Stringy Differential Geometry and Double Field Theory

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Recent Development in Theoretical Physics

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Prologue

- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
  - Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0, \ \Gamma^\lambda_{[\mu\nu]} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$
- On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing, as they form a multiplet of T-duality.
- This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.
- Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.

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- My talk today aims to introduce such a **Stringy Geometry** which is defined in **doubled-yet-gauged** spacetime.

- In four-dimensional spacetime photon has two physical degrees of freedom, but can be best described by a four component vector.

- Similarly, D-dimensional spacetime may be better understood in terms of **doubled-yet-gauged** $(D+D)$ coordinates.

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Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  arXiv:1011.1324 JHEP
- Double field formulation of Yang-Mills theory arXiv:1102.0419 PLB
- Stringy differential geometry, beyond Riemann arXiv:1105.6294 PRD
- Incorporation of fermions into double field theory arXiv:1109.2035 JHEP
- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity
  arXiv:1112.0069 PRD Rapid Comm.
- Ramond-Ramond Cohomology and O(D,D) T-duality arXiv:1206.3478 JHEP
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory** arXiv:1210.5078 PLB
- Comments on double field theory and diffeomorphisms arXiv:1304.5946 JHEP
- Covariant action for a string in doubled yet gauged spacetime arXiv:1307.8377 NPB

Parallel works on U-duality

- U-geometry: SL(5) with Yoonji Suh arXiv:1302.1652 JHEP
- M-theory and F-theory from a Duality Manifest Action with Chris Blair and Emanuel Malek arXiv:1311.5109 JHEP
- U-gravity: SL(N) with Yoonji Suh arXiv:1402.5027 JHEP

Supergravity

- The low energy effective action of $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ is well known in terms of Riemannian geometry

$$S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g}e^{-2\phi}\left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right).$$

- Diffeomorphism and B-field gauge symmetry are manifest,

$$x^\mu \to x^\mu + \delta x^\mu\,, \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu\,.$$

- Though not manifest, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. Buscher

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T-duality

- Redefine the dilaton,

$$e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- Set a $(D+D)\times(D+D)$ symmetric matrix, Duff

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$$

- Hereafter, $A, B, \dots$: 'doubled' $(D+D)$-dimensional vector indices, with $D = 10$ for SUSY.

- T-duality is realized as an $\mathbf{O}(D,D)$ rotation in doubled spacetime Tseytlin, Siegel

$$\mathcal{H}_{AB} \longrightarrow M_A{}^C M_B{}^D \mathcal{H}_{CD}\,, \qquad\qquad d \longrightarrow d\,,$$

where

$$M \in \mathbf{O}(D,D)\,.$$

T-duality

- $\mathbf{O}(D, D)$ metric,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

freely raises or lowers the $(D + D)$-dimensional vector indices.

- Hull and Zwiebach (later with Hohm) reformulated the effective action under the name, 'Double Field Theory', in an $\mathbf{O}(D,D)$ manifest manner:

$$S_{\mathrm{DFT}} = \int_{\Sigma_D} e^{-2d}\, L_{\mathrm{DFT}}(\mathcal{H}, d)\,,$$

where

$$L_{\mathrm{DFT}}(\mathcal{H},d) = \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) \\ +4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.$$

- Spacetime is formally doubled, $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1,2,\cdots, D{+}D$.
- Yet, Double Field Theory (for NS-NS sector) is a D-dimensional theory written in terms of $(D+D)$-dimensional language, i.e. tensors.
- All the fields MUST live on a D-dimensional null hyperplane or 'section', Σ_D.

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Section condition in Double Field Theory

- By stating DFT lives on a D-dimensional null hyperplane, we mean that, the $\mathbf{O}(D,D)$ d'Alembert operator is trivial, acting on arbitrary fields as well as their products:

$$\partial_A\partial^A\Phi = 2\frac{\partial^2}{\partial\tilde{x}_\mu\partial x^\mu}\Phi \equiv 0\,, \qquad \partial_A\Phi_1\partial^A\Phi_2 \equiv 0 \quad : \quad \text{section condition}$$

- The origin of the section condition may be traced to the 'level matching condition' of the massless sector on the worldsheet,

$$p\cdot w \equiv 0 \quad \Longleftrightarrow \quad \partial_A\partial^A = 2\frac{\partial^2}{\partial x^\mu\partial\tilde{x}_\mu} \equiv 0\,.$$

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Double Field Theory (DFT), 2009 - 2010

- Thus, in the DFT formulation of the effective action by Hull, Zwiebach & Hohm the $\mathbf{O}(D, D)$ T-duality structure is manifest,

$$L_{\mathrm{DFT}}(\mathcal{H}, d) = \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) + 4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.$$

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- **Key concepts are**
  - **Projector**
  - **Semi-covariant derivative**
  - **Semi-covariant curvature**
  - **And their complete covariantization via 'projection'**

Geometric Constitution of Double Field Theory

- *Notation*

Capital Latin alphabet letters denote the $\mathbf{O}(D,D)$ vector indices, i.e. $A,B,C,\cdots=1,2,\cdots,D{+}D$, which can be freely raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right).$$

- *Doubled-yet-gauged spacetime*

The spacetime is formally doubled, being $(D{+}D)$-dimensional.

However, **the doubled spacetime is gauged** : the coordinate space is equipped with an *equivalence relation*,

$$x^{A} \;\sim\; x^{A}+\phi\partial^{A}\varphi\,,$$

which we call '*coordinate gauge symmetry*'.

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Each equivalence class, or gauge orbit, represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

- *Realization of the coordinate gauge symmetry.*

The equivalence relation is realized in DFT by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\,, \qquad\qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

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Explicitly, acting on arbitrary functions, Φ, Φ', and their products, we have

$$\partial_A \partial^A \Phi = 0 \qquad \text{(weak constraint)}\,,$$

$$\partial_A \Phi \partial^A \Phi' = 0 \qquad \text{(strong constraint)}\,.$$

- *Diffeomorphism.*

Diffeomorphism symmetry in $\mathbf{O}(D, D)$ DFT is generated by a generalized Lie derivative Siegel, Courant, Grana

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

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where ω_T denotes the weight.

