Casimir Effect: Diffraction and General Boundary Conditions

Dimitra Karabali

City University of New York

Lehman College

Recent Developments in Theoretical Physics

Korea Institute for Advanced Study

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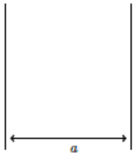
- Casimir effect: emergence of a macroscopic force due to quantum fluctuations in vacuum.
- The original Casimir effect (CASIMIR, 1948) is for two parallel infinitely long conducting plates.
  - Even in pure vacuum, there is an attractive force between the plates given by $F = -\partial E/\partial a$,

$$\frac{E}{A} = -\frac{\pi^2}{720\, a^3}$$

    where A is the area of the plates.



  - Boundary conditions determine the field modes. The zero-point energy contribution of the fields shifts in the presence of the conducting plates. The difference is finite and calculable.

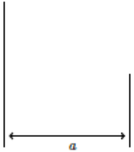
There is renewed interest in Casimir effect:

- Advances in instrumentation have allowed precise measurements of the effect (LAMOREAUX (1997); MOHIDEEN & ROY (1998) + ...)
- Casimir force becomes relevant in the design of nano-scale mechanical devices

On the theory side, a lot of activity in developing approaches to deal with the effects of different geometries, orientation, surface roughness, thermal fluctuations etc., effects relevant in realistic experimental setups.

An important and difficult question to address is diffraction due to edges.

- For example, consider a variant of the parallel plate geometry



- The Casimir energy has an additional term $E \sim L/a^2$, where L is length along the edge
- This is due to the modes undergoing diffraction over the edge

- Diffraction had not been analyzed in previous analytical methods
- Mainly numerical methods using ”world line approach” (Monte Carlo simulations) by GIES & KLINGMULLER

Can we calculate edge diffraction analytically?

Work done with D. KABAT, V.P. NAIR (arXiv: 1002.3575; 1005.3352 ; 1107.0952; 1111.0838; 1304.0511)

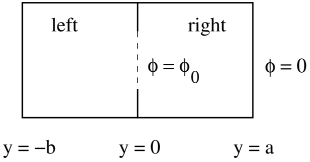
- Start with a scalar field theory
- Path integral approach

$$Z = \int [d\phi]\, e^{-S(\phi)}, \qquad S(\phi) = \frac{1}{2}\int_0^\beta d\tau d^3x\ (\partial\phi\ \partial\phi)$$

- The free energy is

$$E = -\frac{1}{\beta}\log Z, \qquad \beta \to \infty$$

- ϕ propagates in two regions separated by a plate with a hole (at $y = 0$)
- Dirichlet boundary conditions, $\phi = 0$ on all boundaries; $\phi = \phi_0$ on the hole



- STRATEGY: Integrate over ϕ in the bulk left and right regions, leave ϕ_0 integral to the end. This will produce a non-local effective action on the hole.

$$\phi = \phi_{\rm cl} + \eta$$

η vanishes on all boundaries (including hole area), and $\Box \phi_{\rm cl} = 0$, with

$$\phi_{\rm cl} \to \begin{cases} \phi_0 & \text{on hole} \\ 0 & \text{elsewhere on boundary} \end{cases}$$

- The solution is

$$\phi_{\rm cl}(x) = \begin{cases} \int d^{d-1}x' \;\; \phi_0(x') \; n \cdot \partial' G_L(x|x') & \text{on left} \\ \int d^{d-1}x' \;\; \phi_0(x') \; n \cdot \partial' G_R(x|x') & \text{on right} \end{cases}$$

where $G_L, \; G_R$ are Green's functions obeying Dirichlet boundary conditions.

- The action becomes

$$S(\phi) = \frac{1}{2}\int(\partial\eta\,\partial\eta)_L + \frac{1}{2}\int(\partial\eta\,\partial\eta)_R + S_{\text{hole}}$$

$$S_{\text{hole}} = \int_{hole} \frac{1}{2}\phi_0(x)\,M(x|x')\,\phi_0(x'), \qquad M = M_L + M_R$$

where

$$M_{L,R}(x|x') = n\cdot\partial\;\; n\cdot\partial'\, G_{L,R}(x|x')$$

- We can expand as

$$\phi_0 = \sum_\alpha c_\alpha\, u_\alpha(x)$$

$\{u_\alpha(x)\}$ = complete set of modes for functions which are nonzero in the hole with the boundary condition, $u_\alpha(x) \to 0$ at the edge of the hole.

