Semilocal Popov Equations

Recent Development in Theoretical Physics

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Based on

CK, Phys. Lett. B 733 (2014) 253 [arXiv:1404.3695]
CK, arXiv:1405.6651

Popov Equations

- Vortex-type equations on a 2-sphere

Popov (2012)

- Obtained by a dimensional reduction of SU(1,1) YM instanton equations on $S^2 \times H^2$

- Integrable when the scalar curvature vanishes

$$R_M = R_{S^2} + R_{H^2} = 2\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right)$$

- Explicit solutions constructed by Manton (2012)

  ~ Liouville solutions

- S^2 with radius $R_1 = \sqrt{2}$

$$ds^2 = \Omega dz d\bar{z} \qquad \Omega = \frac{8}{(1+|z|^2)^2}$$

- Popov equations

$$D_{\bar{z}}\phi \equiv \partial_{\bar{z}}\phi - i a_{\bar{z}}\phi = 0\,,$$
$$F_{z\bar{z}} = -\frac{2i}{(1+|z|^2)^2}(C^2 - |\phi|^2)\,.$$

$$C = R_1/R_2 = \sqrt{2}/R_2$$ (R_2 = Radius of H^2)

Sign flipped in the 2nd eq.

➔ Usual Bogomolny eq. of abelian Higgs model on S^2

1st eq. → $a_{\bar z} = -i\partial_{\bar z} \ln \phi$

$$F_{z\bar z} = -i\partial_z \partial_{\bar z} \ln |\phi|^2$$

Then, away from the zeros of ϕ,

$$\partial_z \partial_{\bar z} \ln |\phi|^2 = \frac{2}{(1+|z|^2)^2}(C^2 - |\phi|^2)$$

- Popov eqs can be obtained from an energy function

$$E = \frac{1}{2}\int_{S^2} \left[\frac{4}{\Omega}|F_{z\bar z}|^2 - 2(|D_z\phi|^2 + |D_{\bar z}\phi|^2) + \frac{\Omega}{4}(C^2 - |\phi|^2)^2\right] \frac{i}{2} dz \wedge d\bar z$$
$$= \frac{1}{2}\int_{S^2} \left\{-\frac{4}{\Omega}\left[F_{z\bar z} + i\frac{\Omega}{4}(C^2 - |\phi|^2)\right]^2 - 4|D_{\bar z}\phi|^2\right\} \frac{i}{2} dz \wedge d\bar z - \pi C^2 N$$

$$N = \frac{1}{2\pi}\int_{S^2} F_{z\bar z} dz \wedge d\bar z$$: vortex number (1st Chern number)

Nonrel. U(1) CS Matter Theory

Popov eqs can also arise as follows.

- (2+1) dimensional U(1) CS theory on S^2
  - with a nonrelativistic matter
  - In the presence of a uniform external magnetic field

$$S = \int dt \int_{S^2} \left[\frac{\kappa}{2} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \Omega(i\phi^* D_t \phi - V) - (|\tilde{D}_z \phi|^2 + |\tilde{D}_{\bar{z}} \phi|^2) \right] \frac{i}{2} dz \wedge d\bar{z}$$

$$D_t \phi = (\partial_t - i a_t)\phi$$

$$\tilde{D}_z \phi = (\partial_z - i a_z - i A_z^{ex})\phi$$

$$A_{\bar{z}}^{ex} = \frac{i}{2} \frac{g z}{1 + |z|^2}$$

: Uniform external magnetic field on S^2 with magnetic charge g

$$V = -\frac{g}{8}|\phi|^2 + \frac{1}{2\kappa}|\phi|^4$$

- Gauss constraint $$F_{z\bar{z}} = -i\frac{\Omega}{2\kappa}|\phi|^2$$
- Energy

$$E = \int_{S^2} (|\tilde{D}_z\phi|^2 + |\tilde{D}_{\bar{z}}\phi|^2 + \Omega V)\frac{i}{2}dz \wedge d\bar{z}$$
$$= 2\int_{S^2} |\tilde{D}_{\bar{z}}\phi|^2 \frac{i}{2}dz \wedge d\bar{z}$$

where we used, up to a total derivative,

$$|\tilde{D}_z\phi|^2 = |\tilde{D}_{\bar{z}}\phi|^2 - i(F_{z\bar{z}} + F^{ex}_{z\bar{z}})|\phi|^2$$
$$= |\tilde{D}_{\bar{z}}\phi|^2 - \frac{\Omega}{2\kappa}|\phi|^4 + \frac{g}{8}\Omega|\phi|^2$$