In particular, the generalized Lie derivative of the $\mathbf{O}(D,D)$ invariant metric is trivial,

$$\hat{\mathcal{L}}_X \mathcal{J}_{AB} = 0\,.$$

The commutator of the generalized Lie derivatives is closed by C-bracket,

$$\left[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y\right] = \hat{\mathcal{L}}_{[X,Y]_{\mathrm{C}}}\,, \qquad [X,Y]^A_{\mathrm{C}} = X^B\partial_B Y^A - Y^B\partial_B X^A + \tfrac{1}{2}Y^B\partial^A X_B - \tfrac{1}{2}X^B\partial^A Y_B\,.$$

- *Dilaton and a pair of two-index projectors.*

The geometric objects in DFT consist of a dilation, d, and a pair of symmetric projection operators,

$$P_{AB} = P_{BA}\,, \qquad \bar{P}_{AB} = \bar{P}_{BA}\,, \qquad P_{A}{}^{B}P_{B}{}^{C} = \delta_{A}{}^{C}\,, \qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C} = \delta_{A}{}^{C}\,.$$

Further, the projectors are orthogonal and complementary,

$$P_{A}{}^{B}\bar{P}_{B}{}^{C} = 0\,, \qquad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,.$$

- *Dilaton and a pair of two-index projectors.*

The geometric objects in DFT consist of a dilation, d, and a pair of symmetric projection operators,

$$P_{AB} = P_{BA}\,, \qquad \bar{P}_{AB} = \bar{P}_{BA}\,, \qquad P_{A}{}^{B}P_{B}{}^{C} = \delta_{A}{}^{C}\,, \qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C} = \delta_{A}{}^{C}\,.$$

Further, the projectors are orthogonal and complementary,

$$P_{A}{}^{B}\bar{P}_{B}{}^{C} = 0\,, \qquad P_{AB} + \bar{P}_{AB} = \mathcal{J}_{AB}\,.$$

Remark: The difference of the two projectors, $P_{AB} - \bar{P}_{AB} = \mathcal{H}_{AB}$, corresponds to the "generalized metric" which can be also independently defined as a symmetric $\mathbf{O}(D, D)$ element, i.e. $\mathcal{H}_{AB} = \mathcal{H}_{BA}$, $\mathcal{H}_{A}{}^{B}\mathcal{H}_{B}{}^{C} = \delta_{A}{}^{C}$. However, in supersymmetric double field theories it appears that the projectors are more fundamental than the "generalized metric".

- *Integral measure.*

While the projectors are weightless, the dilation gives rise to the $\mathbf{O}(D, D)$ invariant integral measure with weight one, after exponentiation,

$$e^{-2d}\,.$$

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

We define a semi-covariant derivative,

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,,$$

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and a semi-covariant Riemann curvature,

$$S_{ABCD} := \tfrac{1}{2} \left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB} \Gamma_{ECD} \right) .$$

Here R_{ABCD} denotes the ordinary “field strength" of a connection,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} .$$

Geometric Constitution of Double Field Theory

- *Semi-covariant derivative and semi-covariant Riemann curvature.*

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We may choose the (torsionless) connection to be

$$\begin{aligned}\Gamma_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) .\end{aligned}$$

The semi-covariant derivative then obeys the Leibniz rule and annihilates the $\mathbf{O}(D,D)$ invariant constant metric,

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A crucial defining property of the semi-covariant Riemann curvature is that, under arbitrary transformation of the connection, it transforms as total derivative,

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Further, the semi-covariant Riemann curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}\,, \qquad S_{[ABC]D} = 0\,,$$

as well as additional identities concerning the projectors,

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,, \qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,.$$

It follows that

$$S^{AB}{}_{AB} = 0\,.$$

- *The uniqueness of the torsionless connection.*

The connection is the unique solution to the following five constraints:

$$\nabla_A P_{BC} = 0\,, \qquad \nabla_A \bar{P}_{BC} = 0\,,$$

$$\nabla_A d = -\tfrac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \tfrac{1}{2} \Gamma^B{}_{BA} = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{ACB} = 0\,,$$

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- The first two relations are the compatibility conditions with all the geometric objects –or NS-NS sector– in DFT.
- The third constraint is the compatibility condition with the $\mathbf{O}(D,D)$ invariant constant metric, *i.e.* $\nabla_A \mathcal{J}_{BC} = 0$.

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- The next cyclic property makes the semi-covariant derivative compatible with the generalized Lie derivative as well as with the C-bracket,

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla)\,, \qquad [X,Y]_{\rm C}(\partial) = [X,Y]_{\rm C}(\nabla)\,.$$

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- *Six-index projection operators.*

The six-index projection operators are explicitly,

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} \,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D} \,,$$

which satisfy the 'projection' properties,

$$\mathcal{P}_{ABC}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI} \,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI} \,.$$

Further, they are symmetric and traceless,

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{DEFABC} \,, \qquad \mathcal{P}_{ABCDEF} = \mathcal{P}_{A[BC]D[EF]} \,, \qquad P^{AB} \mathcal{P}_{ABCDEF} = 0 \,,$$

$$\bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{DEFABC} \,, \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{A[BC]D[EF]} \,, \qquad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0 \,.$$

Crucially, the projection operator dictates the anomalous terms in the diffeomorphic transformations of the semi-covariant derivative and the semi-covariant Riemann curvature,

$$(\delta_X-\hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n}=\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E X_F\, T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$(\delta_X-\hat{\mathcal{L}}_X)S_{ABCD}=2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F X_G\Big)+2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F X_G\Big).$$

- *Complete covariantizations.*

Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized, through appropriate contractions with the projectors:

$$P_C{}^D\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\,,\qquad \bar{P}_C{}^D P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_D T_{B_1\cdots B_n}\,,$$

$$P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_A T_{BD_1\cdots D_n}\,,\qquad \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_A T_{BD_1\cdots D_n}\qquad\text{(divergences)}\,,$$

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and

$$P_A{}^C\bar{P}_B{}^D S_{CED}{}^E\qquad\text{(Ricci curvature)}\,,$$

$$(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\qquad\text{(scalar curvature)}\,.$$

- *Action.*

The action of $\mathbf{O}(D,D)$ DFT is given by the fully covariant scalar curvature,

$$\int_{\Sigma_D} e^{-2d}(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\,,$$

where the integral is taken over a section, Σ_D.

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The dilation and the projector equations of motion correspond to the vanishing of the scalar curvature and the Ricci curvature respectively.

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Note: It is precisely the above expression that allows the '1.5 formalism' to work in the full order supersymmetric extensions of $\mathcal{N} = 1, 2$, $D = 10$ Jeon-Lee-JHP

- *Section.*

Up to $\mathbf{O}(D,D)$ duality rotations, the solution to the section condition is unique. It is a D-dimensional section, Σ_D, characterized by the independence of the dual coordinates, i.e.

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0\,,$$

while the whole doubled coordinates are given by

$$x^A = (\tilde{x}_\mu, x^\nu)\,,$$

where μ, ν are now D-dimensional indices.

- *Riemannian reduction.*

To perform the Riemannian reduction to the D-dimensional section, Σ_D, we parametrize the dilation and the projectors in terms of D-dimensional Riemannian metric, $G_{\mu\nu}$, ordinary dilaton, ϕ, and a Kalb-Ramond two-form potential, $B_{\mu\nu}$,

$$P_{AB} - \bar{P}_{AB} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|G|}e^{-2\phi}.$$

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The DFT scalar curvature then reduces upon the section to

$$(P^{AC}P^{BD} - \bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\Big|_{\Sigma_D} = R_G + 4\Delta\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu},$$

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where as usual, $H_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]}$.

Up to field redefinitions, the above is the most general parametrization of the "generalized metric", $\mathcal{H}_{AB} = P_{AB} - \bar{P}_{AB}$, when the upper left $D \times D$ block of it is non-degenerate.

- *Non-Riemannian backgrounds.*

When the upper left $D\times D$ block of $\mathcal{H}_{AB}=(P-\bar{P})_{AB}$ is degenerate –where G^{-1} might be positioned – the Riemannian metric ceases to exist upon the section, Σ_D.

