Project on the physics of Goldstone bosons in Condensed Matter Systems

(1) Number of Goldstone modes

The number of Goldstone modes of the symmetry broken system can be counted as the number of lost d.o.f (or symmetry) due to the symmetry breaking from the original symmetry preserved system.

(1) O(N) symmetry case: Imagine a freely rotating spin $\vec{S}$ in N-dimension space which has O(N) symmetry. Count the total number of independent rotational axis (transformation) of a freely rotating spin. Similarly count the number of the remaining independent rotational axis when the spin is pointing to a specific direction (spontaneous symmetry breaking). Show the difference of two numbers is $N-1$.

(2) T(N) symmetry case: Imagine a freely moving particle in N-dimensional space (translational symmetry). Count how many independent translational d.o.f. the particle has. Imagine the particle position is pinned on a specific position in the N-dimensional space. Count the number of remaining independent translational d.o.f of the particle in this case. Find the difference of two numbers (trivial example).

(2) Goldstone modes and Higgs modes

Consider the Lagrangian of the charged scalar field $\psi(x)$ which is a complex number. Besides the standard kinetic term we assume the famous Mexican hat potential $V = r_0|\psi(x)|^2 + \frac{u_0}{2}|\psi(x)|^4$. So with $g_{\mu\nu} = (+,-,-,-)$, we have

$$L = \partial_\mu \psi^*(x)\partial^\mu \psi(x) - r_0[\psi^*(x)\psi(x)] - \frac{u_0}{2}[\psi^*(x)\psi(x)]^2 \tag{1}$$

Derive the Eq. of motion for $\psi(x)$ as

$$[\partial_\mu \partial^\mu + r_0 + u_0|\psi(x)|^2]\psi(x) = 0 \tag{2}$$

Find the mean field solution of $\psi(x)$ (meaning a constant solution $\psi(x) = \psi_0$) of the above eq. both when $r_0 > 0$ and $r_0 < 0$, respectively.

We are interested in the non-zero solution $\psi(x) = \psi_0$ when $r_0 < 0$. Expanding the above Lagrangian with a small fluctuations $\eta(x), \xi(x)$ around the mean field solution as $\psi(x) = [\psi_0 +$

$\eta(x)]e^{i\xi(x)}$, find the equations of motion of the fluctuations as

$$\partial_\mu \partial^\mu \xi(x) = 0 \quad (3)$$

$$(\partial_\mu \partial^\mu - 2r_0)\eta(x) = 0 \quad (4)$$

These results clearly show that $\xi(x)$ is a massless Goldstone mode originating from phase fluctuations and $\eta(x)$ is a massive mode with a mass of $2r_0$. This massive mode itself is not a symmetry restoring mode (not a Goldstone mode) but it is a smoking gun evidence for the presence of the Higgs field $\psi(x)$ which goes to a spontaneous symmetry breaking and called Higgs boson.

(3) Anderson Higgs mechanism

It was discovered by several people in early 60th that the above found massless Goldstone mode can be combined with the massless gauge modes to make the gauge modes massive. In the phenomena of superconductivity, this is the exactly the mechanism for the Meissner effect (perfect diamagnetism), meaning that external magnetic field can not penetrate into the superconducting sample while it can propagate freely into ordinary metals.

In order to study this phenomena, introduce E&M fields to couple to the scalar field $\psi(x)$. This can be done by the minimal coupling $(\partial_\mu + ieA_\mu)$ as follows.

$$L = [(\partial_\mu + ieA_\mu)\psi(x)]^*[(\partial_\mu + ieA_\mu)\psi(x)] - r_0[\psi^*(x)\psi(x)] - \frac{u_0}{2}[\psi^*(x)\psi(x)]^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (5)$$

Again assume $r_0 < 0$ and the non-zero mean field solution $\psi(x) = \psi_0$, then expand the above Lagrangian around $\psi(x) = \psi_0$ substituting $\psi(x) = [\psi_0 + \eta(x)]e^{i\xi(x)}$. Now take a gauge transformation as follows

$$[\psi_0 + \eta(x)]e^{i\xi(x)} \rightarrow [\psi_0 + \eta(x)] \quad (6)$$

$$A^\mu \rightarrow A^\mu + \frac{1}{e}\partial^\mu \xi(x) = \tilde{A}^\mu \quad (7)$$

Then you found that you can always eliminate $\xi(x)$ field in the above Lagrangian. Therefore now you don't have Eq. of Motion for $\xi(x)$, but show that Eq. of Motion for $\eta(x)$ is the same as before (Eq.(4)). Finally find the Eq. of motion for the gauge field $\tilde{A}^\mu$ as

$$[\partial_\mu \partial^\mu - 2e^2\frac{r_0}{u_0}]\tilde{A}^\mu = 0 \quad (8)$$

which clearly shows that the gauge fields $\tilde{A}^\mu$ obtained a mass $\sqrt{2e^2\frac{r_0}{u_0}}$.

(4) Gauge Invariance of the Meissner effect

In the original BCS theory, the Meissner effect is expressed as

$$J_i(q) = K_{ij}(q)A_j(q), \quad i = 1, 2, 3 \tag{9}$$

and any non-zero value of K_{ij} means the Meissner effect and the BCS theory evaluates K_{ij} as

$$K_{ij} = -\frac{e^2 n}{m}\delta_{ij} + < BCS|j_i(q)j_j(-q)|BCS > \tag{10}$$

The first term is the trivial diamagnetic term and the second term is the current-current correlation function. In the BCS theory, the second term is trivially zero, hence the BCS theory immediately achieves the Meissner effect. The problem is that $J_i(q) = -\frac{e^2 n}{m}\delta_{ij}A_j(q)$ is gauge non-invariant. Find out the general form of the $K_{ij}(q)$ which satisfy the gauge invariance as well as the conservation law.

Also think about what is the requirement to change the BCS result $< BCS|j_i(q)j_j(-q)|BCS >= 0$ in order to satisfy the gauge invariance. This problem was elegantly resolved by Nambu among others using the generalized Ward identity.