- Energy vanishes if $\tilde{D}_{\bar{z}}\phi = 0$
- Combining two equations, we have

$$\partial_z\partial_{\bar{z}} \ln|\phi|^2 = -\frac{\Omega}{2\kappa}\left(\frac{\kappa g}{4} - |\phi|^2\right)$$

$\kappa = -2, g = -2C^2$ ➔ Popov equation

Solutions

- Only C=1 case is integrable[Popov 2012]. From now on we let C=1.
- Integrate the 2nd equation:

$$N = -2 + \int_{S^2} \Omega |\phi|^2 \frac{i}{2} dz \wedge d\bar{z} \ \geq \ -2$$

- Note that

$$\partial\bar{\partial} \ln(1+|z|^2) = \frac{1}{(1+|z|^2)^2}$$

- Define $|\phi|^2 = (1+|z|^2)^2 e^v$
- v satisfies the Liouville equation

$$\partial\bar{\partial} v = -2e^v$$

  - Exact solutions:

$$e^v = \frac{|R'(z)|^2}{(1+|R(z)|^2)^2}$$

$R(z)$: rational functions on S^2

- Then, with a suitable local gauge choice,

$$\phi = \frac{(1+|z|^2)R'(z)}{1+|R(z)|^2}$$

Manton (2012)

$$a_{\bar{z}} = i\left[\frac{R(z)\overline{R'(z)}}{1+|R(z)|^2} - \frac{z}{1+|z|^2}\right]$$

- $R(z)$: ratio of polynomials of degree n
  ➔ Vortex number: $N = 2n - 2$ (even)
- The simplest solution:

$$R(z) = z$$
$$\phi = 1$$
$$a = 0$$

- Alternative form: Let

$$R(z) = \frac{Q(z)}{P(z)}$$

$P(z)$, $Q(z)$: polynomials without common zeros (generically of order n)

Then we can rewrite

$$e^v = \frac{|P(z)Q'(z) - Q(z)P'(z)|^2}{(|P(z)|^2 + |Q(z)|^2)^2}$$

A natural choice of gauge is

$$\phi = \frac{(1+|z|^2)(P(z)Q'(z) - Q'(z)P(z))}{(|P(z)|^2 + |Q(z)|^2)}$$

$$a_{\bar{z}} = i\left[\frac{P(z)\overline{P'(z)} + Q(z)\overline{Q'(z)}}{|P(z)|^2 + |Q(z)|^2} - \frac{z}{1+|z|^2}\right]$$

Semilocal Popov Equations

- Two matter fields with global SU(2) symmetry

$$S = \int dt \int_{S^2} \left\{ \frac{\kappa}{2}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \Omega(i\bar{\phi}_1 D_t \phi_1 + i\bar{\phi}_2 D_t \phi_2 - V) - \sum_{i=1}^{2}(|\tilde{D}_z \phi_i|^2 + |\tilde{D}_{\bar{z}} \phi_i|^2) \right\} \frac{i}{2} dz \wedge d\bar{z}$$

$$V = -\frac{g}{8}(|\phi_1|^2 + |\phi_2|^2) + \frac{1}{2\kappa}(|\phi_1|^2 + |\phi_2|^2)^2$$

- The corresponding Bogomolny eqs are

$$D_{\bar{z}}\phi_i \equiv \partial_{\bar{z}}\phi_i - i a_{\bar{z}}\phi_i = 0\,, \qquad (i = 1, 2)$$

$$F_{z\bar{z}} = -\frac{2i}{(1+|z|^2)^2}(1 - |\phi_1|^2 - |\phi_2|^2)\,,$$

- For both ϕ_1 and ϕ_2,

$$a_{\bar{z}} = -i\partial_{\bar{z}} \ln \phi_i \quad \Rightarrow \quad \partial_{\bar{z}} \ln \left(\frac{\phi_2}{\phi_1} \right) = 0$$

Then $w(z) \equiv \dfrac{\phi_2}{\phi_1}$ is locally holomorphic.

- ϕ_i have zeros at discrete points. [Taubs 1980]

  ➔ $w(z)$ should be a rational function.