Nevertheless, the $\mathbf{O}(D,D)$ DFT and a doubled sigma model –which I will discuss later– have no problem with describing such a non-Riemannian background.

An extreme example of such a non-Riemannian background is the flat background where

$$\mathcal{H}_{AB}=(P-\bar{P})_{AB}=\mathcal{J}_{AB}\,.$$

This is a vacuum solution to the bosonic $\mathbf{O}(D,D)$ DFT and the corresponding doubled sigma model reduces to a certain 'chiral' sigma model.

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Allowing non-Riemannian backgrounds, DFT is NOT a mere reformulation of SUGRA.
c.f. 'global aspects' Berman, Cederwall, Perry, Marques, Kanghoon Lee, Grana

Further Remarks

Based on the differential geometry I just described,

after incorporating fermions and R-R sector,

it is possible to construct, to the full order in fermions,

Type II, or $\mathcal{N}=2$, $D=10$ Supersymmetric Double Field Theory

of which the Lagrangian reads

$$\begin{array}{l}\mathcal{L}_{\rm Type\,II}=e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}\\ \quad+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\end{array}$$

Jeon-Lee-Suh-JHP

Symmetries of $\mathcal{N}=2$ $D=10$ SDFT

- **O**(D,D) **T-duality**
- **Gauge symmetries**
  1. **DFT-diffeomorphism (generalized Lie derivative)**
  2. **A *pair* of local Lorentz symmetries, Spin**$(1,D{-}1)_L \times$ **Spin**$(D{-}1,1)_R$
  3. **local** $\mathcal{N}=2$ **SUSY with** 32 **supercharges.**
- **All the bosonic symmetries are realized manifestly and simultaneously.**
- **For this, it is crucial to have the right field variables:**

$$d\,,\quad V_{Ap}\,,\quad \bar{V}_{A\bar{p}}\,,\quad \mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \rho^{\alpha}\,,\quad \rho'^{\bar{\alpha}}\,,\quad \psi^{\alpha}_{\bar{p}}\,,\quad \psi'^{\bar{\alpha}}_{p}$$

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- The section condition is equivalent to the 'coordinate gauge symmetry', 1304.5946

$$x^M \sim x^M + \varphi\partial^M\varphi' \,.$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

- The coordinate gauge symmetry can be realized on worldsheet, 1307.8377

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{2}\sqrt{-h}h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

where

$$D_iX^M = \partial_iX^M - \mathcal{A}_i^M\,, \qquad \mathcal{A}_i^M\partial_M \equiv 0\,.$$

- The Lagrangian is symmetric with respect to the string worldsheet diffeomorphisms, Weyl symmetry, $\mathbf{O}(D,D)$ T-duality, target spacetime generalized diffeomorphisms and the coordinate gauge symmetry, thanks to the auxiliary gauge field, $\mathcal{A}_i^M$.

c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

Comment 1: String propagates in doubled-yet-gauged spacetime

- The section condition is equivalent to the 'coordinate gauge symmetry', 1304.5946

$$x^M \sim x^M + \varphi \partial^M \varphi' \,.$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

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Comment 1: String propagates in doubled-yet-gauged spacetime

- After parametrization and integrating out $\mathcal{A}_i^M$, it can produce either the standard string action for the 'non-degenerate' Riemannian case,

$$\frac{1}{4\pi\alpha'}\mathcal{L}\equiv\frac{1}{2\pi\alpha'}\Big[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i Y^\mu\partial_j Y^\nu G_{\mu\nu}(Y)+\tfrac{1}{2}\epsilon^{ij}\partial_i Y^\mu\partial_j Y^\nu B_{\mu\nu}(Y)+\tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\Big]\,,$$

or chiral actions for 'degenerate' non-Riemannian cases, e.g. for $\mathcal{H}_{AB}=\mathcal{J}_{AB}$,

$$\frac{1}{4\pi\alpha'}\mathcal{L}\equiv\frac{1}{4\pi\alpha'}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_j Y^\mu\,,\qquad\qquad \partial_i Y^\mu+\frac{1}{\sqrt{-h}}\epsilon_i{}^j\partial_j Y^\mu=0\,.$$

c.f. Gomis-Ooguri

- Precisely analogous formalism has been developed for $\mathbf{SL}(N)$, $N \neq 4$.
  - Extended-yet-gauged spacetime ($\equiv$ section condition), $x^{ab} = -x^{ba}$
  - Diffeomorphism generated by a generalized Lie derivative
  - Semi-covariant derivative and semi-covariant curvature
  - Complete covariantizations of them dictated by a projection operator

- The action of $\mathbf{SL}(N)$ U-gravity is given by the fully covariant scalar curvature,

$$\int_{\Sigma} M^{\frac{1}{4-N}} S ,$$

where $M = \det(M_{ab})$ and the integral is taken over a section, Σ.

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Summary

- Riemannian geometry is for *particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- Novel differential geometic ingredients:
  - ▷ Projection
  - ▷ Semi-covariant derivative and curvature
  - ▷ Spacetime being doubled-yet-gauged (section condition).

- $\mathcal{N}=2$ $D=10$ SDFT unifies IIA and IIB, as well as allows non-Riemannian 'metric-less' backgrounds.

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- Further study and classification of the non-Riemannian, ‘metric-less’ backgrounds.
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- The uplift of type II SDFT to $\mathcal{M}$-theory, or the extension of T-duality to U-duality: West (E_{11}); Damour, Henneaux, Nicolai, Riccioni, Steele; Cook; Aldazabal, Graña, Marqués, Rosabal
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Thank you.

The End

Question: Is DFT a mere reformulation of SUGRA?

- YES, if we take the following as a definition of the generalized metric,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix} .$$

- NO, if we define the generalized metric as a symmetric $\mathbf{O}(D, D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA} , \qquad \mathcal{H}_A{}^C \mathcal{H}_B{}^D \mathcal{J}_{CD} = \mathcal{J}_{AB} .$$

- With this abstract definition, DFT as well as a sigma model (which I will discuss later) perfectly make sense.
- It may then describe a novel non-Riemannian string theory backgrounds, e.g.

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which does not admit any Riemannian interpretation!

- *c.f.* Global aspects such as "non-geometry" Berman-Cederwall-Perry, Papadopoulos and Scherk-Schwarz Geissbuhler, Grana-Marques, Aldazabal-Grana-Marques-Rosabal, Dibitetto-Fernandez-Melgarejo-Marques-Roest, Berman-Lee

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- All the fields are required to satisfy the section condition,

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which implies an invariance under a shift set by a 'derivative-index-valued' vector,

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    - DFT-dilaton: d
    - DFT-vielbeins: $V_{Ap}\,, \qquad \bar{V}_{A\bar{p}}$
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Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ & DFT-diffeom. vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ vector	$\eta_{pq} = \mathbf{diag}(- + + \cdots +)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D{-}1)_L$ spinor	$C_{+\alpha\beta}\,,\quad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
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R-R potential and Fermions carry NOT $(D+D)$-dimensional BUT undoubled D-dimensional indices.

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***A priori*, O(D,D) rotates only the O(D,D) vector indices (capital Roman), and the R-R sector and all the fermions are O(D,D) T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after fixing a gauge.

