2014 KIAS-SNU Physics Winter Camp: Mini-Project
"Monopoles in real and momentum spaces of condensed matter systems"

In this camp, I assume that you will learn a lot about "geometric phases in physics". The well-known electromagentic vector potential $\mathbf{A}(\mathbf{r})$ can be understood as a kind of geometric phase of $U(1)$ symmetry in quantum mechanical systems. In condensed matter systems, extra degrees of freedom imbedded in physical systems can be realized in a form of so-called Berry phase, which emerges naturally from the geometric connection of physical objects in either real or momentum spaces. Consequently the effect of Berry phase can be observed in various physical measurements.

To acquire a basic knowledge of geometric phase and investigate its physical implications, let us take an example of 'monopole', a prototypical topological object, in condensed matter systems. In an elementary electromagnetism textbook, the concept of 'magnetic monopole' is raised and discussed. So far its 'physical' existence has not been proved in experiment yet. However, beyond the realm of elementary particle physics, one may witness monopoles in connection with Berry phase, which still bear the physical connection to the original monopoles in context.

☐ Warm-up exercises

1. **Gauge-independence of Berry curvature.** Consider a two-level system which depends on an external field, i.e., a parameter θ. One of the eigenstates has a form:

$$|\psi(\theta)\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \tag{1}$$

   Check the expression for the Berry curvature, $\mathscr{F}(\theta,\phi)$, of both $|\psi(\theta)\rangle$ and $e^{-i\phi}|\psi(\theta)\rangle$, where ϕ is an arbitrary phase factor. Also discuss its implication.

2. **Berry curvature in Cartesian coordinates.** Find the eigenstates of an Hamiltonian

$$\mathscr{H} = \vec{h}\cdot\vec{\sigma} = h_x\sigma_x + h_y\sigma_y + h_z\sigma_z$$

   in terms of $\vec{h} = (h_x, h_y, h_z)$, rather then in the spherical angles (θ,ϕ) in the space of $\vec{h}$. Of course $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ represent (2×2) Pauli matrices.

   Calculate the vectors of Berry curvature $\vec{\mathscr{F}}(\vec{h})$ corresponding to the $|+\rangle$ and $|\rangle$ states, corresponding to the "spin" directed parallel and antiparallel to the field $\vec{h}$. You should be able to recover the field of the monopole located at the origin of the parameter space, i.e., $\vec{h}$-space, with particular values of the monopole strength. How are those related to $|+\rangle$ and $|\rangle$ states, respectively?

3. **Berry phase for an arbitrary spin.** Consider a spin-S particle in a magnetic field, the Hamiltonian of which can be described by $\mathscr{H} = \vec{h}\cdot\vec{S}$, where $\vec{S}$ now is the spin operator of $S \geq 1/2$ (e.g., $\vec{S} = \vec{\sigma}/2$ for $S = 1/2$). This system has $2S+1$ degenerate states for $\vec{h} = 0$. Suppose that the system is initially prepared in a state such that

$$\mathscr{H}|n\rangle = hn|n\rangle,$$

where n measures the component of the spin along the field.

What is the resulting strength of the Berry curvature $\vec{\mathscr{F}}(\vec{h})$ when a monopole is located at the origin of the parameter space, i.e., $\vec{h}$-space. What is the Berry phase accumulated over a closed path when the direction of the magnetic field is varied?

☐ Suggested questions

1. **Quantum Hall effect.** Try to make a geometric interpretation of the quantized Hall effect, which is often described by a simple Hamiltonian $\mathscr{H} = (1/2m)(\mathbf{p}^2 - e\mathbf{A}/c)^2$ with a vector potential $\mathbf{A}(\mathbf{r})$. Discuss about the connection between the measured Hall conductivity and the Berry curvature.

   For more information, for example, see an article by Laurent Michel [3].

2. **Spin Ice.** A group of condensed matter experimentalists claimed that a particular form of so-called "spin-ice" state realized in a highly frustrated pyrochlore lattice manifests the characteristics of magnetic monopoles and Dirac strings, which has so far proven elusive as elementary particles. Examine the evidences and ideas of spin-ice as well as symmetry-breaking magnetic field presented in the experiment, for example, by Morris et al.[4], and discuss its validity and the Berry phase concept behind the physical argument.

3. **Anomalous Hall effect and skyrmions.** Electrons in a magnetic crystal are affected by the real magnetic field as well as the Berry phase, an emergent 'electromagnetic field' in solids. The Berry phase can arise in both real and momentum spaces. In momentum space, a monopole, which can be mimicked by a certain feature of electronic band structure, results in an anomalous Hall effect. On the other hand, in real space, a particle spin texture of so-called skyrmions can be realized. Find out the physical implications of monopoles and skyrmions in real and momentum spaces of magnetic solids.

   For more information, for example, see an article by Nagaosa et al. [5].

☐ References

1. Jeeva Anandan, Joy Christian, and Kazimir Wanelik, *"Geometric Phases in Physics"*, American Journal of Physics **65**, 180 (1997).

2. Ming-Che Chang and Qian Niu, *"Berry curvature, orbital moment, and effective quantum theory of electrons in electromagnetic fields"*, J. Phys.: Condens. Matter **20** 193202 (2008).

3. Laurent Michel, *"Geometric interpretation of the quantum Hall conductivity"*, `http://www.cjcaesar.ch/fribourg/docfrib/final_year_project/berry_qhe.pdf`

4. D. J. P. Morris et al., *"Dirac Strings and Magnetic Monopoles in the Spin Ice $Dy_2Ti_2O_7$"*, Science **326**, 411 (2009).

5. N. Nagaosa, X. Z. Yu and Y. Tokura, *"Gauge fields in real and momentum spaces in magnets: monopoles and skyrmions"*, Phil. Trans. R. Soc. A **370**, 58065819 (2012).