- Eliminate the gauge field:

$$\partial_z \partial_{\bar{z}} \ln |\phi_1|^2 = \partial_z \partial_{\bar{z}} \ln |\phi_2|^2 = \frac{2}{(1+|z|^2)^2} (1 - |\phi_1|^2 - |\phi_2|^2)$$

Except the flipped sign, this is the semilocal abelian Higgs vortex equation on S^2.

- As before, define

$$e^{u_i} = \frac{|\phi_i|^2}{(1+|z|^2)^2}$$

Then we have Toda-type equations

$$\partial_z \partial_{\bar{z}} u_i = -K_{ij} e^{u_j}, \qquad K = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

- This is not integrable but some exact solutions are known. [CK 1992; CK, P. Ko, B.-H. Lee, C. Lee, H. Min 1993]

Some Exact Solutions

- Trivial (Liouville) solution: $\phi_2 = c\phi_1$
  ➔ reduces to the Popov equation with a single ϕ
- To obtain other solutions, we introduce

$$e^{u_1} = \frac{|w'|^2}{(1+|w|^2)^3} e^v$$

- Then we have

$$\partial_w \partial_{\bar{w}} v = \frac{2}{(1+|w|^2)^2} \left(\frac{3}{2} - e^v \right)$$

- Therefore, the Semilocal Popov equation with C=1 reduces to the Popov equation with $C = \sqrt{3/2}$.

- $C \neq 1$ Popov eq is not integrable.
  General solutions are not known.
- Nevertheless, we know one solution:

$$e^{v} = \frac{3}{2}$$

- This (trivial) solution gives a family of exact solutions to the semilocal Popov equation,

$$|\phi_1|^2 = \frac{3}{2}\frac{(1+|z|^2)^2|w'|^2}{(1+|w|^2)^3}$$

- Given $|\phi_1|^2$, we can construct other fields. At this time, $w(z)$ plays the role of $R(z)$.

- Let $w(z) = \frac{Q(z)}{P(z)}$. Then

$$\begin{pmatrix} |\phi_1|^2 \\ |\phi_2|^2 \end{pmatrix} = \frac{3}{2} \frac{(1+|z|^2)^2 |P(z)Q'(z) - Q(z)P'(z)|^2}{(|P(z)|^2 + |Q(z)|^2)^3} \begin{pmatrix} |P(z)|^2 \\ |Q(z)|^2 \end{pmatrix}$$

- With a local gauge choice, the solutions are

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{(1+|z|^2)(P(z)Q'(z) - Q(z)P'(z))}{(|P(z)|^2 + |Q(z)|^2)^{3/2}} \begin{pmatrix} P(z) \\ Q(z) \end{pmatrix}$$

$$a_{\bar{z}} = i \left[\frac{3}{2} \frac{P(z)\overline{P'(z)} + Q(z)\overline{Q'(z)}}{|P(z)|^2 + |Q(z)|^2} - \frac{z}{1+|z|^2} \right].$$

- This is a genuinely different family of solutions from the Liouville solutions.
  - ϕ_1 and ϕ_2 share only a part of vortex points.
  - For a generic rational function $w(z)$ of degree n,

$$N = 3n - 2$$

  - $N = 1$ solution belongs to this family.

Circular Symmetric Solution

Choose $P(z) = c^n$, $Q(z) = z^n$. ($c > 0$ for simplicity)

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{nc^n(1+|z|^2)z^{n-1}}{(c^{2n}+|z|^{2n})^{3/2}} \begin{pmatrix} c^n \\ z^n \end{pmatrix}$$

- Vortices at $z = 0$ with multiplicities $n - 1$ and $2n - 1$ for ϕ_1 and ϕ_2, respectively.
  - c: size parameter
- In terms of the coordinate $\xi = 1/x$,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \left(\frac{|\xi|}{\xi}\right)^{3n-2} \frac{nc^{-n}(1+|\xi|^2)\xi^{n-1}}{(c^{-2n}+|\xi|^{2n})^{3/2}} \begin{pmatrix} \xi^n \\ c^{-n} \end{pmatrix}$$

  - Vortices at $\xi = 0$ ($z = \infty$) with multiplicities $2n - 1$ and $n - 1$, respectively, so that the total vortex number is $3n - 2$.