- The DFT-dilaton gives rise to a scalar density with weight one,

$$e^{-2d}\,.$$

- The DFT-vielbeins satisfy the **four defining properties:**

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,,\qquad \bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,,\qquad V_{Ap}\bar{V}^{A}{}_{\bar{q}}=0\,,\qquad V_{Ap}V_{B}{}^{p}+\bar{V}_{A\bar{p}}\bar{V}_{B}{}^{\bar{p}}=\mathcal{J}_{AB}\,.$$

- For fermions, the gravitinos and the DFT-dilatinos are not twenty, but ten-dimensional Majorana-Weyl spinors,

$$\gamma^{(D+1)}\psi_{\bar{p}}=\mathbf{c}\,\psi_{\bar{p}}\,,\qquad \gamma^{(D+1)}\rho=-\mathbf{c}\,\rho\,,$$

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- Lastly for the R-R sector, we set the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, to be in the **bi-fundamental** spinorial representation of $\mathbf{Spin}(1,D{-}1)_L\times\mathbf{Spin}(D{-}1,1)_R$. It possesses the chirality,

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- That is to say, $\mathcal{N}=2$ $D=10$ SDFT is chiral with respect to both **Pin**$(1,D{-}1)_L$ and **Pin**$(D{-}1,1)_R$, and the theory is unique, unlike IIA/IIB SUGRAs.
- Hence, without loss of generality, we may safely set

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- The DFT-vielbeins generate **a pair of two-index projectors,**

$$P_{AB} := V_A{}^p V_{Bp} \,, \qquad P_A{}^B P_B{}^C = P_A{}^C \,, \qquad \bar{P}_{AB} := \bar{V}_A{}^{\bar{p}} \bar{V}_{B\bar{p}} \,, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C \,,$$

which are symmetric, orthogonal and complementary to each other,

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- Further, we construct **a pair of six-index projectors,**

$$\mathcal{P}_{CAB}{}^{DEF} := P_{C}{}^{D}P_{[A}{}^{[E}P_{B]}{}^{F]} + \tfrac{2}{D-1}P_{C[A}P_{B]}{}^{[E}P^{F]D}\,, \qquad \mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}\,,$$

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- **Having all the ‘right’ field-variables prepared, we now discuss their derivatives or what we call, ‘semi-covariant derivative’.**

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Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - Γ_A for the DFT-diffeomorphism (generalized Lie derivative),
  - Φ_A for the 'unbarred' local Lorentz symmetry, $\textbf{Spin}(1,D{-}1)_L$,
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- Combining all of them, we introduce **master 'semi-covariant' derivative**

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- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A\,, \qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A\,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1A_2\cdots A_n} := \partial_C T_{A_1A_2\cdots A_n} - \omega\Gamma^B{}_{BC} T_{A_1A_2\cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,.$$

- And the latter is the covariant derivative for the $\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R$ local Lorenz symmetries.

- It is also useful to set

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- ***By definition,*** **the master derivative annihilates all the 'constants',**

$$\mathcal{D}_A\mathcal{J}_{BC}=\nabla_A\mathcal{J}_{BC}=\Gamma_{AB}{}^{D}\mathcal{J}_{DC}+\Gamma_{AC}{}^{D}\mathcal{J}_{BD}=0\,,$$

$$\mathcal{D}_A\eta_{pq}=D_A\eta_{pq}=\Phi_{Ap}{}^{r}\eta_{rq}+\Phi_{Aq}{}^{r}\eta_{pr}=0\,,$$

$$\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=D_A\bar{\eta}_{\bar{p}\bar{q}}=\bar{\Phi}_{A\bar{p}}{}^{\bar{r}}\bar{\eta}_{\bar{r}\bar{q}}+\bar{\Phi}_{A\bar{q}}{}^{\bar{r}}\bar{\eta}_{\bar{p}\bar{r}}=0\,,$$

$$\mathcal{D}_AC_{+\alpha\beta}=D_AC_{+\alpha\beta}=\Phi_{A\alpha}{}^{\delta}C_{+\delta\beta}+\Phi_{A\beta}{}^{\delta}C_{+\alpha\delta}=0\,,$$

$$\mathcal{D}_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=D_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=\bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}}\bar{C}_{+\bar{\delta}\bar{\beta}}+\bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}}\bar{C}_{+\bar{\alpha}\bar{\delta}}=0\,,$$

including the gamma matrices,

$$\mathcal{D}_A(\gamma^{p})^{\alpha}{}_{\beta}=D_A(\gamma^{p})^{\alpha}{}_{\beta}=\Phi_A{}^{p}{}_{q}(\gamma^{q})^{\alpha}{}_{\beta}+\Phi_A{}^{\alpha}{}_{\delta}(\gamma^{p})^{\delta}{}_{\beta}-(\gamma^{p})^{\alpha}{}_{\delta}\Phi_A{}^{\delta}{}_{\beta}=0\,,$$

$$\mathcal{D}_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=D_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=\bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}}(\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}+\bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}}(\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}}-(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}}\bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}}=0\,.$$

- It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}\,,$$

$$\Phi_{Apq} = -\Phi_{Aqp}\,, \qquad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}\,, \qquad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}\,,$$

  and as usual,

$$\Phi_{A}{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,, \qquad \bar{\Phi}_{A}{}^{\bar{\alpha}}{}_{\bar{\beta}} = \tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,.$$

- ***Further,*** **the master derivative is compatible with the whole NS-NS sector,**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2} e^{2d} \nabla_A (e^{-2d}) = \partial_A d + \tfrac{1}{2} \Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

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$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

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- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,,$$

$$\Phi_{Apq} = \Phi^0_{Apq} + \Delta_{Apq} \,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = \bar{\Phi}^0_{A\bar{p}\bar{q}} + \bar{\Delta}_{A\bar{p}\bar{q}} \,.$$

Here

$$\begin{aligned} \Gamma^0_{CAB} = \; & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \,, \end{aligned}$$

and, with the corresponding derivative, $\nabla^0_A = \partial_A + \Gamma^0_A$,

$$\Phi^0_{Apq} = V^B{}_p \nabla^0_A V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B{}_p V^C{}_q \,,$$

$$\bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} \,.$$

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- Further, the extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_A d = 0$, must satisfy

$$\Delta_{Apq} V^{Ap} = 0 \,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}} \bar{V}^{A\bar{p}} = 0 \,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_A \,, \qquad \bar{\psi}_{\bar{p}}\gamma_A\psi_{\bar{q}} \,, \qquad \bar{\rho}\gamma_{Apq}\rho \,, \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}} \,,$$

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- The 'torsionless' connection,

$$\Gamma^0_{CAB} = \quad 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC}$$
$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right)\,,$$

further obeys

$$\Gamma^0_{ABC} + \Gamma^0_{BCA} + \Gamma^0_{CAB} = 0\,,$$

and

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,.$$

- **In fact, the torsionless connection,**

$$\Gamma^{0}_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC}$$
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is uniquely determined by requiring

$$\nabla_A \mathcal{J}_{BC} = 0 \iff \Gamma_{CAB} + \Gamma_{CBA} = 0\,,$$

$$\nabla_A P_{BC} = 0\,,$$

$$\nabla_A d = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0\,,$$

$$(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{DEF}\Gamma_{DEF} = 0\,.$$

