Time dependent Hartree-Fock solution
of Gross-Neveu models

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Plan:

1. Introduction
2. How to solve the TDHF equations
3. Selected results

*) References:
Gerald V. Dunne and Michael Thies, PRL 111, 121602 (2013); PRA 88, 062115 (2013);
PRD 89, 025008 (2014)
1. Introduction

Quantum mechanics

Exactly solvable model problems:

δ-potential, harmonic oscillator, H-atom

Very helpful for the development and teaching of QM
Relevant for nature

Quantum field theory

The solution of a QFT is not a physical system, but a whole world
Very few exact, analytical results

Technically involved → frustrating teaching and learning experience
Main roads for solving QFT problems

- Perturbation theory: most impressive quantitative results (QED), but limited number of problems.

- Non-perturbative questions: Mostly numerical – “theoretical experiments” (lattice QCD). Very successful, but a large number of topics cannot be handled with present Monte-Carlo methods in Euclidean space (finite density, structure functions, time dependent problems, scattering or decay).

- Alternative for non-perturbative questions: Semiclassical methods. At the origin of most analytical results in QFT (Higgs mechanism, Seiberg-Witten, SUSY models, $1/N$ expansion).
Quest for exactly solvable quantum field theoretical models

Lagrangian $\rightarrow$ wealth of phenomena (analytically)

How to build a solvable QFT (here: for strongly interacting fermions)

1) Dimensional reduction from 3+1 to 1+1
2) Massless fermions with point interaction
3) Large $N$ limit (number of flavors)

Nambu–Jona-Lasinio model (1963)

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i \slashed{D} \psi_k + \frac{g^2}{2} \left[ \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2 + \left( \sum_{k=1}^{N} \bar{\psi}_k i \gamma_5 \psi_k \right)^2 \right]$$

Gross-Neveu model (1974)

$$\mathcal{L} = \sum_{k=1}^{N} \bar{\psi}_k i \slashed{D} \psi_k + \frac{g^2}{2} \left( \sum_{k=1}^{N} \bar{\psi}_k \psi_k \right)^2$$
Salient features

\[ \mathcal{L} = \bar{\psi} i \gamma_5 \psi + \frac{g^2}{2} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_5 \psi)^2 \right] \]

- renormalizable in 2d \([g] = 1\)
- asymptotic freedom \((\beta = -\frac{Ng^2}{2\pi})\)
- U\((N)\) flavor symmetry
- continuous or discrete chiral symmetry \((\psi \rightarrow e^{i\alpha\gamma_5} \psi \text{ resp. } \psi \rightarrow \gamma_5 \psi)\), spontaneously broken in the vacuum

\[ S = m \cos \varphi = -g^2 \langle \bar{\psi} \psi \rangle \]
\[ P = m \sin \varphi = -g^2 \langle \bar{\psi} \gamma_5 \psi \rangle \]

- dimensional transmutation (dynamical fermion mass)
- fermion-antifermion bound states \((\sigma, \pi)\)
- rich spectrum of multifermion bound states, no confinement
- rich phase diagram as a function of \(T, \mu\)

Semiclassical methods become exact in the large \(N\) limit
Relativistic time-dependent Hartree-Fock  (Dirac 1930, Witten 1979)

\[(i\partial - S - i\gamma_5 P)\psi_\alpha = 0\]

\[S = -g^2 \sum_\beta \bar{\psi}_\beta \psi_\beta, \quad P = -g^2 \sum_\beta \bar{\psi}_\beta i\gamma_5 \psi_\beta\]

Infinite set of coupled, nonlinear PDE’s (Dirac sea+valence levels).

Many exact, static solutions known in closed analytic form:
- vacuum (dynamical fermion mass, SSB of chiral symmetry)
- baryons (topologically nontrivial kinks, kink-antikink baryons)
- multikink and multibaryon bound states (“nuclei”)
- cold dense matter (soliton crystal, akin to Skyrme crystal)
- full phase diagram at finite \(T\) and \(\mu\)

To find time-dependent solutions has proven more difficult. Only one such solution was known before 2010, the breather of the Gross-Neveu model (Dashen, Hasslacher and Neveu 1975).

Is it worth the effort?
The Gross-Neveu kink (Callen, Coleman, Gross, Zee 1975)
Reflection of $Z_2$ chiral symmetry, paradigm for fractional fermion number

$$S(x) = \pm \tanh x$$

Structure functions (Brendel 2010)

Total baryon momentum: Valence quarks $\frac{1}{2} \ln 2 \approx 35\%$, sea quarks $1/2 = 50\%$, antiquarks $\frac{1}{2}(1 - \ln 2) \approx 15\%$
Déjà vu

2) Machida and Nakanishi (1984) Superconductivity under a ferromagnetic molecular field (ErRh$_4$B$_4$)
2. How to solve the TDHF equations

Gross-Neveu model

\[(i\partial - S)\psi_\alpha = 0, \quad S = -g^2 \sum_{\beta} \bar{\psi}_\beta \psi_\beta\]

