Irregular conformal block

Chaiho Rim (Sogang University)

Arxiv:1207.4480 JHEP 10 (2012) 138 with T. Nishinaka

Arxiv:1312.5535 JHEP 04(2014) 106 with S. Choi

Arxiv:1405.3141 with Y. Matsuo, H. Zhang

Plan of talk

1. Brief introduction: Regular and irregular conformal state

2. Irregular conformal block and colliding limit

3. Recent results and future works

REGULAR AND IRREGULAR CONFORMAL STATE

Virasoro primary states

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Primary state:

$$L_0|\Delta >= \Delta|\Delta >, \qquad L_k|\Delta >= 0 \text{ when } k > 0$$

$$|\Delta >= \lim_{z\to 0} V_\alpha(z)|0 >$$

Descendent state:

$$L_0\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta >\Big)$$

$$= (n_1 + n_2 + \cdots n_k)\Big(L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\Delta >\Big)$$

AGT conjecture (2009)

- Nekrasov partition function of $\mathcal{N}=2$, su(2), Nf=4 gauge theory with Coulomb branch parameter a, masses m of hypermultiplet and Ω-background parameter ϵ_1 and ϵ_2 introduced for regulation.

- Liouville conformal block with vertex operator with dimension $\Delta = \alpha(Q-\alpha)$ and background charge $Q=b + 1/b$.

Regular conformal block

$$\langle V_{\alpha_\infty + \frac{Q}{2}}(\infty) V_{\alpha_1}(1) V_{\alpha_2}(q) V_{\alpha_0 + \frac{Q}{2}}(0) \rangle_{\text{Liouville}}$$

$$\propto \left| q^{Q^2/4 - \Delta_{\widetilde{m}_2} - \Delta_{\widetilde{m}_0 + \frac{Q}{2}}} \right|^2 \int a^2 da \left| Z^{SU(2)}_{\text{Nekrasov}}(a, \mu_I, \epsilon_i) \right|^2$$

- $Z^{SU(2)}_{\text{Nekrasov}}(a, \mu_I, \epsilon_i)$ Nekrasov partition function : 1-loop and instanton contribution

$$\mu_1 = m_1 + m_\infty, \quad \mu_2 = m_1 - m_\infty, \quad \mu_3 = m_2 + m_0, \quad \mu_4 = m_2 - m_0$$

$$\epsilon_1 = b\,, \quad \epsilon_2 = 1/b\,, \quad Q = (\epsilon_1 + \epsilon_2)\,, \quad \alpha_i = m_i\,, \quad \alpha = a + Q/2$$

- $V_\alpha(x) = e^{2\alpha\phi(x)}$ Liouville conformal block of primary fields

Super-conformal quivers

- Energy momentum tensor and Virasoro generator

$$T(z) = \sum_k \frac{L_k}{z^{k+2}}$$

- In-state or out-state as the primary state
- The poles has degree 2 or smaller

$$T(z)|\mathrm{in}\rangle = \sum_{k\leq 0} \frac{L_k}{z^{k+2}}|\mathrm{in}\rangle$$

Asymptotically free quivers

According to AGT conjecture, one can try to find a new conformal theory by taking the colliding limit of primary fields to obtain 'irregular conformal block' associated with Argyres-Douglas type gauge theory

Irregular module

- What about degree greater than 2 ?

$$T(z)|\mathrm{in}\rangle = \sum_{k\le m} \frac{L_k}{z^{k+2}}|\mathrm{in}\rangle \ ; \qquad m>0$$

- Irregular conformal block, Gaiotto (0908.0307)
- Seiberg-Witten curve shows the singularity structure

$$\phi_2 =<\mathrm{out}|\mathrm{in}>= \frac{\Lambda^2}{z^3} + \frac{2u}{z^2} + \frac{\Lambda^2}{z}$$

$$|\mathrm{in}>=|\Delta,\Lambda^2>, \quad <\mathrm{out}| =<\Delta,\Lambda^2|$$

$$L_1|\Delta,\Lambda^2>=\Lambda^2|\Delta,\Lambda^2>$$

Case n=1 with Nf=1

$$L_1|\Delta,\Lambda^2\rangle = \Lambda^2|\Delta,\Lambda^2\rangle\,, \qquad L_2|\Delta,\Lambda^2\rangle = 0$$

$$c = 1 + 6Q^2, \quad Q = b + 1/b$$

$$|\Delta,\Lambda^2\rangle = \sum_\ell \Lambda^{2\ell} u_n$$

$$u_0 = |\Delta\rangle$$

$$u_1 = \frac{1}{2\Delta} L_{-1}|\Delta\rangle$$

$$u_2 = \frac{(c+8\Delta)L_{-1}^2 - 12\Delta L_{-2}}{4\Delta(2\Delta + c + 16\Delta^2 - 10\Delta)}|\Delta\rangle$$