- Having the two symmetric properties, $\Gamma_{A(BC)}=0$, $\Gamma_{[ABC]}=0$, we may safely replace ∂_A by $\nabla^0_A=\partial_A+\Gamma^0_A$ in $\hat{\mathcal{L}}_X$ and also in $[X,Y]^A_{\mathbf{C}}$,

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n}=X^B\nabla^0_B T_{A_1\cdots A_n}+\omega\nabla^0_B X^B T_{A_1\cdots A_n}+\textstyle\sum_{i=1}^n(\nabla^0_{A_i}X_B-\nabla^0_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n},$$

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Semi-covariant curvature

- The usual curvatures for the three connections,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

$$F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr}\Phi_{B}{}^{r}{}_{q} - \Phi_{Bpr}\Phi_{A}{}^{r}{}_{q}\,,$$

$$\bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}}\bar{\Phi}_{B}{}^{\bar{r}}{}_{\bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}}\bar{\Phi}_{A}{}^{\bar{r}}{}_{\bar{q}}\,,$$

are, from $[\mathcal{D}_A, \mathcal{D}_B]V_{Cp} = 0$ and $[\mathcal{D}_A, \mathcal{D}_B]\bar{V}_{C\bar{p}} = 0$, related to each other,

$$R_{ABCD} = F_{CDpq}V_{A}{}^{p}V_{B}{}^{q} + \bar{F}_{CD\bar{p}\bar{q}}\bar{V}_{A}{}^{\bar{p}}\bar{V}_{B}{}^{\bar{q}}\,.$$

- **However, the crucial object in DFT turns out to be**

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right),$$

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Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}\left(S_{[AB][CD]} + S_{[CD][AB]}\right) ,$$

$$S^0_{[ABC]D} = 0 .$$

- Projection property,

$$P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \equiv 0 .$$

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S_{ABCD} = \mathcal{D}_{[A}\delta\Gamma_{B]CD} + \mathcal{D}_{[C}\delta\Gamma_{D]AB} - \tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^E{}_{CD} - \tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^E{}_{AB} ,$$

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- Generically, under DFT-diffeomorphisms, $\delta_X P_{AB} = \hat{\mathcal{L}}_X P_{AB},\ \delta_X d = \hat{\mathcal{L}}_X d$, the variation of the semi-covariant derivative contains an anomalous non-covariant part dictated by the six-index projectors,

$$\delta_X \left(\nabla_C T_{A_1\cdots A_n}\right) \equiv \hat{\mathcal{L}}_X \left(\nabla_C T_{A_1\cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_{\cdots B\cdots}\,.$$

- Hence, it is not DFT-diffeomorphism covariant,

$$\delta_X \neq \hat{\mathcal{L}}_X\,.$$

- However, the characteristic property of our 'semi-covariant' derivative is that, **combined with the projectors it can generate various fully covariant quantities,** as listed below.

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$$\delta_X \left(\nabla_C T_{A_1\cdots A_n}\right) \equiv \hat{\mathcal{L}}_X \left(\nabla_C T_{A_1\cdots A_n}\right) + \sum_i 2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BFDE}\partial_F\partial_{[D}X_{E]}\,T_{\cdots B\cdots}\,.$$

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- **For O(D, D) tensors:**

$$P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

$$\bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}P_{A_2}{}^{B_2}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

$$\left.\begin{array}{l}P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\,,\\ \bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\end{array}\right\}\quad\text{Divergences}\,,$$

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- **For Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$ tensors:**

$$\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$
$$\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p}q_1q_2\cdots q_n}\,,$$
$$\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$

where we set

$$\mathcal{D}_p := V^A{}_p\mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}}\mathcal{D}_A\,.$$

These are the pull-back of the previous results using the DFT-vielbeins.

Projector-aided, fully covariant derivatives

- **Dirac operators for fermions,** $\rho^{\alpha},\ \psi_{\bar{p}}^{\alpha},\ \rho'^{\bar{\alpha}},\ \psi_{p}'^{\bar{\alpha}}$:

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\psi}^{A}\gamma_{p}(\mathcal{D}_{A}\psi_{\bar{q}}-\tfrac{1}{2}\mathcal{D}_{\bar{q}}\psi_{A})\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

$$\bar{\psi}'^{A}\bar{\gamma}_{\bar{p}}(\mathcal{D}_{A}\psi'_{q}-\tfrac{1}{2}\mathcal{D}_{q}\psi'_{A})\,.$$

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- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ bi-fundamental spinorial fields,** $\mathcal{T}^{\alpha}{}_{\bar{\beta}}$ **:**

$$\mathcal{D}_+\mathcal{T} := \gamma^A\mathcal{D}_A\mathcal{T} + \gamma^{(D+1)}\mathcal{D}_A\mathcal{T}\bar{\gamma}^A\,,$$

$$\mathcal{D}_-\mathcal{T} := \gamma^A\mathcal{D}_A\mathcal{T} - \gamma^{(D+1)}\mathcal{D}_A\mathcal{T}\bar{\gamma}^A\,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent**

$$(\mathcal{D}^0_+)^2\mathcal{T} \equiv 0\,, \qquad\qquad (\mathcal{D}^0_-)^2\mathcal{T} \equiv 0\,,$$

and hence, they define **O(D, D) covariant cohomology.**

- **The field strength of the R-R potential,** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$**, is then defined by**

$$\mathcal{F} := \mathcal{D}^0_+\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^0_+\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^0_+(\delta\mathcal{C}) = (\mathcal{D}^0_+)^2\Delta \equiv 0\,.$$

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- **Scalar curvature:**

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \,.$$

- **"Ricci" curvature:**

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Combining all the results above, we are now ready to spell

- **Type II *i.e.* $\mathcal{N}=2$ $D=10$ Supersymmetric Double Field Theory**

Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

- **Lagrangian** :

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
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where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_{+}^{-1}\mathcal{F}^{T}C_{+}$.

- As they are contracted with the DFT-vielbeins properly,
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- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC}=\quad\Gamma^{0}_{ABC}+i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho-2i\bar{\rho}\gamma_{BC}\psi_{A}-i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}+4i\bar{\psi}_{B}\gamma_{A}\psi_{C}$$

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which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:
$\mathcal{D}^{\star}_{A}$ for the unprimed fermions and $\mathcal{D}'^{\star}_{A}$ for the primed fermions, set by

$$\Gamma^{\star}_{ABC}=\;\Gamma_{ABC}-i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho+i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_{A}+i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}-2i\bar{\psi}_{B}\gamma_{A}\psi_{C}+i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A}\,,$$

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- **The $\mathcal{N}=2$ supersymmetry transformation rules are**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon\mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon\rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon\rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon\psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon\psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,,$$

where

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- The Lagrangian is **pseudo** **:** It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}'+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right)\equiv 0\,.$$

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- Under the $\mathcal{N}=2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_{\varepsilon}\mathcal{L}_{\rm Type\,II} \simeq -\tfrac{1}{8}e^{-2d}\,\bar{V}^{A}{}_{\bar{q}}\delta_{\varepsilon}V_{Ap}\mathrm{Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right)\,,$$

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This verifies, to the full order in fermions, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a **nontrivial consistency check,** the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

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Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

- Under the $\mathcal{N}=2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_{\varepsilon}\mathcal{L}_{\rm Type\,II} \simeq -\tfrac{1}{8}e^{-2d}\,\bar{V}^{A}{}_{\bar{q}}\delta_{\varepsilon}V_{Ap}{\rm Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right)\,,$$

  where

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}^{\prime}+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}^{\prime}_{p}\bar{\gamma}^{\bar{q}}\right)\,.$$

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- DFT-vielbein:

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  This is DFT-generalization of Einstein equation.