$N = 1$ solution with reflection symmetry ($n = c = 1$).

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \sqrt{\frac{3}{2}} \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 \\ z \end{pmatrix}$$

- This is precisely the CP^1 lump configuration with unit size satisfying

$$|\phi_1|^2 + |\phi_2|^2 = \frac{3}{2}$$

  which defines S^3 fibered as a circle bundle over CP^1.

- Gauge field

$$a_{\bar{z}} = \frac{i}{2} \frac{z}{1+|z|^2}$$

$$F_{z\bar{z}} = \frac{i}{(1+|z|^2)^2}$$

: Uniform magnetic field with g=1

More Matter Fields

- M matter fields with SU(M) global symmetry

$$F_{z\bar{z}} = -i\frac{\Omega}{2\kappa}\sum_{k=1}^{M}|\phi_k|^2$$

$$\tilde{D}_{\bar{z}}\phi_k = 0, \qquad (k = 1, \ldots, M)$$

$$a_{\bar{z}} + A_{\bar{z}}^{ex} = -i\partial_{\bar{z}}\ln\phi_k, \qquad (k = 1, \ldots, M)$$

$$\partial_{\bar{z}}\ln\left(\frac{\phi_k}{\phi_1}\right) = 0; \quad \rightarrow \quad w_k(z) \equiv \frac{\phi_k}{\phi_1}$$: rational functions

$$F_{z\bar{z}} = -i\partial_z\partial_{\bar{z}}\ln|\phi_k|^2 - i\frac{g}{8}\Omega$$

$$\partial_z\partial_{\bar{z}}\ln|\phi_k|^2 = -\frac{\Omega}{2\kappa}\left(\frac{\kappa g}{4} - \sum_{k=1}^{M}|\phi_k|^2\right)$$

$(\kappa = -2, g = -2C^2)$

Solutions

- Embeding of fewer flavor case:

  Given solutions $\tilde{\phi}_k \quad (k = 1, \ldots, m)$,

$$\phi_k = \sum_{l=1}^{m} U_{kl}\tilde{\phi}_l, \qquad (U \in \mathrm{SU}(M))$$

  – Example: For $m = 2$, the previous $N = 1$ solution is embedded as a CP^{M-1} lump configuration

$$\phi_k = \sqrt{\frac{3}{2}} \frac{U_{k1} + U_{k2} z}{\sqrt{1 + |z|^2}},$$

$$a_z = \frac{3i}{2} \frac{z}{1 + |z|^2}.$$

$$\sum_{k=1}^{M} |\phi_k|^2 = \frac{3}{2}$$

- To find other solutions, we rewrite the eq. as

$$\partial_z \partial_{\bar{z}} u_1 = -2 \sum_{k=1}^{M} |w_k(z)|^2 e^{u_1}$$

- Find solutions for some simple $w_k(z)$'s.

$$w_k(z) = \sqrt{\binom{M-1}{k-1}} w^{k-1}(z)$$

Ghosh(1996)

$$\partial_z \partial_{\bar{z}} u_1 = -2(1+|w(z)|^2)^{M-1} e^{u_1}$$

$$e^{u_1} = \frac{|w'|^2}{(1+|w|^2)^{M+1}} e^{v} \quad \rightarrow \quad \partial_w \partial_{\bar{w}} v = \frac{2}{(1+|w|^2)^2} \left(\frac{M+1}{2} - e^{v} \right)$$

: $C = \sqrt{\frac{M+1}{2}}$ Popov eq.

Corresponds to the external magnetic charge $g = -(M+1)$

- $e^v = \frac{M+1}{2}$ gives the solution

$$\phi_k = \sqrt{\frac{M+1}{2}\binom{M-1}{k-1}}\frac{(1+|z|^2)P^{M-k}Q^{k-1}(PQ'-QP')}{(|P|^2+|Q|^2)^{\frac{M+1}{2}}}$$

$$a_{\bar{z}} = i\frac{M+1}{2}\frac{P\overline{P'}+Q\overline{Q'}}{|P|^2+|Q|^2},$$

