Time dependent Hartree-Fock solution of Gross-Neveu models*

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Plan:

1. Introduction
2. How to solve the TDHF equations
3. Selected results

*) References:

Gerald V. Dunne and Michael Thies, PRL **111**, 121602 (2013); PRA **88**, 062115 (2013); PRD **89**, 025008 (2014)

1. Introduction

Quantum mechanics

Exactly solvable model problems:

δ-potential, harmonic oscillator, H-atom

Very helpful for the development and teaching of QM
Relevant for nature

Quantum field theory

The solution of a QFT is not a physical system, but a whole world
Very few exact, analytical results

Technically involved $\rightarrow$ frustrating teaching and learning experience

Main roads for solving QFT problems

• Perturbation theory: most impressive quantitative results (QED), but limited number of problems.

• Non-perturbative questions: Mostly numerical – “theoretical experiments” (lattice QCD). Very successful, but a large number of topics cannot be handled with present Monte-Carlo methods in Euclidean space (finite density, structure functions, time dependent problems, scattering or decay).

• Alternative for non-perturbative questions: Semiclassical methods. At the origin of most analytical results in QFT (Higgs mechanism, Seiberg-Witten, SUSY models, $1/N$ expansion).

Quest for exactly solvable quantum field theoretical models

Lagrangian $\longrightarrow$ wealth of phenomena
(analytically)

How to build a solvable QFT (here: for strongly interacting fermions)

1) Dimensional reduction from 3+1 to 1+1
2) Massless fermions with point interaction
3) Large N limit (number of flavors)

Nambu–Jona-Lasinio model (1963)

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i\partial\!\!\!/ \psi_k + \frac{g^2}{2}\left[\left(\sum_{k=1}^{N} \bar{\psi}_k \psi_k\right)^2 + \left(\sum_{k=1}^{N} \bar{\psi}_k i\gamma_5 \psi_k\right)^2\right]$$

Gross-Neveu model (1974)

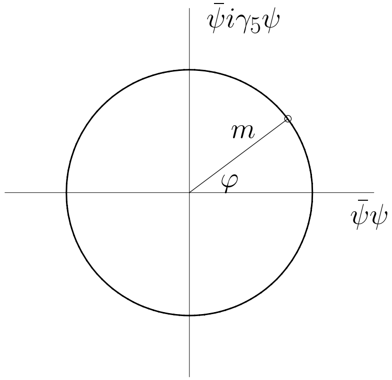
$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i\partial\!\!\!/ \psi_k + \frac{g^2}{2}\left(\sum_{k=1}^{N} \bar{\psi}_k \psi_k\right)^2$$

Salient features

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i \gamma_5 \psi\right)^2\right]$$

- renormalizable in 2d ($[g] = 1$)
- asymptotic freedom ($\beta = -\frac{Ng^2}{2\pi}$)
- U(N) flavor symmetry
- continuous or discrete chiral symmetry ($\psi \to e^{i\alpha\gamma_5}\psi$ resp. $\psi \to \gamma_5\psi$), spontaneously broken in the vacuum

$$\begin{aligned} S &= m \cos\varphi = -g^2\langle\bar{\psi}\psi\rangle \\ P &= m \sin\varphi = -g^2\langle\bar{\psi} i\gamma_5\psi\rangle \end{aligned}$$



- dimensional transmutation (dynamical fermion mass)
- fermion-antifermion bound states (σ, π)
- rich spectrum of multifermion bound states, no confinement
- rich phase diagram as a function of T, μ

Semiclassical methods become exact in the large N limit

Relativistic time-dependent Hartree-Fock (Dirac 1930, Witten 1979)

$$\left(i\partial\!\!\!/ - S - i\gamma_5 P\right)\psi_\alpha = 0$$

$$S = -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta \psi_\beta, \quad P = -g^2 \sum_\beta^{\rm occ} \bar{\psi}_\beta i\gamma_5 \psi_\beta$$

Infinite set of coupled, nonlinear PDE's (Dirac sea+valence levels).