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

  Namely, the on-shell Lagrangian vanishes!

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$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

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Truncation to $\mathcal{N}=1$ $D=10$ SDFT [1112.0069]

- Turning off the primed fermions and the R-R sector truncates the $\mathcal{N}=2$ $D=10$ SDFT to $\mathcal{N}=1$ $D=10$ SDFT,

$$\mathcal{L}_{\mathcal{N}=1}=e^{-2d}\Big[\tfrac{1}{8}\left(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD}\right)S_{ACBD}+i\tfrac{1}{2}\bar{\rho}\gamma^{A}\mathcal{D}^{\star}_{A}\rho-i\bar{\psi}^{A}\mathcal{D}^{\star}_{A}\rho-i\tfrac{1}{2}\bar{\psi}^{B}\gamma^{A}\mathcal{D}^{\star}_{A}\psi_{B}\Big]\,.$$

- $\mathcal{N}=1$ **Local SUSY:**

$$\begin{aligned}
\delta_{\varepsilon}d &= -i\tfrac{1}{2}\bar{\varepsilon}\rho\,,\\
\delta_{\varepsilon}V_{Ap} &= -i\bar{\varepsilon}\gamma_{p}\psi_{A}\,,\\
\delta_{\varepsilon}\bar{V}_{A\bar{p}} &= i\bar{\varepsilon}\gamma_{A}\psi_{\bar{p}}\,,\\
\delta_{\varepsilon}\rho &= -\gamma^{A}\hat{\mathcal{D}}_{A}\varepsilon\,,\\
\delta_{\varepsilon}\psi_{\bar{p}} &= \bar{V}^{A}{}_{\bar{p}}\hat{\mathcal{D}}_{A}\varepsilon-i\tfrac{1}{4}(\bar{\rho}\psi_{\bar{p}})\varepsilon+i\tfrac{1}{2}(\bar{\varepsilon}\rho)\psi_{\bar{p}}\,.
\end{aligned}$$

- Commutator of supersymmetry reads

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \equiv \hat{\mathcal{L}}_{X_3} + \delta_{\varepsilon_3} + \delta_{\mathbf{so}(1,9)_L} + \delta_{\mathbf{so}(9,1)_R} + \delta_{\text{trivial}} \,.$$

where

$$X_3^A = i\bar{\varepsilon}_1\gamma^A\varepsilon_2 \,, \qquad \varepsilon_3 = i\tfrac{1}{2}\left[(\bar{\varepsilon}_1\gamma^p\varepsilon_2)\gamma_p\rho + (\bar{\rho}\varepsilon_2)\varepsilon_1 - (\bar{\rho}\varepsilon_1)\varepsilon_2\right] \,, \qquad \text{etc.}$$

and δ_{trivial} corresponds to the fermionic equations of motion.

Now we are going to

- parametrize the DFT-field-variables in terms of Riemannian variables,
- discuss the 'unification' of IIA and IIB,
- choose a diagonal gauge of Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$,
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Parametrization: Reduction to Generalized Geometry

- As stressed before, one of the characteristic features in our construction of $\mathcal{N}=2$ $D=10$ SDFT is the usage of the $\mathbf{O}(D,D)$ covariant, genuine DFT-field-variables.
- However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, *i.e.* zehnbeins and B-field.
- Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the general form,

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^{\mu} \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}.$$

Here $e_\mu{}^p$ and $\bar{e}_\nu{}^{\bar{p}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_\mu{}^p e_\nu{}^q \eta_{pq} = -\bar{e}_\mu{}^{\bar{p}} \bar{e}_\nu{}^{\bar{q}} \bar{\eta}_{\bar{p}\bar{q}} = g_{\mu\nu}\,,$$

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- Yet if we consider dimensional reductions from D to lower dimensions, there is no longer preferred parametrization $\Longrightarrow$ "Non-geometry"

  c.f. Other parametrizations: Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun *et al.*

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- In connection to the section condition, $\partial^A\partial_A \equiv 0$, the former matches well with the choice, $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$, while the latter is natural when $\frac{\partial}{\partial x^\mu} \equiv 0$.

- Yet if we consider dimensional reductions from D to lower dimensions, there is no longer preferred parametrization $\Longrightarrow$ "Non-geometry"

  c.f. Other parametrizations: Lust, Andriot, Betz, Blumenhagen, Fuchs, Sun *et al.*

Parametrization: Reduction to Generalized Geometry

- Two parametrizations:

$$V_{Ap} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_p{}^{\mu} \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B+\bar{e})_{\nu \bar{p}} \end{pmatrix}$$

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- However, let me emphasize that to maintain the clear $\mathbf{O}(D,D)$ covariant structure, it is necessary to work with the parametrization-independent, and $\mathbf{O}(D,D)$ covariant, DFT-vielbeins, V_{Ap}, $\bar{V}_{A\bar{p}}$, rather than the Riemannian variables, $e_{\mu}{}^{p}$, $B_{\mu\nu}$.
- *Furthermore, 'degenerate' cases are also allowed which lead to genuinely non-Riemannian 'metric-less' backgrounds* $\Longrightarrow$ *New type of string theory backgrounds 1307.8377*

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- This reduces (S)DFT to Generalized Geometry

  Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

$$\sqrt{2}\gamma^A\mathcal{D}_A\rho \equiv \gamma^m\left(\partial_m\rho + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\rho + \tfrac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_m\phi\rho\right),$$

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- Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

$$(e^{-1}\bar{e})_{p}{}^{\bar{p}}(e^{-1}\bar{e})_{q}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq}\,.$$

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$$\mathbf{cc}'\,\det(e^{-1}\bar{e}) = +1 \quad : \quad \text{type IIA}\,,$$
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- This identification with the ordinary IIA/IIB SUGRAs can be established, if we 'fix' the two zehnbeins equal to each other,

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- Setting the diagonal gauge,

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

with $\eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}$, $\bar{\gamma}^{\bar{p}} = \gamma^{(D+1)}\gamma^p$, $\bar{\gamma}^{(D+1)} = -\gamma^{(D+1)}$, breaks the local Lorentz symmetry,

$$\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1, D-1)_D\,.$$

- And it reduces SDFT to SUGRA:

$\mathcal{N} = 2$ $D = 10$ **SDFT** $\Longrightarrow$ $10D$ **Type II democratic SUGRA**

Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

$\mathcal{N} = 1$ $D = 10$ **SDFT** $\Longrightarrow$ $10D$ **minimal SUGRA** Chamseddine; Bergshoeff *et al.*