Case n=1 with Nf=2

$$L_1|\Delta,\Lambda,m\rangle = -\Lambda^2|\Delta,\Lambda,m\rangle\,, \qquad L_2|\Delta,\Lambda,m\rangle = -2m\Lambda|\Delta,\Lambda,m\rangle$$

$$c = 1+6Q^2, \quad Q = b+1/b$$

$$|\Delta,\Lambda^2\rangle = \sum_\ell \Lambda^\ell w_n$$

$$w_0 = |\Delta\rangle$$

$$w_1 = -\frac{m}{\Delta}L_{-1}|\Delta\rangle$$

$$w_2 = \frac{(cm^2+\Delta(3+8m^2))L_{-1}^2 - 2\Delta(1+2\Delta+6m^2)L_{-2}}{4\Delta(2\Delta+c+16\Delta^2-10\Delta)}|\Delta\rangle$$

Gaiotto (2009), Marshak, Mironov, Morozov (2009)

Irregular module

Consider simultaneous eigenstate of positive Virasoro generators but is not the eigenstate of L_0

$$L_k|\mathrm{in}\rangle = 0 \quad \text{for} \quad 2n < k$$
$$L_k|\mathrm{in}\rangle = \Lambda_k|\mathrm{in}\rangle \quad \text{for} \quad n \leq k \leq 2n$$

$$L_k|\mathrm{in}\rangle = (\Lambda_k + v_k)|\mathrm{in}\rangle \quad \text{for} \quad 1 \leq k \leq n-1$$
$$L_0|\mathrm{in}\rangle = (\alpha(Q-\alpha) + v_0)|\mathrm{in}\rangle$$

v_k is an appropriate operator

Coherent state representation

$$a_k|\text{in}\rangle = c_k|\text{in}\rangle \quad \text{for} \quad 1 \le k \le n$$
$$a_k|\text{in}\rangle = 0 \qquad \text{for} \quad k > n$$

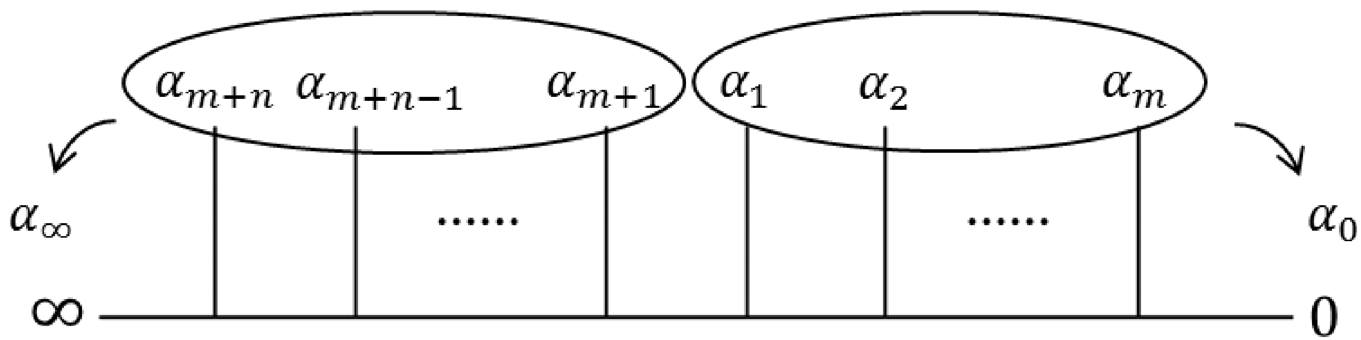
c_k is the coherent coordinates of bosonic annihilation operator (Gaiotto&Teschner,2012)

$$L_k|\text{in}\rangle = \Lambda_k|\text{in}\rangle \quad \text{for} \quad n \le k \le 2n$$
$$\Lambda_k = (k+1)Q\, c_k - \sum_\ell c_k c_{k-\ell}$$
$$L_k|\text{in}\rangle = (\Lambda_k + v_k)|\text{in}\rangle \quad \text{for} \quad 1 \le k \le n-1$$
$$L_0|\text{in}\rangle = (\alpha(Q-\alpha) + v_0)|\text{in}\rangle$$
$$v_k = \sum_\ell c_{\ell+k} \frac{\partial}{\partial c_\ell}$$