Many exact, static solutions known in closed analytic form:

- vacuum (dynamical fermion mass, SSB of chiral symmetry)
- baryons (topologically nontrivial kinks, kink-antikink baryons)
- multikink and multibaryon bound states ("nuclei")
- cold dense matter (soliton crystal, akin to Skyrme crystal)
- full phase diagram at finite T and μ

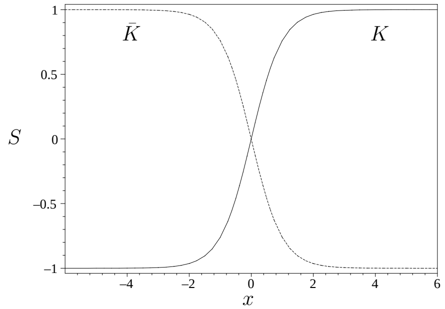
To find time-dependent solutions has proven more difficult. Only one such solution was known before 2010, the *breather* of the Gross-Neveu model (Dashen, Hasslacher and Neveu 1975).

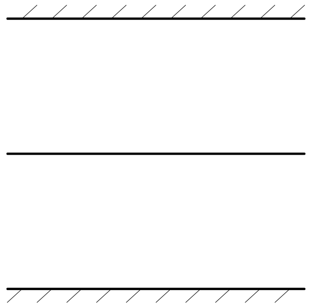
Is it worth the effort?

The Gross-Neveu kink (Callen, Coleman, Gross, Zee 1975)

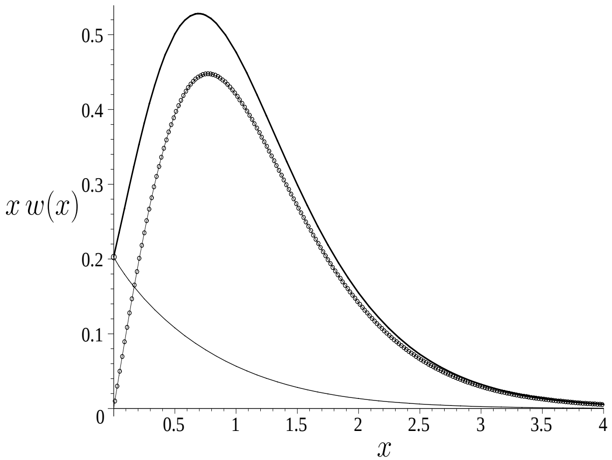
Reflection of Z_2 chiral symmetry, paradigm for fractional fermion number

$$S(x) = \pm \tanh x$$





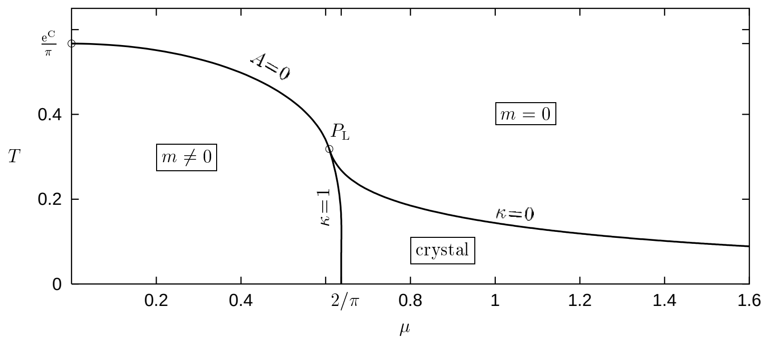
Structure functions (Brendel 2010)



Total baryon momentum: Valence quarks $\frac{1}{2}\ln 2 \approx 35\%$, sea quarks $1/2 = 50\%$, antiquarks $\frac{1}{2}(1-\ln 2) \approx 15\%$

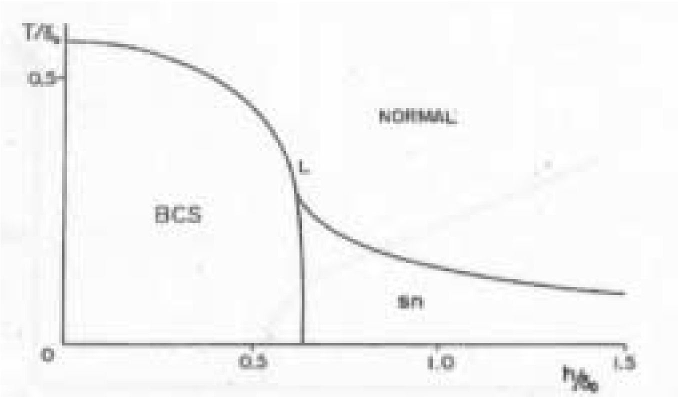
Déjà vu

1)