- Static solutions: Complete picture from inverse scattering theory. Transparent potentials
- Time dependent solutions: no systematic method available.
  Only one solution was known: DHN breather

NJL model

\[(i\partial - S - i\gamma_5 P)\psi_\alpha = 0, \quad S = -g^2 \sum_{\beta} \bar{\psi}_\beta \psi_\beta, \quad P = -g^2 \sum_{\beta} \bar{\psi}_\beta i\gamma_5 \psi_\beta\]

- Only one static solution was known: twisted kink (Shei 1976)
Twisted kink of NJL model (Shei 1976)

Reflection of U(1) chiral symmetry

Self-consistency relates occupation fraction of valence state to chiral twist

\[ S - iP = \frac{1 + e^{-2i\theta} e^{2x \sin \theta}}{1 + e^{2x \sin \theta}}, \quad \nu = \frac{\theta}{\pi} \]

Total fermion number vanishes due to U(1) \( \otimes \) U(1) chiral symmetry

\[ \partial_\mu j_\mu^V = 0, \quad \partial_\mu j_\mu^A = 0 \]

1+1 dimensions

\[ j_\mu^V = \epsilon^{\mu\nu} \partial_\nu \Phi, \quad j_\mu^A = \epsilon^{\mu\nu} j_\nu^V, \quad \Box \Phi = 0 \]
Recent progress on time-dependent issues in the Gross-Neveu model

- Solve multi-kink dynamics (sinh-Gordon theory)
- Solve few baryon scattering problems (ansatz method)
- Solve breather-breather scattering (ansatz method)

Problem: Intransparent results, limited to few hadrons; mysterious hints of factorization; pushing Maple to its limits

Apply similar methods to NJL model: Striking simplification, factorization in terms of twisted kinks. Static multikink solutions found independently by Takahashi and Nitta in condmatt (Bogoliubov–de Gennes equation).

Both NJL model and Gross-Neveu model solved in full generality
Two-strike procedure:

1) Find the most general transparent, time-dependent, scalar/pseudoscalar Dirac potential

\[(i\partial - S - i\gamma_5 P) \psi_\alpha = 0\]

Generalizes transparent, static Schrödinger potentials (Kay, Moses 1956)

Light cone variables \((m = 1)\)

\[\zeta = p - E, \quad z = x - t, \quad \bar{z} = x + t\]

(Reflectionless) continuum spinor,

\[\psi_\zeta = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} \zeta \chi_1 \\ -\chi_2 \end{pmatrix} e^{i(\zeta \bar{z} - z/\zeta)/2}\]

Full result \((N \times N)\) matrices

\[S - iP = \frac{\det(\omega + A)}{\det(\omega + B)}, \quad \chi_1 = \frac{\det(\omega + C)}{\det(\omega + B)}, \quad \chi_2 = \frac{\det(\omega + D)}{\det(\omega + B)}\]

\[B_{nm} = i\frac{e_n e^*_m}{\zeta_n - \zeta^*_m}, \quad e_n = e^{i(\zeta^*_n \bar{z} - z/\zeta^*_n)/2}, \quad n = 1, ..., N\]
Other matrices

\[ A = (Z^\dagger)^{-1} B Z, \quad C = (\zeta - Z^\dagger) B (\zeta - Z)^{-1}, \quad D = (Z^\dagger)^{-1} C Z \]

with

\[ Z = \text{diag} (\zeta_1, ..., \zeta_N) \]

Can verify Dirac equation analytically in a few lines.

What is behind this solution? “Dressing method”

Chiral representation of Dirac matrices

\[ \gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma^5 = \gamma^0 \gamma^1 = -\sigma_3 \]

Dirac equation \((\Delta = S - iP)\)

\[ 2i \bar{\psi}_2 = \Delta \psi_1, \quad 2i \bar{\psi}_1 = -\Delta^* \psi_2 \]

Suppose we have found \(N\) solutions \((\rightarrow \psi_1, \psi_2 \text{ are } N\)-component vectors). Perform a “gauge transformation”

\[ \psi_1 = \Omega \psi'_1, \quad \psi_2 = \Omega \psi'_2 \]
Dirac equation

\begin{align*}
2i\bar{\partial}\psi'_2 &= \Delta\psi'_1 - 2i\Omega^{-1}\bar{\partial}\Omega\psi'_2 \\
2i\partial\psi'_1 &= -\Delta^*\psi'_2 - 2i\Omega^{-1}\partial\Omega\psi'_1
\end{align*}

Provided that

\[ \bar{\partial}\Omega = c\psi_1\psi_1^\dagger, \quad \partial\Omega = c^*\psi_2\psi_2^\dagger, \quad \Omega = \Omega^\dagger \]

this is again a Dirac equation for \(\psi'_1, \psi'_2\) with the new potential

\[ \Delta' = \Delta - 2i\psi_1^\dagger\psi_2' \]

This reproduces the above results for the following choice ("vacuum")

\[ \Delta = 1, \quad \psi_1 = e, \quad \psi_2 = -(Z^\dagger)^{-1}e, \quad \Omega = \omega + B, \quad c = 1/2 \]

General transparent potential depends on \(\zeta_n = -e^{i\theta_n/2}e^{-\xi_n}\) (chiral twist, rapidity of each kink) and \(\omega_{nm}\) (diagonal: spatial structure of composite states, initial conditions, off-diagonal: oscillation modes of breathers). Kinks with the same rapidity belong to a bound cluster and must have different chiral twist.
General solution contains a subset of solutions with \( P = 0 \), candidates for self-consistent Gross-Neveu potentials. Two possibilities: Either chiral twist \( \pi \), or pairs of kinks with total twist \( 2\pi \).