IRREGULAR CONFORMAL BLOCK , THE COLLIDING LIMIT AND PENNER-TYPE MATRIX MODEL

Colliding limit

- Vertex operators at the same point with infinite virasoro charge so that appropriate finiteness is maintained



Penner-type matrix model

$$\left\langle \left(\prod_{I=1}^{N} \int d\lambda_I d\overline{\lambda}_I \; e^{2b\phi(\lambda_I)} \right) V_{\alpha_\infty + \frac{Q}{2}}(\infty) V_{\alpha_1}(1) V_{\alpha_2}(q) V_{\alpha_0 + \frac{Q}{2}}(0) \right\rangle_0$$

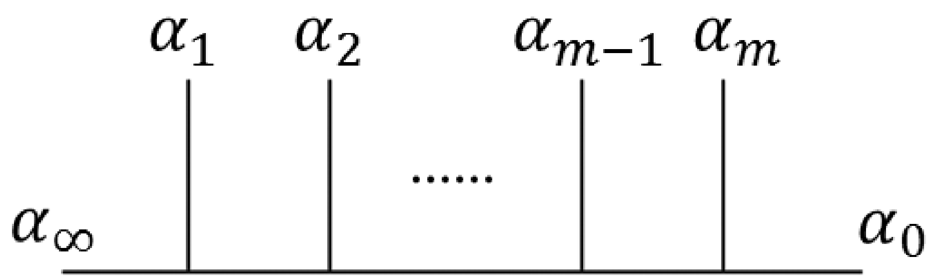
$$\propto \left| q^{-\frac{2(m_0+\epsilon_+/2)m_2}{\hbar^2}} (1-q)^{-\frac{2m_1 m_2}{\hbar^2}} \;\; I_4 \right|^2$$

$$I_4 = \int \left[\prod_{I=1}^{N} d\lambda_I \right] \; \prod_{I<J} (\lambda_I - \lambda_J)^{-2b^2} \;\; \exp\left(-\frac{2b}{\hbar} \sum_I V(\lambda_I) \right)$$

$$\frac{V(z)}{\hbar} = \left(\alpha_0 + \frac{Q}{2} \right) \log z + \alpha_1 \log(z-1) + \alpha_2 \log(z-q).$$

$$\langle e^{2\alpha_1 \phi(z)} e^{2\alpha_2 \phi(w)} \rangle = |z-w|^{-4\alpha_1 \alpha_2} \,, \qquad \sum_{i=0}^{2} \alpha_i + \alpha_\infty + bN = 0$$

Penner-type matrix model



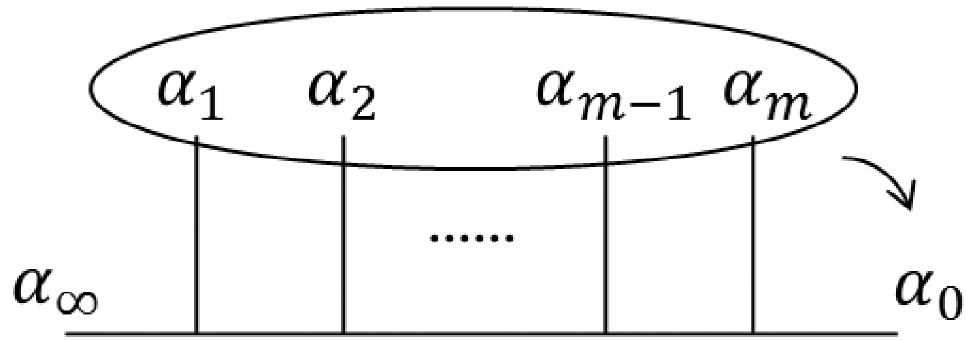
$$Z_{\beta-\text{Penner}} = Z(\{N_i\}, \{\alpha_r, z_r\}, g, \beta)$$

$$\equiv \left[\prod_{I=1}^{N} \int d\lambda_I\right] \Delta^{2\beta} \exp\left(\frac{\sqrt{\beta}}{g} \sum_I V(\lambda_I)\right)$$

$$V(z) = \sum_r \alpha_r \log(z - z_r)\,. \qquad \Delta = \prod_{I<J} (\lambda_I - \lambda_J)$$

$$b = i\sqrt{\beta},\ \hbar = -2ig$$

Colliding limit



Eguchi& Maruyoshi (2010), Bonelli, Maruyoshi, Tanzini (2011), Gaiotto & Teschner (2012),