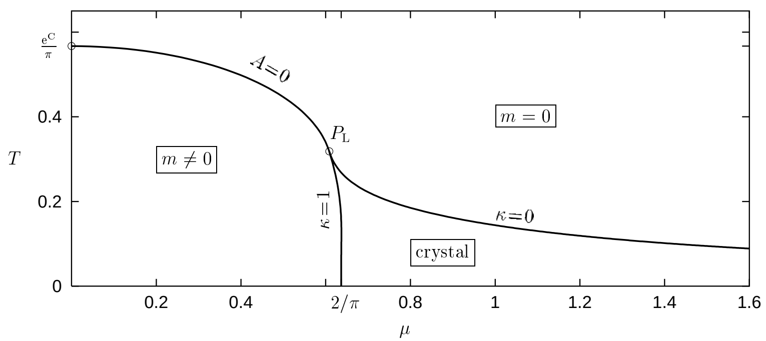
$\frac{e^C}{\pi}$
0.4
0.2
0
T
$\Delta = 0$
P_L
$m = 0$
$m \neq 0$
$\kappa = 1$
$\kappa = 0$
crystal
0.2
0.4
$2/\pi$
0.8
1
1.2
1.4
1.6
μ

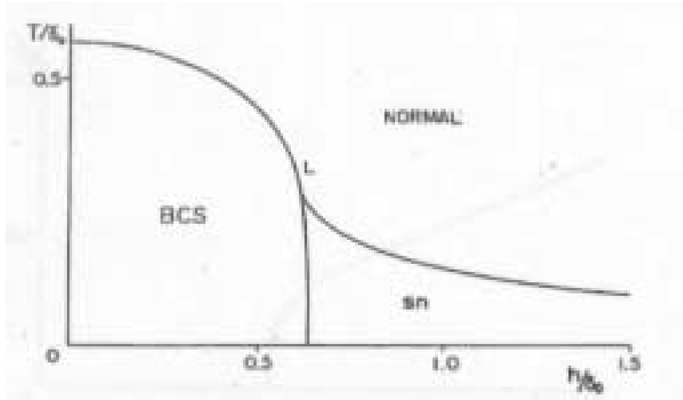
2)



0.5
NORMAL
L
BCS
sn
0
0.5
1.0
1.5

Déjà vu

1) 

2) 

1) Schnetz, Thies and Urlichs (2004) Phase diagram of the Gross-Neveu model: exact results and condensed matter precursors

2) Machida and Nakanishi (1984) Superconductivity under a ferromagnetic molecular field ($ErRh_4B_4$)

2. How to solve the TDHF equations

Gross-Neveu model

$$(i\partial\!\!\!/ - S)\,\psi_\alpha = 0, \quad S = -g^2 \sum_\beta^{\text{occ}} \bar{\psi}_\beta \psi_\beta$$

- Static solutions: Complete picture from inverse scattering theory.

Transparent potentials

- Time dependent solutions: no systematic method available.