- The effective action on the hole becomes

$$S_{\text{hole}} = \sum c_\alpha\,\mathcal{O}_{\alpha\beta}\,c_\beta$$

$$\mathcal{O}_{\alpha\beta} = \int d^{d-1}x\, d^{d-1}x'\;\; u_\alpha(x)\,M(x|x')\,u_\beta(x')$$

- Integrating over c_α we get

$$Z = \det{}^{-1/2}(-\Box_L)\,\det{}^{-1/2}(-\Box_R)\,\det{}^{-1/2}(\mathcal{O})$$

- The bulk determinants capture the Casimir energy that would be present if there was no hole. Corrections are given by the non-local field theory S_{hole} that lives on the hole.
- The explicit form of $\mathcal{O}$ depends on the arrangement of plates and holes. We find

$$\begin{aligned} M_R(x|x') &= \; < x|\sqrt{-\nabla^2}\,\coth\left(b_1\sqrt{-\nabla^2}\right)|x'> \\ M_L(x|x') &= \; < x|\sqrt{-\nabla^2}\,\coth\left(b_2\sqrt{-\nabla^2}\right)|x'> \end{aligned}$$

where ∇^2 is the Laplacian on the middle plate.

- Consider different limits of b_1, b_2 to study different geometries.

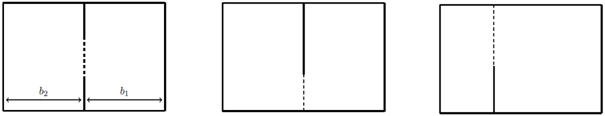
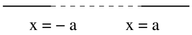


Plate with a Slit

- Consider a single slit on a plate of width $w = 2a$ and length L at inverse temperature β



- Functions on the hole are expanded in terms of modes given by

$$u_m^{\text{odd}} \sim \sin\left(m\pi x/a\right) \ , \ m = 1, 2, \cdots$$

$$u_p^{\text{even}} \sim \cos\left(p\pi x/a\right) \ , \ p = 1/2, 3/2, \cdots$$

- For the odd modes, we get (similarly for even modes)

$$\mathcal{O}_{mn}^{\text{odd}} = \frac{2a}{\pi} \int_{-\infty}^{\infty} dk \left(\sin^2 ka\right) \frac{m\pi}{k^2 a^2 - m^2 \pi^2} \sqrt{k^2 + \mu^2} \, \frac{n\pi}{k^2 a^2 - n^2 \pi^2}$$

where $\mu^2 = (2\pi n/\beta)^2 + (2\pi l/L)^2$.

This separates naturally into a pole contribution and a cut contribution.

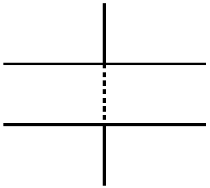
- $\mathcal{O}_{mn} = \mathcal{O}_{mn}^{\text{direct}} + \mathcal{O}_{mn}^{\text{diffractive}}$

$$\mathcal{O}_{mn}^{\text{odd,direct}} = \sqrt{(m\pi/a)^2 + \mu^2}\ \delta_{mn} \qquad \text{diagonal}$$

$$\mathcal{O}_{mn}^{\text{odd,diffractive}} = -\frac{2\mu^2 a}{\pi}\int_1^\infty dy\,\sqrt{y^2-1}\left(1-e^{-2\mu a y}\right)\frac{m\pi}{m^2\pi^2+\mu^2 a^2 y^2}\,\frac{n\pi}{n^2\pi^2+\mu^2 a^2 y^2}$$

There is a similar result for the even-parity modes.

