically provide the Meissner effect
(Perfect diamagnetism)

J = σ E = - σ ∂A/∂t : Ohm's law for normal metals

Combine with Maxwell eq. $\nabla \times B = J + \frac{\partial E}{\partial t}; \quad \nabla \times E = -\frac{\partial B}{\partial t}$

$$\nabla \times \nabla \times B = \nabla \times J + \nabla \times \frac{\partial E}{\partial t}$$

$$-\nabla^2 B = KB - \frac{\partial^2 B}{\partial^2 t} \qquad \leftarrow B \sim e^{-i\omega t + ikx}$$

For SC, $-k^2 = K - \omega^2$ with $J = KA$
when $\omega \to 0, \quad B \sim e^{-\sqrt{K}x}$

(1) Exponentially decaying EM wave

For Ohm's law, with $J = \sigma E$, $k^2 = \omega^2 - i\sigma\omega$
$B \sim e^{(1+i)\sigma\omega}$

(2) Usual skin depth with ω→ 0 limit

BCS theory in a nutshell

$$H' = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{l}} V_{\mathbf{k}\mathbf{l}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}.$$

This Hamiltonian is unstable when V <0 interaction.

Assume non-zero of $< c_{k\uparrow} c_{-k\downarrow} >= \Delta$ and $< c^\dagger_{k\downarrow} c^\dagger_{-k\uparrow} >= \Delta^*$

$$H = \sum_{k\sigma} \epsilon(k) c^\dagger_{k\sigma} ck\sigma + \sum_k [c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \Delta + c_{-k\downarrow} c_{k\uparrow} \Delta^*]$$

$$H = (c^\dagger_{k,\uparrow} c_{-k,\downarrow}) \begin{pmatrix} \epsilon(k) & \Delta \\ \Delta & -\epsilon(k) \end{pmatrix} \begin{pmatrix} c_{k,\uparrow} \\ c^\dagger_{-k,\downarrow} \end{pmatrix}$$

Doubling in p-h space

$$H = \sum_k \xi(k) (\gamma^\dagger_{k1} \gamma_{k1} - \gamma^\dagger_{-k0} \gamma_{-k0})$$

$$\xi^2(k) = [\epsilon^2(k) + \Delta^2] \text{ , so } \xi(k) = \pm\sqrt{\epsilon^2(k) + \Delta^2}$$

$$c_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1}$$
$$c^\dagger_{\mathbf{k}\downarrow} = -v^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + u_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1}$$

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) |0\rangle .$$

SC ground sate

$$\gamma^\dagger_{\mathbf{k}_1\sigma_1} \gamma^\dagger_{\mathbf{k}_2\sigma_2} \cdots \gamma^\dagger_{\mathbf{k}_n\sigma_n} |\mathrm{BCS}\rangle .$$

All excited states has a minimum energy gap Δ.

$$J_{n,s} = n_{n,s} v_{n,s} = n_{n,s}\frac{p_{n,s}}{m} = n_{n,s}\frac{i\hbar\nabla}{m}$$

With an external force/fields,

$$< J_{n,s}(\vec{A},\phi) >= n_{n,s}\frac{<[i\hbar\nabla - \frac{e}{c}\vec{A}]>}{m}$$

$$H = \frac{[i\hbar\nabla - \frac{e}{c}\vec{A}]^2}{2m} + \ldots$$

$$H_I = \int d\mathbf{r}\,\psi^*(\mathbf{r})\left[\frac{-ie\hbar}{2mc}(\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A} \quad + \frac{e^2}{2mc^2}\mathbf{A}^2(\mathbf{r})\right]\psi(\mathbf{r}). \quad = J\cdot A$$

$$\mathfrak{J}_P(\mathbf{r}) = \frac{e\hbar}{2m\Omega}\sum_{\mathbf{k},\mathbf{q},\sigma} c_{\mathbf{k}+\mathbf{q},\sigma}{}^* c_{\mathbf{k},\sigma} e^{-i\mathbf{q}\cdot\mathbf{r}}(2\mathbf{k}+\mathbf{q}),$$

$$\mathfrak{J}_D(\mathbf{r}) = -\frac{e^2}{mc}\frac{1}{\Omega}\sum_{\mathbf{k},\mathbf{q},\sigma} c_{\mathbf{k}+\mathbf{q},\sigma}{}^* c_{\mathbf{k},\sigma} e^{-i\mathbf{q}\cdot\mathbf{r}}\mathbf{A}(\mathbf{r}).$$