Only one solution was known: DHN breather

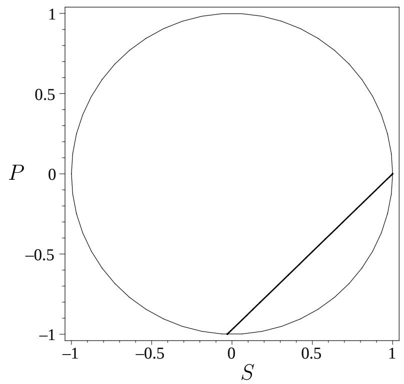
NJL model

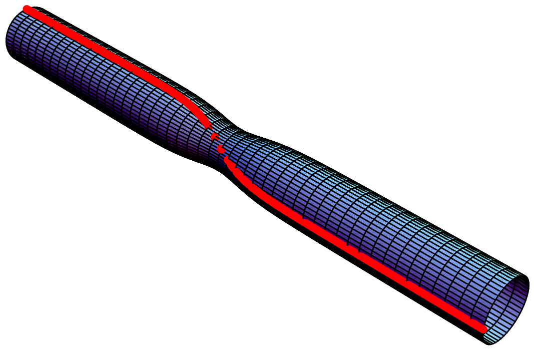
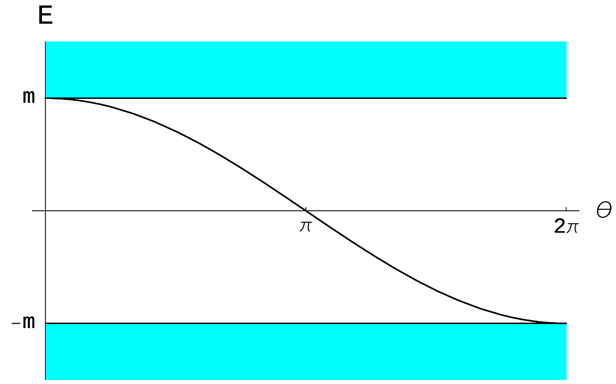
$$(i\partial\!\!\!/ - S - i\gamma_5 P)\,\psi_\alpha = 0, \quad S = -g^2 \sum_\beta^{\text{occ}} \bar{\psi}_\beta \psi_\beta, \quad P = -g^2 \sum_\beta^{\text{occ}} \bar{\psi}_\beta i\gamma_5 \psi_\beta$$

- Only one static solution was known: twisted kink (Shei 1976)

Twisted kink of NJL model (Shei 1976)

Reflection of U(1) chiral symmetry



Self-consistency relates occupation fraction of valence state to chiral twist

$$S - iP = \frac{1 + e^{-2i\theta} e^{2x \sin\theta}}{1 + e^{2x \sin\theta}}, \quad \nu = \frac{\theta}{\pi}$$

Total fermion number vanishes due to U(1)⊗U(1) chiral symmetry

$$\partial_\mu j_V^\mu = 0, \qquad \partial_\mu j_A^\mu = 0$$

1+1 dimensions

$$j_V^\mu = \epsilon^{\mu\nu} \partial_\nu \Phi, \quad j_A^\mu = \epsilon^{\mu\nu} j_{V,\nu} = \partial^\mu \Phi, \quad \Box \Phi = 0$$

Recent progress on time-dependent issues in the Gross-Neveu model

- Solve multi-kink dynamics (sinh-Gordon theory)
- Solve few baryon scattering problems (ansatz method)
- Solve breather-breather scattering (ansatz method)

Problem: Intransparent results, limited to few hadrons; mysterious hints of factorization; pushing Maple to its limits

Apply similar methods to NJL model: Striking simplification, factorization in terms of twisted kinks. Static multikink solutions found independently by Takahashi and Nitta in condmatt (Bogoliubov–de Gennes equation).

Both NJL model and Gross-Neveu model solved in full generality

Two-strike procedure:

1) Find the most general transparent, time-dependent, scalar/pseudoscalar Dirac potential

$$\left(i\partial\!\!\!/ - S - i\gamma_5 P\right)\psi_\alpha = 0$$

Generalizes transparent, static Schrödinger potentials (Kay, Moses 1956)

Light cone variables ($m = 1$)

$$\zeta = p - E, \quad z = x - t, \quad \bar{z} = x + t$$

(Reflectionless) continuum spinor,

$$\psi_\zeta = \frac{1}{\sqrt{1+\zeta^2}} \begin{pmatrix} \zeta\chi_1 \\ -\chi_2 \end{pmatrix} e^{i(\zeta\bar{z} - z/\zeta)/2}$$

Full result ($N \times N$ matrices)

$$S - iP = \frac{\det(\omega + A)}{\det(\omega + B)}, \quad \chi_1 = \frac{\det(\omega + C)}{\det(\omega + B)}, \quad \chi_2 = \frac{\det(\omega + D)}{\det(\omega + B)}$$

$$B_{nm} = i\frac{e_n e_m^*}{\zeta_n - \zeta_m^*}, \quad e_n = e^{i(\zeta_n^* \bar{z} - z/\zeta_n^*)/2}, \quad n = 1, ..., N$$

Other matrices

$$A = \left(Z^\dagger\right)^{-1} BZ, \quad C = (\zeta - Z^\dagger) B (\zeta - Z)^{-1}, \quad D = \left(Z^\dagger\right)^{-1} CZ$$

with

$$Z = \text{diag}\,(\zeta_1, ..., \zeta_N)$$

Can verify Dirac equation analytically in a few lines.