Direct term:



- The strategy now is to expand in powers of the diffractive contribution.

$$-\log Z_B \quad = \quad \frac{1}{2}\mathrm{Tr}\,\log \mathcal{O}_{\text{direct}} + \frac{1}{2}\mathrm{Tr}\,\mathcal{O}^{-1}_{\text{direct}}\mathcal{O}_{\text{diffr}} - \frac{1}{4}\mathrm{Tr}\,\mathcal{O}^{-1}_{\text{direct}}\mathcal{O}_{\text{diffr}}\mathcal{O}^{-1}_{\text{direct}}\mathcal{O}_{\text{diffr}} + \cdots$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$E^{\text{direct}} \qquad\qquad E^{(1),\text{diffr}} \qquad\qquad E^{(2),\text{diffr}}$$

- Renormalization: There will be divergent terms; all of them can be absorbed as renormalization of parameters $(\sigma,\ \alpha, ...)$ of the slit

$$S_{slit} = \sigma(\text{Area}) \ + \ \alpha(\text{Perimeter}) \ + \cdots$$

- Integrate over μ. In 4 dim and zero temperature $(\beta \to \infty)$,

$$-\log Z_{4d} = \beta L \int \frac{d^2\mu}{(2\pi)^2}\ (-\log Z_{2d})$$

- Using the series expansion we find

$$E^{\text{direct}} \quad = \quad -\frac{\zeta(3)L}{128\pi a^2} = (-2.99 \times 10^{-3})\ \frac{L}{a^2} \qquad \text{(2 + 1) dim Casimir energy}$$

$$E^{(1),\text{diffr}} \quad = \quad (2.15 \times 10^{-3})\ \frac{L}{a^2}, \quad E^{(2),\text{diffr}} = (0.14 \times 10^{-3})\ \frac{L}{a^2}, \quad E^{(3),\text{diffr}} = (0.02 \times 10^{-3})\ \frac{L}{a^2}$$

- The geometry we consider is

- We obtain results from previous by restricting to odd modes which vanish at mid-point.

$$E_{\perp}^{\text{direct}} = (-11.96 \times 10^{-3})\,\frac{L}{a^2}$$

$$E_{\perp}^{(1)} = (5.01 \times 10^{-3})\,\frac{L}{a^2}, \quad E_{\perp}^{(2)} = (0.66 \times 10^{-3})\,\frac{L}{a^2} \quad E_{\perp}^{(3)} = (0.16 \times 10^{-3})\,\frac{L}{a^2}$$

$$E_{\perp}^{(4)} = (0.05 \times 10^{-3})\,\frac{L}{a^2} \quad E_{\perp}^{(5)} = (0.01 \times 10^{-3})\,\frac{L}{a^2}$$

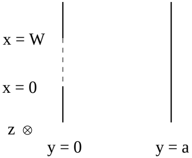
- The total to this order is

$$E_{\perp} = (-6.07 \times 10^{-3})\,\frac{L}{a^2}$$

- In good agreement with worldline Monte Carlo calculations by GIES & KLINGMULLER and recent multiple-scattering results by MIT GROUP (MAGHREBI ET AL).

$$E_{\perp}^{GK} = (-6.00 \times 10^{-3})\,\frac{L}{a^2}$$

- Similar procedure for parallel plates with a large slit



- Bulk plus direct contribution: $E^{(0)} = -\frac{\pi^2 L(L_1 - W)}{1440 a^3} - \frac{\zeta(3) L}{32\pi a^2}$

  Bulk same as the standard Casimir formula, with $A \to$ the facing area of plates.

- The diffractive contributions are

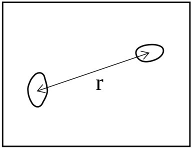
$$E^{(1)} = \frac{L}{a^2}(5.54 \times 10^{-3}), \qquad E^{(2)} = \frac{L}{a^2}(0.80 \times 10^{-3}), \qquad E^{(3)} = \frac{L}{a^2}(0.19 \times 10^{-3})$$

$$E^{(4)} = \frac{L}{a^2}(0.05 \times 10^{-3}), \qquad E^{(5)} = \frac{L}{a^2}(0.01 \times 10^{-3})$$

- The total contribution per edge is $E_{edge} = -\frac{\gamma}{2}\frac{L}{a^2}, \qquad \gamma = 0.00537$

- To be compared to $\gamma \approx 0.00523$(GIES & KLINGMULLER), ≈ 0.0050(GRAHAM *et al* (MIT))

- An interesting new result we can derive is the Casimir interaction between holes on a plate.
- The geometry is



- The functional integral is now

$$-\log Z_B = \beta \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \frac{1}{2} \,\mathrm{Tr}\, \log \begin{pmatrix} \mathcal{O}_{11} & \mathcal{O}_{12} \\ \mathcal{O}_{21} & \mathcal{O}_{22} \end{pmatrix}$$

where $\mathcal{O}_{ij} \sim \langle$ modes on hole i$|\mathcal{O}|$modes on hole j $\rangle$