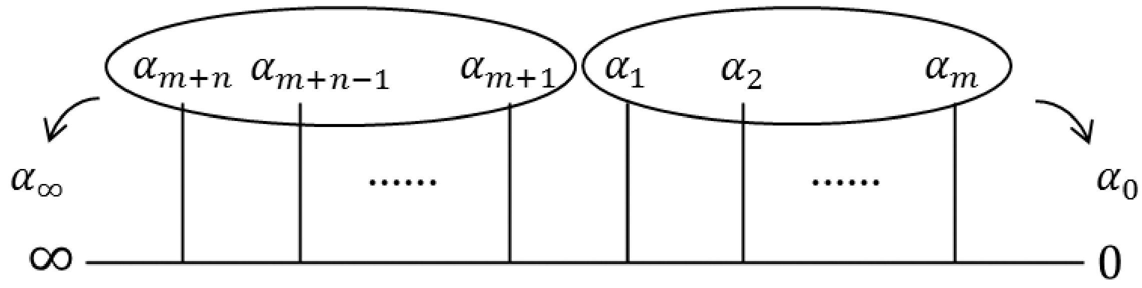
$$\frac{V(z)}{\hbar} = \sum_{r=0}^{n} \alpha_r \log(z - z_r) \quad \overset{\alpha_r \to \infty}{\xrightarrow{z_r \to 0}} \quad \alpha \log z - \sum_{k=1}^{n} \frac{c_k}{k z^k}$$

$$\alpha = \sum_r \alpha_r = c_0$$

$$c_k = \sum_{r=0}^{n} \sum_{0 \le s_1 < \cdots < s_k \le n \, (s_i \ne r)} \alpha_r \prod_{i=1}^{k} (-z_{s_i})$$

RECENT RESULTS AND SOME REMARKS

Inner product of irregular states



- Inner product of irregular states is obtained from the Partition function = ZM= <in | out>
- The colliding limit has the finite limit of the Nekrasov partition function.
- Penner-type matrix model provides the partition with the coherent state parameters

Inner product <R|I2>

$$Z_M^{(2)}(c_1, c_2; N_1, N_2) = (c_2)^{-\frac{bN(bN+2\alpha-Q)}{2}}$$

$$\times \left(\frac{c_2}{c_1^2}\right)^{-b\left\{\frac{N_1(bN_1+2\alpha-Q)}{2} - N_2(bN_2+\alpha-Q)+bN_1N_2\right\}}$$

$$\times \quad e^{-\frac{bN_2c_1^2}{c_2}+O\left(\frac{c_2}{c_1^2}\right)} .$$

$$\frac{V(z)}{\hbar} = \alpha \log z - \sum_{k=1}^{n} \frac{c_k}{kz^k}$$

<I1|I2>

$$Z_N^{(1;2)}(c_{-1},c_0,c_1,c_2;N_1,N_2,N_3) = \Big(c_{-1}/c_0\Big)^{h_1/\hbar^2}$$

$$\Big(\eta_0\Big)^{-b(N_1+N_2)\left(b(N_1+N_2)+2\frac{c_0}{\hbar}\right)}\Big(\eta_1\Big)^{-\frac{bN_2}{2}\left(3bN_2+4\frac{c_0}{\hbar}\right)}$$