What is behind this solution? "Dressing method"

Chiral representation of Dirac matrices

$$\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma_5 = \gamma^0\gamma^1 = -\sigma_3$$

Dirac equation ($\Delta = S - iP$)

$$2i\bar{\partial}\psi_2 = \Delta\psi_1, \quad 2i\partial\psi_1 = -\Delta^*\psi_2$$

Suppose we have found N solutions ($\to \psi_1, \psi_2$ are N-component vectors). Perform a "gauge transformation"

$$\psi_1 = \Omega\psi_1', \quad \psi_2 = \Omega\psi_2'$$

Dirac equation

$$
\begin{aligned}
2i\bar{\partial}\psi_2' &= \Delta\psi_1' - 2i\Omega^{-1}\bar{\partial}\Omega\psi_2' \\
2i\partial\psi_1' &= -\Delta^*\psi_2' - 2i\Omega^{-1}\partial\Omega\psi_1'
\end{aligned}
$$

Provided that

$$\bar{\partial}\Omega = c\psi_1\psi_1^\dagger, \quad \partial\Omega = c^*\psi_2\psi_2^\dagger, \quad \Omega = \Omega^\dagger$$

this is again a Dirac equation for ψ_1', ψ_2' with the new potential

$$\Delta' = \Delta - 2ic\psi_1^\dagger\psi_2'$$

This reproduces the above results for the following choice (“vacuum”)

$$\Delta = 1, \quad \psi_1 = e, \quad \psi_2 = -(Z^\dagger)^{-1}e, \quad \Omega = \omega + B, \quad c = 1/2$$

General transparent potential depends on $\zeta_n = -e^{i\theta_n/2}e^{-\xi_n}$ (chiral twist, rapidity of each kink) and ω_{nm} (diagonal: spatial structure of composite states, initial conditions, off-diagonal: oscillation modes of breathers). Kinks with the same rapidity belong to a bound cluster and must have different chiral twist.

General solution contains a subset of solutions with $P = 0$, candidates for self-consistent Gross-Neveu potentials. Two possibilities: Either chiral twist π, or pairs of kinks with total twist 2π.

Fully factorized, unitary transmission amplitude

$$T(\zeta) = \prod_{n=1}^{N} \frac{\zeta - \zeta_n^*}{\zeta - \zeta_n}, \quad |T(\zeta)| = 1$$

2) Check self-consistency of the transparent Dirac potential

$$S = -g^2 \sum_{\beta}^{\text{occ}} \bar{\psi}_\beta \psi_\beta, \quad P = -g^2 \sum_{\beta}^{\text{occ}} \bar{\psi}_\beta i\gamma_5 \psi_\beta$$

Yields bound state occupation fractions, restricts parameters.

$$S - iP = -2Ng^2 \left(\langle \psi_1^* \psi_2 \rangle_{\text{sea}} + \langle \psi_1^* \psi_2 \rangle_{\text{b}} \right),$$

$$\langle \psi_1^* \psi_2 \rangle_{\text{sea}} = -\frac{1}{2} \int_{1/\Lambda}^{\Lambda} \frac{d\zeta}{2\pi} \frac{1}{\zeta} \chi_1^* \chi_2, \quad \langle \psi_1^* \psi_2 \rangle_{\text{b}} = \sum_n \nu_n \hat{\phi}_{1,n}^* \hat{\phi}_{2,n}$$

Pole at $\zeta = 0$: divergent contribution

$$\langle \psi_1^* \psi_2 \rangle_{\text{sea}}\big|_{\text{div}} = -\frac{(S - iP)}{2\pi} \ln \Lambda.$$

Self-consistent by itself owing to vacuum gap equation

$$\frac{Ng^2}{\pi} \ln \Lambda = 1$$

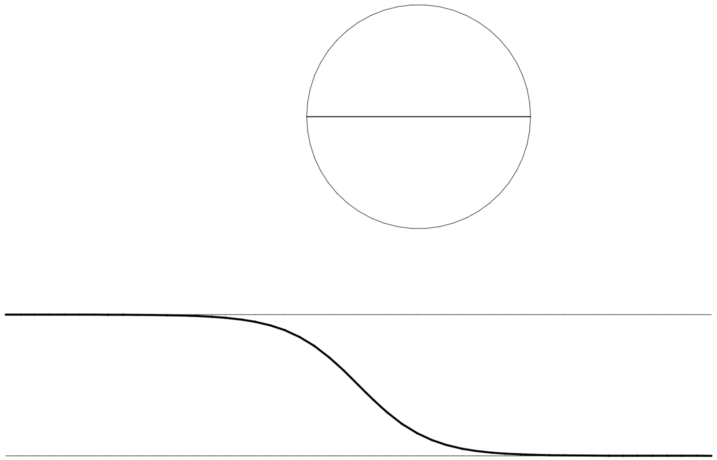
$$\langle \psi_1^* \psi_2 \rangle_{\text{sea}}\big|_{\text{conv}} + \langle \psi_1^* \psi_2 \rangle_{\text{b}} = 0$$

Can be dealt with by algebraic means. The self-consistency condition also guarantees that the fermion density vanishes in the NJL model.