- For separations large compared to the size of the holes,

$$\mathcal{O}_{12} \approx \langle 0| \sqrt{-\nabla^2 + \mu^2}\, |r\rangle = -\frac{1}{2\pi r^3}(1 + \mu r)e^{-\mu r}$$

- For $r >>$ (hole size) we can expand in powers of the off-diagonal entries.
- The interaction energy is

$$\begin{aligned} E_{\text{int}} &= -\frac{1}{2\pi}\int_0^\infty d\mu \, \text{Tr}\left[(\mathcal{O}_{11})^{-1}\mathcal{O}_{12}(\mathcal{O}_{22})^{-1}\mathcal{O}_{21}\right] \\ &= -\frac{5}{32\pi^3}\frac{Q_1 Q_2}{r^7} \qquad \text{(like van der Waals interaction)} \end{aligned}$$

where the charge associated with hole i is

$$\begin{aligned} Q_i &= \int_{\text{hole } i} d^2x \, d^2x' \, \langle \mathbf{x}| \, (\mathcal{O}_{ii})^{-1} \, |\mathbf{x}'\rangle \\ &\approx 1.28R^3 \text{ (round)}, \; 0.228L^3 \text{(square)} \end{aligned}$$

- Similar results are available for infinitely long slits

$$\frac{E_{\rm int}}{L} = -\frac{1}{2}\int \frac{d^2\mu}{2\pi^2}\,{\rm Tr}\left[(\mathcal{O}_{11})^{-1}\mathcal{O}_{12}(\mathcal{O}_{22})^{-1}\mathcal{O}_{21}\right]$$
$$= -5.375\times 10^{-3}\,\frac{Q_1 Q_2}{r^6}$$

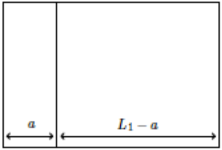
where the charge associated with slit 1 is

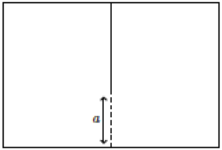
$$Q_1 = \int dx dx' \langle x|\mathcal{O}_{11}^{-1}|x'\rangle$$
$$= 2.88\times 10^{-2}(\text{slit width})^2$$

- Important aspect of Casimir effect is its dependence on temperature

$$F_T = \begin{cases} \sim T & \text{for high temperature} \\ \text{geometry} - \text{dependent} & \text{for low temperature} \end{cases}$$

- At low temperatures, thermal effects are dominated by long-range fluctuations





$$F_T = \frac{aA\zeta(4)}{\pi^2} T^4$$

$$F_T \sim T^b, \; b < 4$$

- Low-T results in open geometries mainly numerical (GIES, WEBER).
- Our formalism provides an analytic way to calculate thermal effects and account for thermal diffractive contributions
- In all geometries we studied

$$F = F^{\text{bulk}} + F^{\text{direct}} + F^{\text{diffractive}}$$

$F^{\text{bulk}} \Rightarrow$ free energy of $4d$ ideal Bose gas

$F^{\text{direct}} \Rightarrow$ free energy of $3d$ ideal Bose gas $\sim$ exponentially suppressed for low T ($T \ll 1/(\text{hole width})$)

$$F_T^{\text{diffractive}} = -\frac{L}{\pi} \sum_{l=1}^{\infty} \int_0^{\infty} d\mu\, \mu\, J_0(\mu l \beta)\, \log[Z_{2d}^{\text{diffractive}}(a\mu)]$$

At low $T(\beta \to \infty)$, finite thermal contributions come from non-analytic behavior of $\log Z_{2d}$ in μ^2 as $\mu \to 0$ (terms $\sim \mu^{2n}(\log(a\mu))^l$)

- *Slit* (of width a)

$$F_{\text{slit,T}} = \frac{\zeta(3)}{4\pi} LaT^3 - \frac{7\zeta(3)}{90\pi} La^2 T^4 + \mathcal{O}(T^6)$$

- *Perpendicular plates*

$$F_{\perp,T} = \frac{\zeta(3)}{4\pi} LaT^3 - \frac{16\pi\zeta(3)}{945} La^4 T^6 + \mathcal{O}(T^8)$$