$$J_{tot} = J_D + J_P = -\frac{e^2}{mc} n_{n,s}\vec{A} + J_P(A)$$

$$J_i(q) = \sum_{j=1}^{3} K_{ij}(q) A_j(q),$$

(J = K A)

with

$$K_{ij}(q) = -\frac{e^2}{m}\langle 0|\rho|0\rangle\delta_{ij} + \sum_n\left(\frac{\langle 0|j_i(q)|n\rangle\langle n|j_j(-q)|0\rangle}{E_n} + \frac{\langle 0|j_j(-q)|n\rangle\langle n|j_i(q)|0\rangle}{E_n}\right).$$

The 2nd term cancels 1st term in normal metal
But, in SC the 2nd term =0, hence J = K A

What was wrong ?

$$< J_s(\vec{A},\phi) >= \frac{en_s}{m} < [i\hbar\nabla - \frac{e}{c}\vec{A}] > \Rightarrow \frac{n_s}{m} < -\frac{e^2}{c}\vec{A} >$$

BCS theory always gives $\frac{en_s}{m} < [i\hbar\nabla] >_{BCS} = 0$

$J' = K\,(A + \nabla\chi) = K_1 A + K_2 \nabla\chi$ and $K_2 = 0$

$= K_{\text{gauge indep}} \bullet A + K_{\text{gauge dep}} \bullet \nabla\chi$ and $K_{\text{gauge dep}} = 0$

How to extract **"gauge dependent part"** out of **K ?**

Using **$\nabla\chi \rightarrow q\,\chi(q)$** in mom space.

So we need to satisfy $K_{ij}(q)\cdot q_j = 0$

if $K_{ij} = K_{ji}$, this is the same as $q_i \cdot K_{ij}(q) = 0$

, which is nothing but the charge conservation law as $q_i J_i = 0$.

$$J_i(q) = \sum_{j=1}^{3} K_{ij}(q) A_j(q),$$

Answer:

$$K_{ij} = K\left[1 - \frac{q_i q_j}{q^2}\right] = K_{ij}^{trans}$$

with

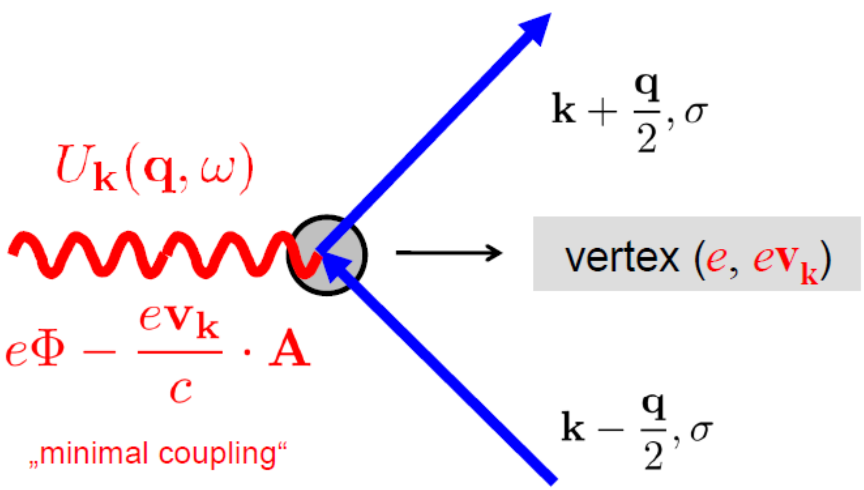
$$K_{ij}(q) = -\frac{e^2}{m}\langle 0|\rho|0\rangle\delta_{ij} + \sum_n \left(\frac{\langle 0|j_i(q)|n\rangle\langle n|j_j(-q)|0\rangle}{E_n} + \frac{\langle 0|j_j(-q)|n\rangle\langle n|j_i(q)|0\rangle}{E_n} \right).$$

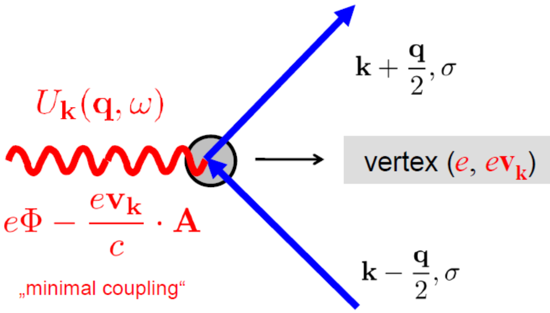
K• q =0 → (1st term) • q + (2nd term) • q =0

While <(2nd term)>$_{BCS}$ =0

What was wrong in BCS theory or BCS calculation ?