- Vortex number $N = n(M+1) - 2$
- Example: $P = 1$, $Q = z$

$$\phi_k = \sqrt{\frac{M+1}{2}\binom{M-1}{k-1}}\frac{z^{k-1}}{(1+|z|^2)^{\frac{M-1}{2}}} \quad \rightarrow \quad \sum_{k=1}^{M}|\phi_k|^2 = \frac{M+1}{2}$$

$$a_{\bar{z}} = i\frac{M+1}{2}\frac{z}{1+|z|^2}.$$

: CP^{M-1} config. with bigger radius

- To obtain other solutions, note the identity

$$\partial_z \partial_{\bar{z}} \ln(1 + |w(z)|^2) = \frac{|w'(z)|^2}{(1+|w(z)|^2)^2}$$

- This can be rewritten as

$$\partial_z \partial_{\bar{z}} \ln \frac{1}{(1+|w(z)|^2)^{M+1}} = -2\frac{M+1}{2} \frac{|w'(z)|^2(1+|w(z)|^2)^{M-1}}{(1+|w(z)|^2)^{M+1}}$$

This form gives the previous solution.

- Other utilization

$$\partial_z \partial_{\bar{z}} \ln \prod_{k=1}^{n} \frac{1}{(c_k^2 + |w(z)|^2)^2} = -2\sum_{k=1}^{n} \frac{c_k^2 |w'(z)|^2}{(c_k^2+|w(z)|^2)^2}$$

$$\equiv -2\frac{|w'(z)|^2}{\prod_{k=1}^{n}(c_k^2+|w(z)|^2)^2} \sum_{k=0}^{2n-2} p_k |w(z)|^{2k}$$

→ $e^{u_k} = \dfrac{p_k |w'(z)|^2 |w(z)|^{2k}}{\prod_{k=1}^{n}(c_k^2+|w(z)|^2)^2}$ solves the eq. with $M = 2n-1$ and $w_k(z) = \sqrt{p_k} w^k(z)$

- Further generalization

$$\partial_z \partial_{\bar{z}} \ln \prod_{k=1}^{n} \frac{1}{(c_k^2 + |w(z)|^2)^{n_k+1}} = -2 \sum_{k=1}^{n} \frac{n_k+1}{2} \frac{c_k^2 |w'(z)|^2 (c_k^2 + |w(z)|^2)^{n_k-1}}{(c_k^2 + |w(z)|^2)^{n_k+1}}$$

$$\equiv -2 \frac{|w'(z)|^2}{\prod_{k=1}^{n} (c_k^2 + |w(z)|^2)^{n_k+1}} \sum_{k=0}^{M-1} p_k |w(z)|^{2k},$$

➔ we obtain the solution (with $M = \sum_k n_k + n - 1$)

$$e^{u_k} = \frac{p_k |w'(z)|^2 |w(z)|^{2k}}{\prod_{k=1}^{n} (c_k^2 + |w(z)|^2)^{n_k+1}}$$

Up to SU(M) transformation, this is the most general solution we constructed with one rational function $w(z)$.

Solutions with more than one rational functions

- Generalized identity

Dunne, Jackiw, Pi, Trugenberger (1991)

$$\partial_z \partial_{\bar{z}} \ln \left(\sum_{i=1}^{n} |f_i(z)|^2 \right) = \frac{\sum_{i<j} |f_{ij}(z)|^2}{(\sum_{i=1}^{n} |f_i(z)|^2)^2}$$

where $f_i(z)$: arbitrary rational functions

$$f_{ij} = f_i f_j' - f_j f_i'$$

- Then

$$e^{u_{ij}} = \frac{|f_{ij}(z)|^2}{(\sum_{i=1}^{n} |f_i(z)|^2)^2}$$

satisfies

$$\partial_z \partial_{\bar{z}} u_{ij} = -2 \sum_{i<j} e^{u_{ij}}$$

- Obviously, further generalization similar to the case of one rational function is possible.

Conclusion

- We derived (semilocal) Popov equations from U(1) CS theory with nonrelativistic matter fields on S^2
- Two matter fields with SU(2) symm.
  - shown to be equivalent to the Popov eq with $C = \sqrt{3/2}$
  - constructed two families of exact solutions
    - Liouville-type sol: $N = 2n - 2$
    - The other family: $N = 3n - 2$
  - $N = 1\ CP^1$ lump configuration solves the equation
  - For $N = 6k - 2$, we have two distinct families of solutions generated by rational functions of different degrees
    - unknown whether they are smoothly connected to each other
- More solutions found for more than two matter fields