(GIES, WEBER world-line formalism : $F_{\perp,T} \sim T^3$)

- *Infinite, Semi-infinite parallel plates*

The thermal free energy can be decomposed into an excluded volume contribution

$$F^{\text{ex}}_{||,T} = \frac{\zeta(4)}{\pi^2} V_{\text{ex}} T^4 - \frac{\zeta(3)}{8\pi} A_{\text{ex}} T^3 + \frac{\zeta(2)}{16\pi} P_{\text{ex}} T^2$$

and a diffractive edge contribution

$$F^{\text{edge}}_{||,T} = -\frac{2\zeta(4)}{\pi^3} Lb^2 T^4 \left(\log(2bT) + \frac{\zeta'(4)}{\zeta(4)}\right) + \frac{3\zeta(5)}{4\pi} Lb^3 T^5 + \cdots$$

(GIES, WEBER world-line formalism : $F^{\text{edge}}_{||,T} \sim T^{3.74}$)

- Previous analysis was for Dirichlet boundary conditions, $\phi = 0$ on the plates
- The most general boundary conditions (consistent with self-adjointness of Laplacian) are of the form

$$\partial_n \phi = -\mathcal{K}\,\phi$$

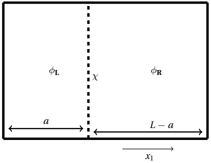
  where $\partial_n \phi$ is the normal derivative and $\mathcal{K}$ is a hermitian operator on the boundary values of the field
- Familiar boundary conditions are special cases:

$$\begin{array}{llll} \mathcal{K} \to 0 & \Longrightarrow & \partial_n \phi = 0 & \text{(Neumann)} \\ \mathcal{K} \to \infty & \Longrightarrow & \phi = 0 & \text{(Dirichlet)} \\ \mathcal{K} = \kappa \text{ (constant)} & \Longrightarrow & \partial_n \phi + \kappa\,\phi = 0 & \text{(Robin)} \end{array}$$

- Our formalism can accommodate all these different boundary conditions.

- The partition function can be thought of in terms of wave functionals if we think of x as time. Consider the space to be divided as shown.

  Integrate over all fields on the left to obtain



$$\Psi_L(\varphi) = \int [D\phi_L]\, e^{-S}$$

  with ϕ_L defined on the interval $0 \le x \le a$

  and $\phi_L = \varphi$ on boundary at $x = a$.

- Likewise, define

$$\widetilde{\Psi}_R(\varphi) = \int [D\phi_R]\, e^{-S}$$

  with ϕ_R defined on the interval $a \le x \le L$ and $\phi_R = \varphi$ on boundary at $x = a$.

- Then

$$Z_B = \int [D\varphi]\, \Psi_L\, \widetilde{\Psi}_R$$

- There are some advantages to this way of thinking about the problem
- If there is an actual Dirichlet plate with an aperture at $x = a$, we need to impose $\varphi = 0$ only on the plate, not the aperture, and we can write

$$Z_B = \int [d\varphi]\, \Psi_L(\varphi)\, \mathcal{C}_{plate}(\varphi)\, \widetilde{\Psi}_R(\varphi)$$

where $\mathcal{C}_{plate} = \delta[\varphi]$ on plate. This will give the previous results for Dirichlet bc.

- What is the analog of $\mathcal{C}_{plate}$ for more general boundary conditions such as $\partial_n \varphi = -\mathcal{K}\,\varphi$?
- If we think of x as "time", we can slice up the path integral along x-direction. The action can be written as

$$S = S(\{\phi_i\}) = \frac{1}{2}\int d^3x^T \left[\frac{(\phi_N - \phi_{N-1})^2}{x_N - x_{N-1}} + \frac{(\phi_{N-1} - \phi_{N-2})^2}{x_{N-1} - x_{N-2}} + \cdots + (\nabla^T \phi)^2\right]$$

where ϕ_N is the boundary value φ.

$$\frac{\delta}{\delta \phi_N}\, e^{-S} = -\frac{(\phi_N - \phi_{N-1})}{x_N - x_{N-1}}\, e^{-S} \to -\partial_n \varphi\, e^{-S}$$