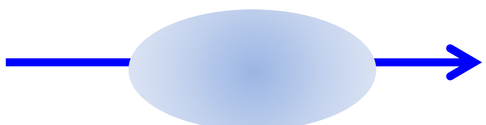
Vertex correction and Ward identity:





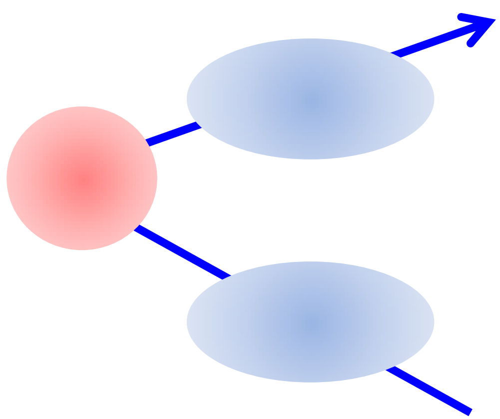
Vertex : the special form to couple with external perturbations (E&M fields)

In real interacting system: the vertex is surrounded by the **backflow or clouds.** even a single particle is surrounded by **clouds**



Self-energy correction.

Vertex correction.



Self-energy correction + Vertex correction should be Consistent → to preserve Gauge inv. & Conservation law

↓

Ward identity

with

$$K_{ij}(q) = -\frac{e^2}{m}\langle 0|\rho|0\rangle\delta_{ij} + \sum_n \left(\frac{\langle 0|j_i(q)|n\rangle\langle n|j_j(-q)|0\rangle}{E_n} + \frac{\langle 0|j_j(-q)|n\rangle\langle n|j_i(q)|0\rangle}{E_n}\right).$$

In this BCS calculation, no vertex correction but fundamentally unique self-energy

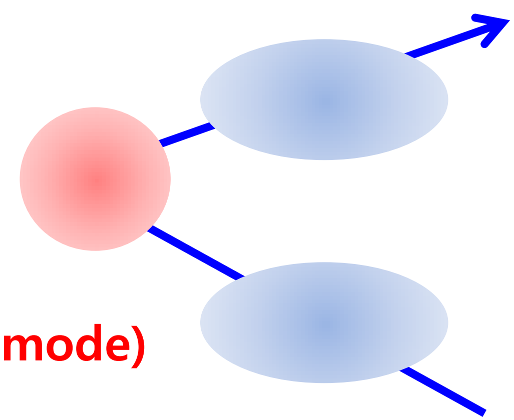
vertex $(e, e\mathbf{v}_{\mathbf{k}})$

$$\sum_k [c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\Delta + c_{-k\downarrow}c_{k\uparrow}\Delta^*]$$

Vertex correction.

What kind of Backflow

→ **Goldstone boson (Phase mode)**

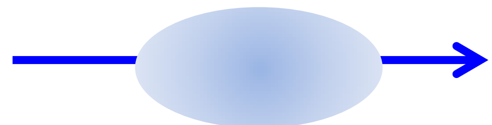


Goldstone boson is important to correct the gauge invariance of the theory.

$\sum_k [c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \Delta + c_{-k\downarrow} c_{k\uparrow} \Delta^*]$ This is U(1) symmetry breaking self-energy correction.

$$H' = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k},\mathbf{l}} V_{\mathbf{kl}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow}.$$

← $< c_{k\uparrow} c_{-k\downarrow} >= \Delta$ and $< c^\dagger_{k\downarrow} c^\dagger_{-k\uparrow} >= \Delta^*$



Global U(1) symmetry

$$c^\dagger_{k,\sigma} \rightarrow c^\dagger_{k,\sigma} e^{i\theta}$$
$$c_{k,\sigma} \rightarrow c_{k,\sigma} e^{-i\theta}$$

H is invariant. But

$$\Delta^* \rightarrow \Delta^* e^{i2\theta}$$
$$\Delta \rightarrow \Delta e^{-i2\theta}$$

SC means non zero value of $< c_{k\uparrow} c_{-k\downarrow} >= \Delta$, it means the ***θ*** angle fixed.

SC is an U(1) symmetry broken phenomena.