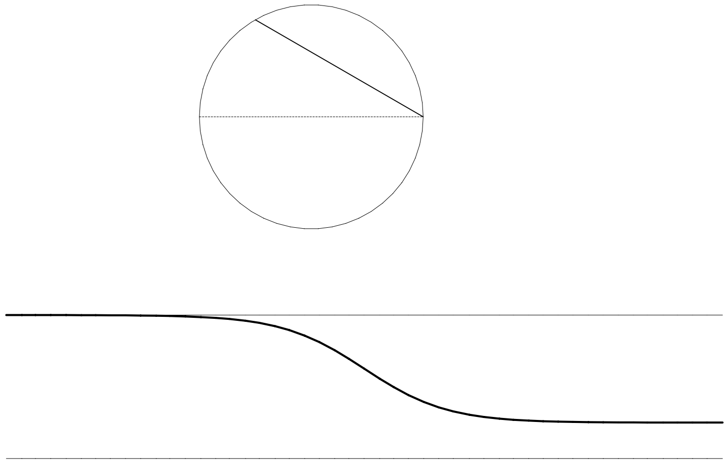
Gross-Neveu model: The self-consistency condition changes (only S). Therefore the occupation fractions change and non-zero fermion density can be described.

3. Selected results

Twisted kinks as confined constituents of Gross-Neveu hadrons



Gross-Neveu kink

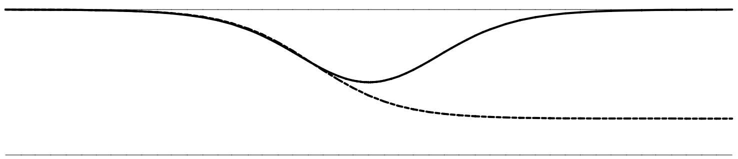


NJL kink (only S)

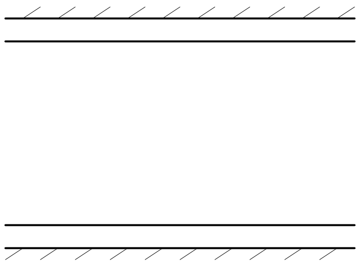
DHN baryon as a bound state of twisted kinks:

Original parametrization: Angle θ without geometrical interpretation — can be identified with chiral twist angle of constituent kinks