Diagonal gauge fixing and Reduction to SUGRA

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Diagonal gauge fixing and Reduction to SUGRA

- **To the full order in fermions, $\mathcal{N}=1$ SDFT reduces to $10D$ minimal SUGRA:**

$$\begin{aligned}\mathcal{L}_{10D} = \det e \times e^{-2\phi}\Big[& R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \\ & + i2\sqrt{2}\bar{\rho}\gamma^m[\partial_m\rho + \tfrac{1}{4}(\omega+\tfrac{1}{6}H)_{mnp}\gamma^{np}\rho] - i4\sqrt{2}\bar{\psi}^p[\partial_p\rho + \tfrac{1}{4}(\omega+\tfrac{1}{2}H)_{pqr}\gamma^{qr}\rho] \\ & - i2\sqrt{2}\bar{\psi}^p\gamma^m[\partial_m\psi_p + \tfrac{1}{4}(\omega+\tfrac{1}{6}H)\gamma^{np}\psi_p + \omega_{mpq}\psi^q - \tfrac{1}{2}H_{mpq}\psi^q] \\ & + \tfrac{1}{24}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\psi}^r\gamma^{mnp}\psi_r) - \tfrac{1}{48}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\rho}\gamma^{mnp}\rho)\ \Big]\,.\end{aligned}$$

$$\delta_\varepsilon\phi = i\tfrac{1}{2}\bar{\varepsilon}(\rho+\gamma^a\psi_a)\,,\qquad \delta_\varepsilon e_\mu^a = i\bar{\varepsilon}\gamma^a\psi_\mu\,,\qquad \delta_\varepsilon B_{\mu\nu} = -2i\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\,,$$

$$\begin{aligned}\delta_\varepsilon\rho = & -\tfrac{1}{\sqrt{2}}\gamma^a[\partial_a\varepsilon + \tfrac{1}{4}(\omega+\tfrac{1}{6}H)_{abc}\gamma^{bc}\varepsilon - \partial_a\phi\varepsilon] \\ & + i\tfrac{1}{48}(\bar{\psi}^d\gamma_{abc}\psi_d)\gamma^{abc}\varepsilon + i\tfrac{1}{192}(\bar{\rho}\gamma_{abc}\rho)\gamma^{abc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[a}\psi_{b]})\gamma^{ab}\rho\,,\end{aligned}$$

$$\begin{aligned}\delta_\varepsilon\psi_a = & \tfrac{1}{\sqrt{2}}[\partial_a\varepsilon + \tfrac{1}{4}(\omega+\tfrac{1}{2}H)_{abc}\gamma^{bc}\varepsilon] \\ & - i\tfrac{1}{2}(\bar{\rho}\varepsilon)\psi_a - i\tfrac{1}{4}(\bar{\rho}\psi_a)\varepsilon + i\tfrac{1}{8}(\bar{\rho}\gamma_{bc}\psi_a)\gamma^{bc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[b}\psi_{c]})\gamma^{bc}\psi_a\,.\end{aligned}$$

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C}\equiv\left(\tfrac{1}{2}\right)^{\frac{D+2}{4}}\textstyle\sum_{p}^{\prime}\,\tfrac{1}{p!}\,\mathcal{C}_{a_1a_2\cdots a_p}\gamma^{a_1a_2\cdots a_p}$$

and obtain the field strength,

$$\mathcal{F}:=\mathcal{D}^{0}_{+}\mathcal{C}\equiv\left(\tfrac{1}{2}\right)^{\frac{D}{4}}\textstyle\sum_{p}^{\prime}\,\tfrac{1}{(p+1)!}\,\mathcal{F}_{a_1a_2\cdots a_{p+1}}\gamma^{a_1a_2\cdots a_{p+1}}$$

where $\sum_{p}^{\prime}$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1a_2\cdots a_p}=p\left(D_{[a_1}\mathcal{C}_{a_2\cdots a_p]}-\partial_{[a_1}\phi\,\mathcal{C}_{a_2\cdots a_p]}\right)+\tfrac{p!}{3!(p-3)!}\,H_{[a_1a_2a_3}\mathcal{C}_{a_4\cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^{0}_{+}$ and $\mathcal{D}^{0}_{-}$, reduce to a 'twisted K-theory' exterior derivative and its dual, after the diagonal gauge fixing,

$$\mathcal{D}^{0}_{+}\quad\Longrightarrow\quad\mathrm{d}+(H-\mathrm{d}\phi)\wedge$$

$$\mathcal{D}^{0}_{-}\quad\Longrightarrow\quad *\left[\,\mathrm{d}+(H-\mathrm{d}\phi)\wedge\ \right]*$$

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \textstyle\sum_p' \frac{1}{p!}\,\mathcal{C}_{a_1 a_2\cdots a_p}\gamma^{a_1 a_2\cdots a_p}$$

  and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \textstyle\sum_p' \frac{1}{(p+1)!}\,\mathcal{F}_{a_1 a_2\cdots a_{p+1}}\gamma^{a_1 a_2\cdots a_{p+1}}$$

  where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1 a_2\cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2\cdots a_p]} - \partial_{[a_1}\phi\,\mathcal{C}_{a_2\cdots a_p]}\right) + \frac{p!}{3!(p-3)!}\,H_{[a_1 a_2 a_3}\mathcal{C}_{a_4\cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to a 'twisted K-theory' exterior derivative and its dual, after the diagonal gauge fixing,

$$\mathcal{D}^0_+ \quad\Longrightarrow\quad \mathrm{d} + (H - \mathrm{d}\phi)\wedge$$

$$\mathcal{D}^0_- \quad\Longrightarrow\quad *\,[\,\mathrm{d} + (H - \mathrm{d}\phi)\wedge\ \,]\,*$$

- **In this way, ordinary SUGRA $\equiv$ gauge-fixed SDFT,**

$$\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1,D{-}1)_D\,.$$

Modifying $\mathbf{O}(D, D)$ transformation rule

- The diagonal gauge, $e_{\mu}{}^{p} \equiv \bar{e}_{\mu}{}^{\bar{p}}$, is incompatible with the vectorial $\mathbf{O}(D, D)$ transformation rule of the DFT-vielbein.