$$e^{-\frac{bN_2 c_0/\hbar}{\eta_1}+O(\eta_0,\eta_1)}$$

$$h_1 = \hbar bN(\hbar bN + 2c_0), \qquad \eta_k = \frac{c_{k-1}c_{k+1}}{c_k}$$

$$\frac{V(z)}{\hbar} = c_0 \log z - \sum_{k=-m}^{n} \frac{c_k}{kz^k}$$

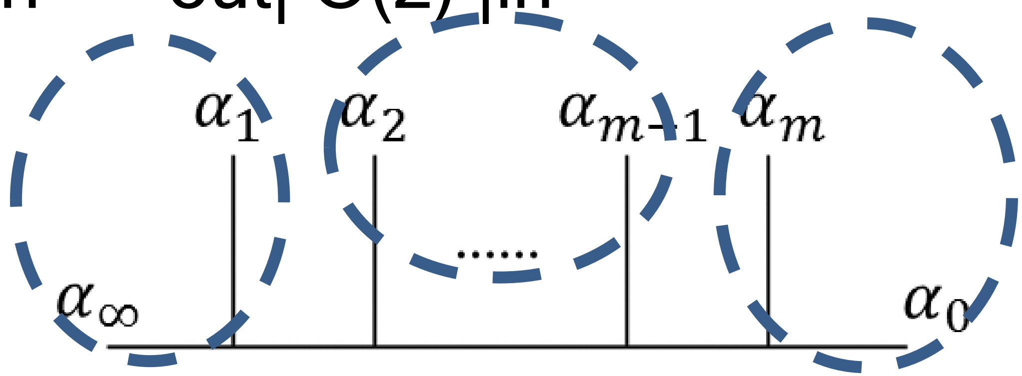
<I2|I2>

$$Z_N^{(2;2)}(c_i;N_i) = \Big(c_{-1}/c_0\Big)^{h_1/\hbar^2}$$
$$\Big(\eta_0\Big)^{-b(N_1+N_2)(b(N_1+N_2)+2c_0/\hbar)}$$
$$\Big(\eta_{-1}\Big)^{\frac{bN_{-1}}{2}(-3bN_{-1}+4(c_0/\hbar+bN))}$$
$$\Big(\eta_1\Big)^{-\frac{bN_2}{2}(3bN_2+4c_0/\hbar)} e^{-\frac{bN_2 c_0/\hbar}{\eta_1}+\frac{bN_{-1}c_0/\hbar}{\eta_{-1}}+O(\eta_{-1},\eta_0,\eta_1)}$$

$$h_1 = \hbar bN(\hbar bN + 2c_0), \qquad \eta_k = \frac{c_{k-1}c_{k+1}}{c_k}$$

Further directions

- Inner product (partition function)
  - Extension to SU(N) quivers (Kanno&Taki2012, Kanno,Maruyoshi, Shiba &Maruyoshi 2013)
  - Multi-component matrix model
  - W-algebra or SH algebra
- One point function = <out| O(z) |in>
  - 3 point vertex



Thank you

Loop equation

$$\frac{f(z)}{4} = W(z)^2 + V'(z)W(z) + g\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)W'(z) + g^2 W(z,z)$$

$$W(z_1, \cdots, z_s) = \beta\left(\frac{g}{\sqrt{\beta}}\right)^{2-s}\left\langle \sum_{I_1}\frac{1}{z_1 - \lambda_{I_1}} \cdots \sum_{I_s}\frac{1}{z_s - \lambda_{I_s}}\right\rangle_{\mathrm{conn}}$$

$$\frac{f(z)}{4} \equiv g\sqrt{\beta}\sum_{I=1}^{N}\left\langle\frac{V'(z) - V'(\lambda_I)}{z - \lambda_I}\right\rangle = g^2\sum_{k=0}^{n}\frac{1}{z - z_k}\frac{\partial \log Z_{\beta-\mathrm{Penner}}}{\partial z_k}$$

Filling fraction Ni is given as the contour integral

$$\frac{1}{2\pi i}\oint_{\Gamma_i} W(z)dz = g\sqrt{\beta}\; N_i$$

Reinterpretation of Loop equation

$$\frac{f(z)}{4} = W(z)^2 + V'(z)W(z) + \frac{\hbar Q}{2}W'(z) - \frac{\hbar^2}{4}g^2 W(z,z)$$

$$b = i\sqrt{\beta},\ \hbar = -2ig, \quad Q = b + 1/b$$

$$\frac{f(z) + V'^2 + \hbar Q V''}{4} = (W(z) + \frac{V'}{2})^2 + \frac{\hbar Q}{2}(W(z) + \frac{V'}{2})' - \frac{\hbar^2}{4}W(z,z)$$

Use the definition of f(z) and V' to get

$$LHS = g^2 \left(\sum_k \frac{\Delta_k}{(z - z_k)^2} + \frac{1}{z - z_k}\frac{\partial}{\partial z_k}\log Z_{\text{eff}} \right)$$

$$Z_{\text{eff}} = Z_{\beta-\text{Penner}} \prod_{0 \le a < b \le n} (z_a - z_b)^{-2\alpha_a \alpha_b}$$

Reinterpretation of Loop equation

$$\begin{aligned}\Phi &\equiv g^2 \left(\sum_k \frac{\Delta_k}{(z-z_k)^2} + \frac{1}{z-z_k} \frac{\partial}{\partial z_k} \log Z_{\text{eff}} \right) \\ &= g^2 \frac{\langle \text{out} | T | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle} \quad \text{if} \quad Z_{\text{eff}} = \langle \text{out} | \text{in} \rangle \,.\end{aligned}$$

$$\text{Solve} \quad \Phi = (W(z) + \frac{V'}{2})^2 + \frac{\hbar Q}{2}(W(z) + \frac{V'}{2})' - \frac{\hbar^2}{4} W(z,z)$$

$$\text{using} \quad \Phi = \frac{P_{2n}(z)}{\prod_{k=0}^{n}(z-z_k)^2} \ \text{and} \quad \frac{1}{2\pi i}\oint_{\Gamma_i} W(z) dz = g\sqrt{\beta}\ N_i$$