The Dirac Monopole and Motion of an Electrically Charged Particle

Units and conventions in this problem may look unfamiliar. Probably the simplest solution is to pretend that $\epsilon_0 = 1$ and $\mu_0 = 1$, which also sets $c = 1$. However, you need to be careful about factor 2's and π's.

Dirac Monopole

Dirac's magnetic monopole is a hypothetical object that carries a magnetic charge. Usual particle we know of, such as electrons and nucleons, carries electric charges only, and Maxwell's equations incorporate only electric charges and electric currents. However, one could imagine a modified Maxwell's equation as follows

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \vec{\nabla} \cdot \vec{B} = g\delta^{(3)}(\vec{r})$$

which gives

$$\vec{B} = -\frac{g\vec{r}}{4\pi r^2}$$

just as in ordinary electric Coulomb problem.

Write this magnetic field in terms of a vector potential $\vec{A}(\vec{r})$ such that

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\vec{A}$ cannot be well-defined everywhere, since $\vec{\nabla} \cdot \vec{\nabla} \times \vec{A} = 0$, which contradict the above modified Maxwell equation at origin. Write down $\vec{A}$ which is well-defined except along z-axis. The singularity along z-axis is called the Dirac string, and is required to be unphysical. The latter asserts that magnetic charge and electric charge are both quantized. Explain this.

Angular Momentum of A Charged Particle Near a Dirac Monopole

Consider such a magnetic monopole, and an particle of electric charge q and mass m. The latter's equation of motion due to the former is

$$m\frac{d^2\vec{r}}{dt^2} = q\vec{r} \times \vec{B}$$

Show that this equation of motion not only conserve the total energy but also some form of angular momentum. Write down these conserved quantities explicitly. Look for conserved angular momentum of the form

$$\vec{J} = \vec{r} \times m\vec{v} + \beta\vec{r}/r$$

and determine the constant β in terms of m, q, g.

This last additional piece, originally due to Poincaré, can also be understood as the angular momentum contribution of the electromagnetic fields. Show this explicitly, by computing angular momentum density of the EM field due to these two particles and integrating the density over the whole space. **Hint: Poynting Vector**

Trajectories

Use this angular momentum to characterize trajectories of the electrically charged particle completely. Are there any bound orbit?

Lagrangian and Rotational Symmetry

Lagrangian Formulation The above motion of electrically charged particle can be studied starting with a Lagrangian,

$$L = \frac{1}{2}m\left(\frac{d\vec{r}}{dt}\right)^2 + q\frac{d\vec{r}}{dt} \cdot \vec{A}$$

Show that the action with the above Lagrangian is invariant under the spatial rotation. This is why there is a conserved angular momentum. Derive conserved angular momentum via Noether procedure, and write it in terms of canonical variables. Finally derive the equation of motion and show it is identical to the usual one, already given above.

Quantum Scattering

This is not an easy problem to solve exactly. Either try to solve it approximately by using a perturbation method, or dig up existing paper on the subject and try to compare the result to the classical result above. Compute scattering amplitude of an electrically charged particle off the vector potential by solving Schroedinger equation,

$$(\vec{\nabla} - iq\vec{A}(r))^2\Psi = i\frac{\partial}{\partial t}\Psi$$

Or try the following two-component Schroedinger equation,

$$\left(\vec{\sigma} \cdot (\vec{\nabla} - iq\vec{A}(r))\right)^2 \tilde{\Psi} = i\frac{\partial}{\partial t}\tilde{\Psi}$$

with three Paulifs matrices, $\vec{\sigma}$, and two-component wavefunction $\tilde{\Psi}$. Use approximation if you must, such as the partial wave analysis, or even truncate the problem to S-wave sector alone.