Shape



Spectrum



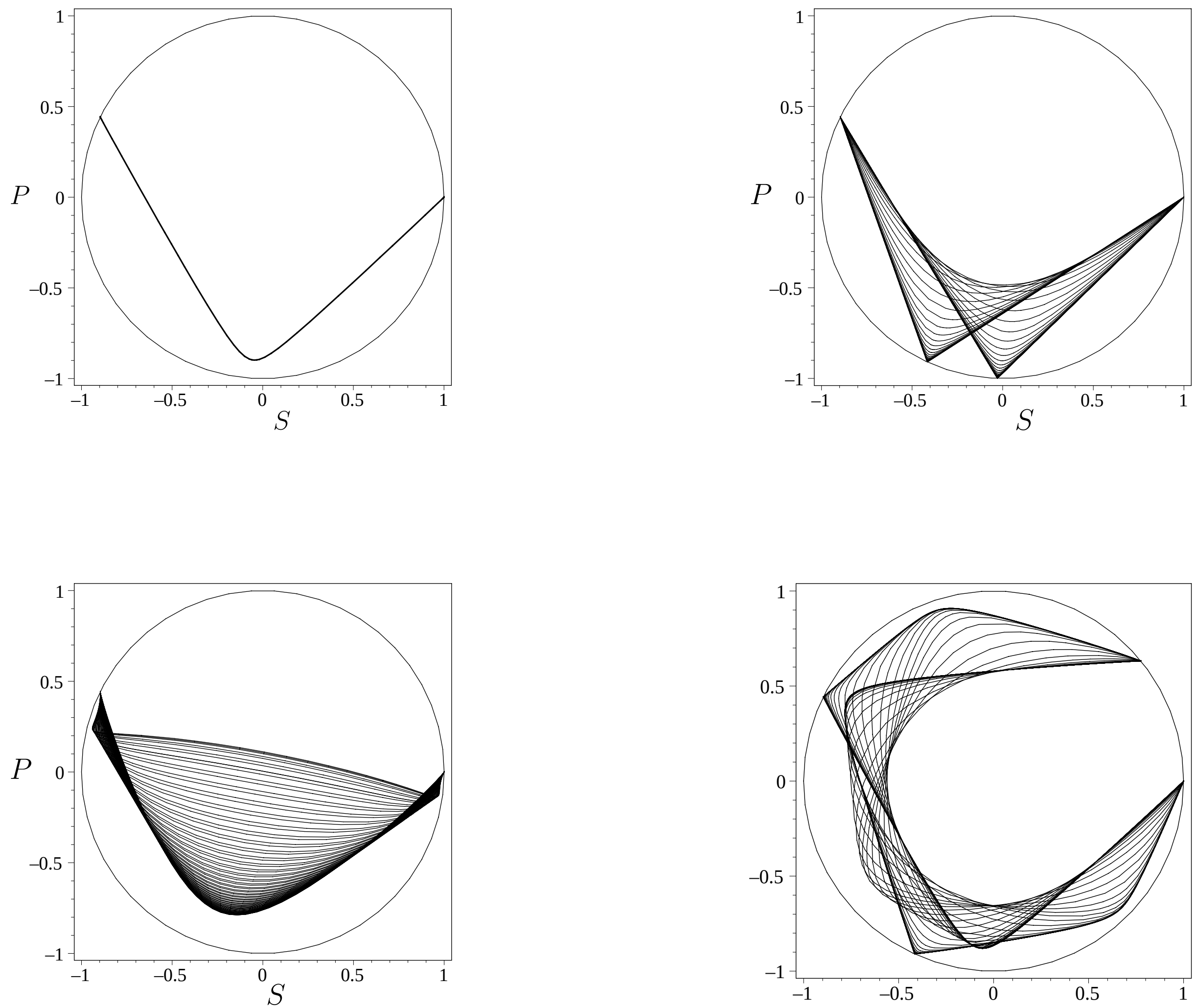
Additive mass

$$M_{\mathsf{DHN}}/N = \frac{2}{\pi}\sin\theta = \frac{1}{\pi}\sin\theta + \frac{1}{\pi}\sin(\pi-\theta) = M_{\mathsf{kink}}/N + M_{\mathsf{kink}}/N$$