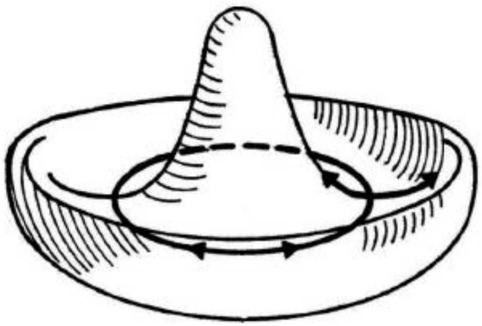
And Goldstone theorem says that the ***θ*** angle should fluctuates, and it is massless Goldstone boson or phase mode.

Simplest toy model showing U(1) symmetry breaking. It contains only Δ variable. Δ =>ψ.

L = T -V

$$L = \partial_\mu \psi^*(x)\partial^\mu \psi(x) - r_0[\psi^*(x)\psi(x)] - \frac{u_0}{2}[\psi^*(x)\psi(x)]^2$$

$$[\partial_\mu \partial^\mu + r_0 + u_0|\psi(x)|^2]\psi(x) = 0$$



$r_0 < 0$

Mean field solution: $\psi(x) = \psi_0 e^{i\theta}$

$r_0 < 0$ $\psi_0 = \frac{r_0}{u_0}$ the $\boldsymbol{\theta}$ angle fixed.

Allow a small fluctuations around gs. $\psi(x) = [\psi_0 + \eta(x)]e^{i\xi(x)}$

$$\partial_\mu \partial^\mu \xi(x) = 0$$
$$(\partial_\mu \partial^\mu - 2r_0)\eta(x) = 0$$

Massless phase mode (Goldstone) **ξ(x): ω^2 = k^2 → ω = k**

Coupling with E&M fields

$$L = [(\partial_\mu + ieA_\mu)\psi(x)]^*[(\partial_\mu + ieA_\mu)\psi(x)] - r_0[\psi^*(x)\psi(x)] - \frac{u_0}{2}[\psi^*(x)\psi(x)]^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Gauge transform

$$[\psi_0 + \eta(x)]e^{i\xi(x)} \rightarrow [\psi_0 + \eta(x)]$$

$$A^\mu \rightarrow A^\mu + \frac{1}{e}\partial^\mu\xi(x) = \tilde{A}^\mu$$

$$[\partial_\mu\partial^\mu - 2e^2\frac{r_0}{u_0}]\tilde{A}^\mu = 0$$

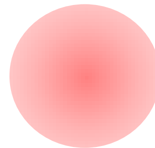
~~$\partial_\mu\partial^\mu\xi(x) = 0$~~

$$(\partial_\mu\partial^\mu - 2r_0)\eta(x) = 0$$

1. Goldstone boson ξ disappears.
2. Photon (gauge boson) becomes massive. → **Meissner effect**
3. The whole system is gauge invariant. →why ?

4. 1+2 ➔ called Higgs mechanism.
5. η mode is called Higgs boson.

Self-consistent vertex correction correctly found Phase mode form the main cloud in vertex



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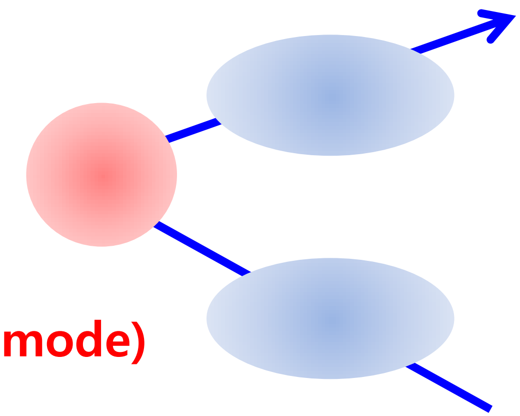
vertex $(e, e\mathbf{v}_\mathbf{k})$

$$\sum_k [c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \Delta + c_{-k\downarrow} c_{k\uparrow} \Delta^*]$$

Vertex correction.