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- The $\mathbf{O}(D, D)$ rotation must accompany a compensating $\mathbf{Pin}(D{-}1, 1)_R$ local Lorentz rotation, $\bar{L}_{\bar{q}}{}^{\bar{p}}$, $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}}$ which we can construct explicitly,

$$\bar{L} = \bar{e}^{-1}\left[\mathbf{a}^t - (g+B)\mathbf{b}^t\right]\left[\mathbf{a}^t + (g-B)\mathbf{b}^t\right]^{-1}\bar{e}\,, \qquad \bar{\gamma}^{\bar{q}}\bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1}\bar{\gamma}^{\bar{p}}S_{\bar{L}}\,,$$

where $\mathbf{a}$ and $\mathbf{b}$ are parameters of a given $\mathbf{O}(D, D)$ group element,

$$M_A{}^B = \begin{pmatrix} \mathbf{a}^{\mu}{}_{\nu} & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_{\rho}{}^{\sigma} \end{pmatrix}.$$

Modified $\mathbf{O}(D, D)$ Transformation Rule After The Diagonal Gauge Fixing

$$
\begin{array}{ccc}
d & \longrightarrow & d \\
V_A{}^p & \longrightarrow & M_A{}^B\, V_B{}^p \\
\bar{V}_A{}^{\bar{p}} & \longrightarrow & M_A{}^B\, \bar{V}_B{}^{\bar{q}}\, \bar{L}_{\bar{q}}{}^{\bar{p}} \\
\mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,, \quad \mathcal{F}^{\alpha}{}_{\bar{\alpha}} & \longrightarrow & \mathcal{C}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,, \quad \mathcal{F}^{\alpha}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}} \\
\rho^{\alpha} & \longrightarrow & \rho^{\alpha} \\
\rho'^{\bar{\alpha}} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\rho'^{\bar{\beta}} \\
\psi^{\alpha}_{\bar{p}} & \longrightarrow & (\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^{\alpha}_{\bar{q}} \\
\psi'^{\bar{\alpha}}_{p} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\psi'^{\bar{\beta}}_{p}
\end{array}
$$

- **All the barred indices are now to be rotated.** Consistent with Hassan
- **The R-R sector can be also mapped to $\mathbf{O}(D, D)$ spinors.**

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

Flipping the chirality: IIA $\Leftrightarrow$ IIB

- **If and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D,D)$ rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L})\, S_{\bar{L}} \bar{\gamma}^{(D+1)}\,.$$

- Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

- However, since $\bar{L}$ explicitly depends on the parametrization of V_{Ap} and $\bar{V}_{A\bar{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $\mathbf{O}(D,D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.

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Comment 1: Double field Yang-Mills theory 1102.0419

- With the semi-covariant derivative, we may construct YM-DFT :

$$\mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B]\,, \qquad \mathcal{V}_A = \begin{pmatrix} \phi^\lambda \\ A_\mu + B_{\mu\nu}\phi^\nu \end{pmatrix},$$

$$\begin{aligned} S_{\rm YM} &= \int_{\Sigma_D} e^{-2d}\,{\rm Tr}\Big(P^{AB}\bar{P}^{CD}\mathcal{F}_{AC}\mathcal{F}_{BD}\Big) \\ &\equiv \int {\rm d}x^D \sqrt{-g}e^{-2\phi}{\rm Tr}\Big(f_{\mu\nu}f^{\mu\nu} + 2D_\mu\phi_\nu D^\mu\phi^\nu + 2D_\mu\phi_\nu D^\nu\phi^\mu + 2i\,f_{\mu\nu}[\phi^\mu, \phi^\nu] \\ &\quad - [\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu] + 2\,(f^{\mu\nu} + i[\phi^\mu, \phi^\nu])\,H_{\mu\nu\sigma}\phi^\sigma + H_{\mu\nu\sigma}H^{\mu\nu}{}_\tau\phi^\sigma\phi^\tau\Big)\,. \end{aligned}$$

- Similar to topologically twisted Yang-Mills, but differs in detail.
- Curved D-branes are known to convert adjoint scalars into one-form,

$$\phi^a \rightarrow \phi_\mu, \qquad \text{Bershadsky}$$

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$$x^M \sim x^M + \varphi\partial^M\varphi' .$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

- The diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the 'gauge orbits'.
- Consequently, finite transformation rules are not unique.

For example, *the exponentiation of the generalized Lie derivative* and a simple *ansatz* proposed by Hohm-Zwiebach. These two appear different but are fully equivalent to each other, up to the coordinate gauge symmetry.

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Comment 2: Doubled-yet-gauged spacetime 1304.5946 1307.8377

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- **String propagates in a doubled-yet-gauged spacetime, 1307.8377**

$$\mathcal{S}=\frac{1}{4\pi\alpha'}\int\mathrm{d}^2\sigma\ \mathcal{L}\,,\qquad\qquad \mathcal{L}=-\tfrac{1}{2}\sqrt{-h}h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X)-\epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

where

$$D_iX^M=\partial_iX^M-\mathcal{A}_i^M\,,\qquad\qquad \mathcal{A}_i^M\partial_M\equiv 0\,.$$

- The Lagrangian is symmetric with respect to the string worldsheet diffeomorphisms, Weyl symmetry, $\mathbf{O}(D,D)$ T-duality, target spacetime generalized diffeomorphisms and the coordinate gauge symmetry, thanks to the auxiliary gauge field, $\mathcal{A}_i^M$.

  c.f. Hull; Tseytlin; Copland, Berman, Thompson; Nibbelink, Patalong; Blair, Malek, Routh

- Further, after parametrization and integrating out $\mathcal{A}_i^M$, it can produce either the standard string action for the 'non-degenerate' Riemannian case,

$$\frac{1}{4\pi\alpha'}\mathcal{L}\equiv\frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_iY^\mu\partial_jY^\nu G_{\mu\nu}(Y)+\tfrac{1}{2}\epsilon^{ij}\partial_iY^\mu\partial_jY^\nu B_{\mu\nu}(Y)+\tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{Y}_\mu\partial_jY^\mu\right],$$

or novel chiral actions for 'degenerate' non-Riemannian cases, e.g. for $\mathcal{H}_{AB}=\mathcal{J}_{AB}$,

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  - Semi-covariant derivative and semi-covariant curvature
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  where $M = \det(M_{ab})$ and the integral is taken over a section, Σ.

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Conclusion

Summary

- Riemannian geometry is for *particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector and underlies DFT.

- The fundamental field-variables of $\mathcal{N}=2$ $D=10$ SDFT are, besides the fermions, the DFT-dilaton, d, DFT-vielbeins, V_{Ap}, $\bar{V}_{A\bar{p}}$, and the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$.

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- *A priori*, in the covariant formalism, the R-R sector and the fermions are $\mathbf{O}(D,D)$ singlet.
  Yet, the diagonal gauge fixing, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, modifies the $\mathbf{O}(D,D)$ transformation rule to call for a compensating $\mathbf{Pin}(D{-}1,1)_R$ rotation, which may flip the chirality of the theory, resulting in the known exchange of type IIA and IIB SUGRAs.

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- $\mathcal{N} = 2$ $D = 10$ SDFT contains not only Riemannian SUGRA backgrounds but also non-Riemannian 'metric-less' backgrounds.

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- YES, if we take the following as a definition of the generalized metric,

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}.$$

- NO, if we define the generalized metric as a symmetric $\mathbf{O}(D,D)$ element,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}\,, \qquad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}\,.$$

- With this abstract definition, DFT as well as a sigma model (which I will discuss later) perfectly make sense.
- It may then describe a novel non-Riemannian string theory backgrounds, e.g.

$$\mathcal{H}_{AB} = \mathcal{J}_{AB}\,,$$

which does not admit any Riemannian interpretation!

- *c.f.* Global aspects such as "non-geometry" Berman-Cederwall-Perry, Papadopoulos and Scherk-Schwarz Geissbuhler, Grana-Marques, Aldazabal-Grana-Marques-Rosabal, Dibitetto-Fernandez-Melgarejo-Marques-Roest, Berman-Lee

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