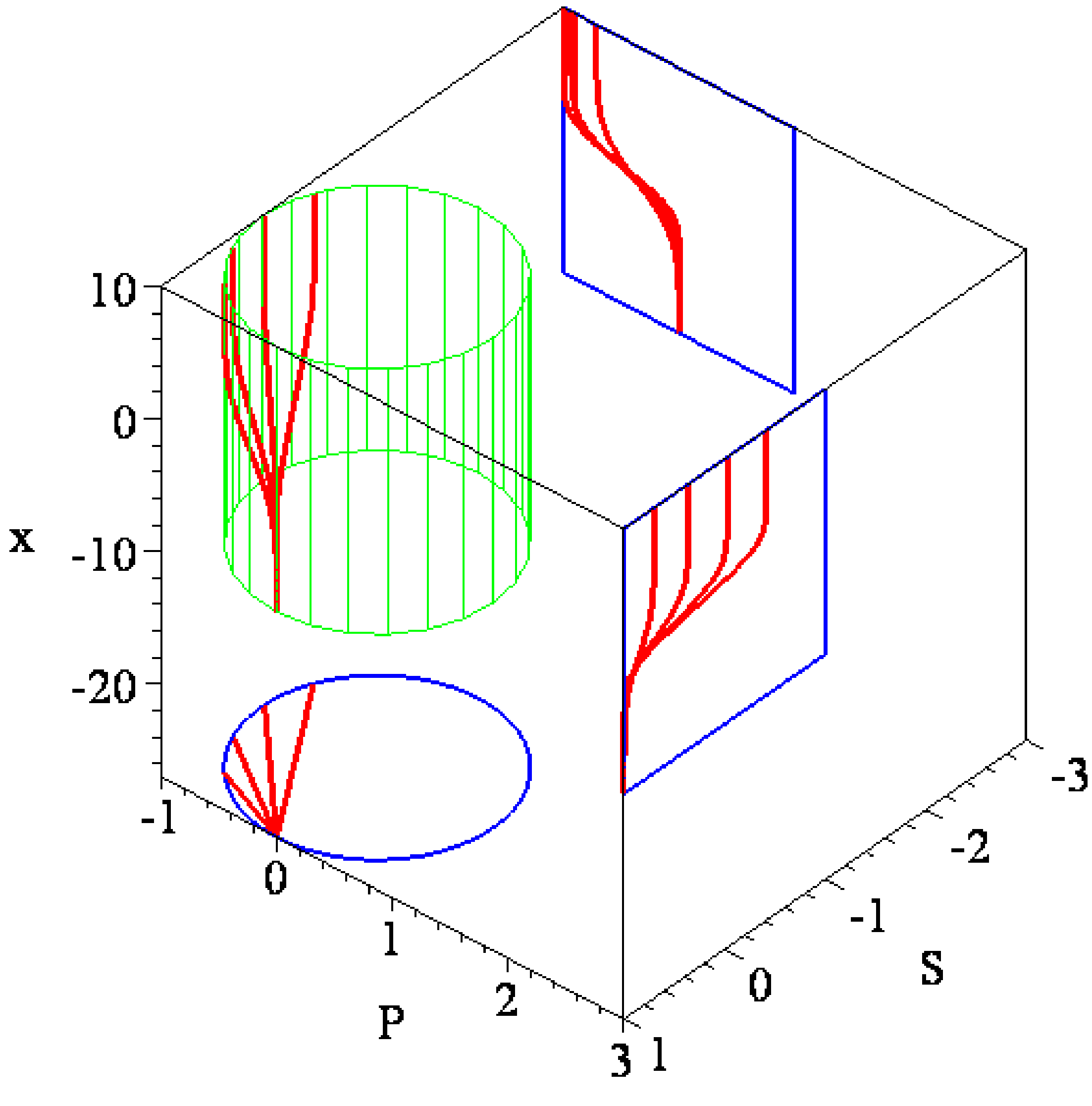
Factorized transmission amplitude ($\zeta = p - \sqrt{p^2+1}$)

$$T_{\mathsf{DHN}} = \frac{p + i\sin\theta}{p - i\sin\theta} = \left(\frac{\zeta + e^{i\theta}}{\zeta + e^{-i\theta}}\right)\left(\frac{\zeta - e^{-i\theta}}{\zeta - e^{i\theta}}\right) = \left(\frac{\zeta - \zeta_1^*}{\zeta - \zeta_1}\right)\left(\frac{\zeta - \zeta_2^*}{\zeta - \zeta_2}\right)$$

How to present results for time-dependent solutions?



Full information: Static, twisted kinks as building blocks of animations



Another kind of dressing

An iteration of our dressing procedure leads back to the original solution

$$(\psi_1, \psi_2, \Delta) \to (\psi_1', \psi_2', \Delta') \to (\psi_1, \psi_2, \Delta)$$

However, the Dirac equation

$$2i\bar{\partial}\psi_2 = \Delta\psi_1, \quad 2i\partial\psi_1 = -\Delta^*\psi_2$$

also supports the following kind of dressing

$$(\psi_1, \psi_2, \Delta) \to (e^{i\alpha(\bar{z})}\psi_1, e^{i\beta(z)}\psi_2, e^{i(\beta(z)-\alpha(\bar{z}))}\Delta)$$

Self-consistency is also preserved. Interpretation: Macroscopic numbers of left- and right-moving pions.

Physically interesting special case:

$$\alpha(\bar{z}) = a\bar{z}, \quad \beta(z) = bz$$

Δ has vanishing fermion density ρ and current density j. Due to the chiral anomaly, $\Delta' = e^{i(bz-a\bar{z})}\Delta$ has

$$\rho = \frac{b-a}{2\pi}, \quad j = \frac{b+a}{2\pi}$$

This enables us to immerse any TDHF solution into a system with constant fermion density (spacelike chiral spiral) or current density (timelike chiral spiral)