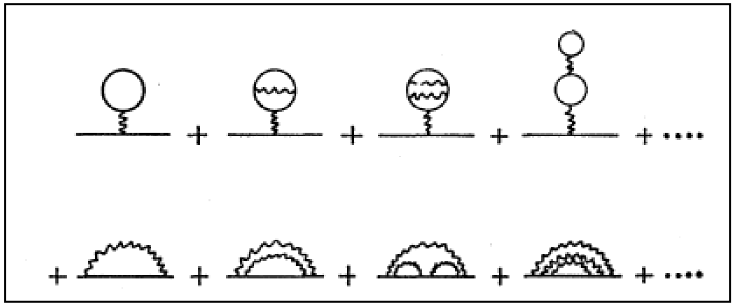
What kind of Backflow

→ **Goldstone boson (Phase mode)**

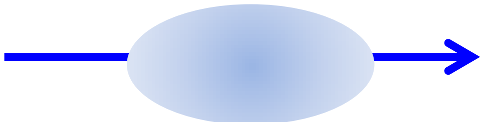


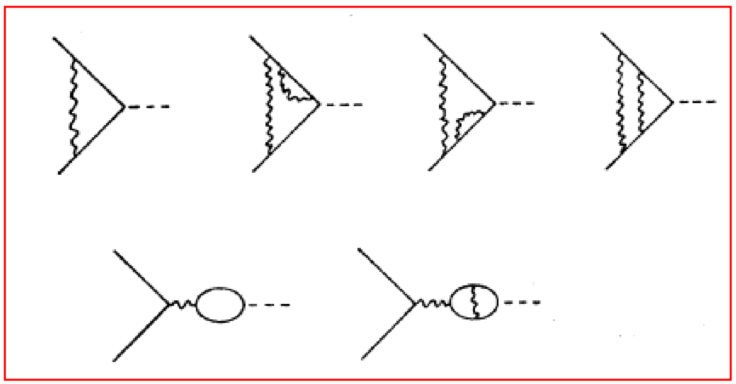
Goldstone boson is important to correct the gauge invariance of the theory.

Nambu's ('60) vertex correction

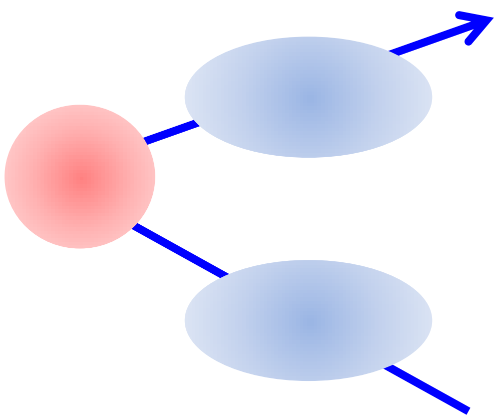


Self-energy $\Sigma \rightarrow \Delta$



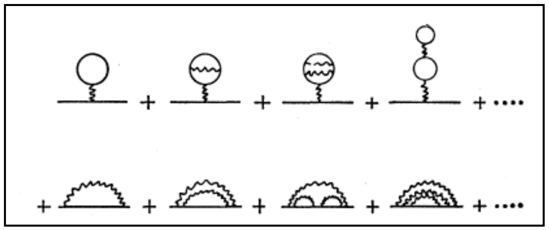


Consistent vertex Γ_i correction



Ward identity:

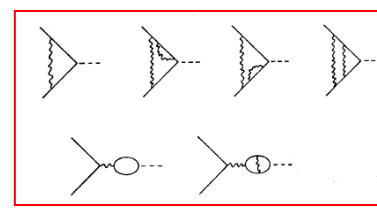
$$\Gamma_i \cdot q_i = [(\omega + q_0) - \epsilon_{k+q} - \Sigma(\omega + q_0, k + q)] - [\omega - \epsilon_k - \Sigma(\omega, k)]$$



→ Γ_i from **WI** or

$$\Gamma_i(p',p) = \gamma_i(p',p) - g^2 \int \tau_3 G(p'-k)\Gamma_i(p'-k, p-k) \times G(p-k)\tau_3 h(k)^2 \Delta(k) d^4k,$$

You can solve an integral Eq. if you can. Nambu did it.



Nambu found:

(1) Γ_i (ω, q) has collective mode.

(2) This collective mode is phase mode (GS mode; Anderson-Bogoliubov mode)

(3) It satisfies $\mathbf{K_{ij}(q) \cdot q_j = 0}$

Summary (**GS boson in SC**)

1. Superconductivity breaks global **U(1)** symmetry.
2. Therefore, Goldstone boson should appear.
3. This GS boson is the phase fluctuations of $\boldsymbol{\Delta e^{i\theta}}$
4. This GS boson makes photon massive (Meissner effect, Higgs mechanism)
5. This GS boson completes the gauge invariance.
6. This GS boson finally disappears.
7. Higgs boson η mode remains to be detected.