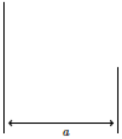
- $\partial_n \varphi + \mathcal{K}\,\varphi = 0$ on the plate $\Longleftrightarrow$ $\left[-\frac{\delta}{\delta\varphi} + \mathcal{K}\,\varphi\right]\ \Psi(\varphi) = 0$

$$\int [d\varphi]\ \exp\left(-\frac{1}{2}\int \varphi\,\mathcal{K}\,\varphi\right)\ (-\frac{\delta}{\delta\varphi} + \mathcal{K}\varphi)\ \Psi[\varphi] = 0$$

  This implies that for general boundary conditions $\mathcal{C}_{plate}(\varphi) = \exp\left(-\frac{1}{2}\int \varphi\,\mathcal{K}\,\varphi\right)$

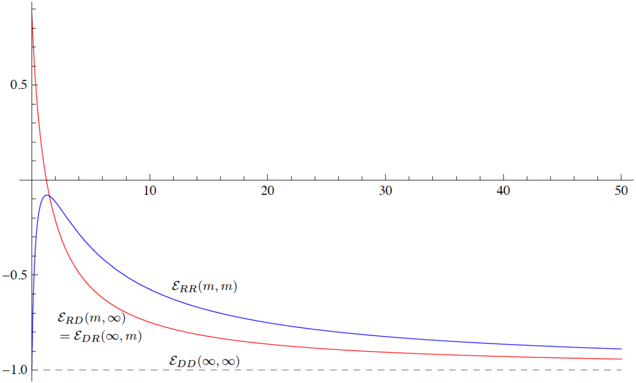
- One can also choose independent boundary fields $\varphi_L,\ \varphi_R$ on the left and right sides of the same plate
- The boundary part of the partition function becomes

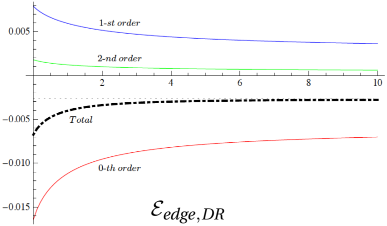
$$\begin{aligned} Z_B &= \int [d\varphi_L\, d\varphi_R]\ \delta_{aperture}(\varphi_L - \varphi_R)\ e^{-S_B} \\ S_B &= \frac{1}{2}\int_{plate,L} \varphi_L\,\mathcal{K}_L\,\varphi_L + \frac{1}{2}\int_{plate,R} \varphi_R\,\mathcal{K}_R\,\varphi_R + \frac{1}{2}\int_{boundary} \varphi\, M\, \varphi \end{aligned}$$



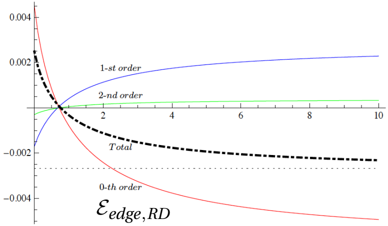
$$E = E_{bulk} + E_{edge}$$
$$E_{bulk} = \frac{\pi^2}{1440a^3} A\,\mathcal{E}(a\kappa_I, a\kappa_{II})$$
$$E_{edge} = \frac{L}{a^2}\,\mathcal{E}_{edge}(a\kappa_I, a\kappa_{II})$$

$\mathcal{E}$ depends on κ as shown in graph ($m = a\kappa$)

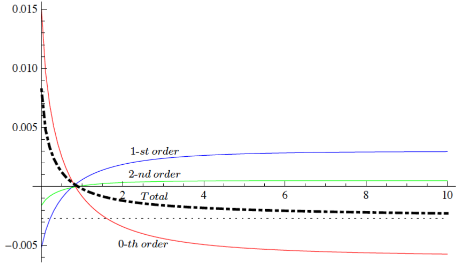




$\mathcal{E}_{edge,DR}$



$\mathcal{E}_{edge,RD}$



$\mathcal{E}_{edge,RR}$

- Contributions to Casimir energy due to boundary openings are described in terms of a non-local field theory defined on the hole.
- Our method of calculating Casimir energy in geometries with edges and apertures gives a systematic way of analyzing diffractive contributions in a perturbative expansion. It can be easily generalized to include finite temperature effects, higher dimensions, general boundary conditions, etc
- How do these results depend on spin, curvature?
- Calculate higher point functions (important for near field transmission effects).
- Generalization of BFK formula