Fully factorized, unitary transmission amplitude

\[
T(\zeta) = \prod_{n=1}^{N} \frac{\zeta - \zeta_{n}^{*}}{\zeta - \zeta_{n}}, \quad |T(\zeta)| = 1
\]

2) Check self-consistency of the transparent Dirac potential

\[
S = -g^{2} \sum_{\beta}^{\text{occ}} \bar{\psi}_{\beta} \psi_{\beta}, \quad P = -g^{2} \sum_{\beta}^{\text{occ}} \bar{\psi}_{\beta} i \gamma_{5} \psi_{\beta}
\]

Yields bound state occupation fractions, restricts parameters.

\[
S - iP = -2Ng^{2} \left( \langle \psi_{1}^{*} \psi_{2} \rangle_{\text{sea}} + \langle \psi_{1}^{*} \psi_{2} \rangle_{b} \right),
\]

\[
\langle \psi_{1}^{*} \psi_{2} \rangle_{\text{sea}} = -\frac{1}{2} \int_{1/\Lambda}^{\Lambda} \frac{d\zeta}{2\pi} \chi_{1}^{*} \chi_{2}, \quad \langle \psi_{1}^{*} \psi_{2} \rangle_{b} = \sum_{n} \nu_{n} \phi_{1,n}^{*} \phi_{2,n}
\]
Pole at $\zeta = 0$: divergent contribution

$$\langle \psi_1^* \psi_2 \rangle_{\text{sea}}|_{\text{div}} = -\frac{(S - iP)}{2\pi} \ln \Lambda.$$ 

Self-consistent by itself owing to vacuum gap equation

$$\frac{Ng^2}{\pi} \ln \Lambda = 1$$

$$\langle \psi_1^* \psi_2 \rangle_{\text{sea}}|_{\text{conv}} + \langle \psi_1^* \psi_2 \rangle_b = 0$$

Can be dealt with by algebraic means. The self-consistency condition also guarantees that the fermion density vanishes in the NJL model.

Gross-Neveu model: The self-consistency condition changes (only $S$). Therefore the occupation fractions change and non-zero fermion density can be described.
3. Selected results

Twisted kinks as confined constituents of Gross-Neveu hadrons

DHN baryon as a bound state of twisted kinks:

Original parametrization: Angle $\theta$ without geometrical interpretation — can be identified with chiral twist angle of constituent kinks
Additive mass

\[ M_{\text{DHN}}/N = \frac{2}{\pi} \sin \theta = \frac{1}{\pi} \sin \theta + \frac{1}{\pi} \sin(\pi - \theta) = M_{\text{kink}}/N + M_{\text{kink}}/N \]

Factorized transmission amplitude \((\zeta = p - \sqrt{p^2 + 1})\)

\[ T_{\text{DHN}} = \frac{p + i \sin \theta}{p - i \sin \theta} = \left( \frac{\zeta + e^{i\theta}}{\zeta + e^{-i\theta}} \right) \left( \frac{\zeta - e^{-i\theta}}{\zeta - e^{i\theta}} \right) = \left( \frac{\zeta - \zeta_1^*}{\zeta - \zeta_1} \right) \left( \frac{\zeta - \zeta_2^*}{\zeta - \zeta_2} \right) \]
How to present results for time-dependent solutions?
Full information: Static, twisted kinks as building blocks of animations
Another kind of dressing
An iteration of our dressing procedure leads back to the original solution

$$(\psi_1, \psi_2, \Delta) \rightarrow (\psi'_1, \psi'_2, \Delta') \rightarrow (\psi_1, \psi_2, \Delta)$$

However, the Dirac equation

$$2i\bar{\partial}\psi_2 = \Delta \psi_1, \quad 2i\partial\psi_1 = -\Delta^*\psi_2$$

also supports the following kind of dressing

$$(\psi_1, \psi_2, \Delta) \rightarrow (e^{i\alpha(\bar{z})}\psi_1, e^{i\beta(z)}\psi_2, e^{i(\beta(z)-\alpha(\bar{z}))}\Delta)$$

Self-consistency is also preserved. Interpretation: Macroscopic numbers of left- and right-moving pions.

Physically interesting special case:

$$\alpha(\bar{z}) = a\bar{z}, \quad \beta(z) = bz$$

$\Delta$ has vanishing fermion density $\rho$ and current density $j$. Due to the chiral anomaly, $\Delta' = e^{i(bz-a\bar{z})}\Delta$ has

$$\rho = \frac{b-a}{2\pi}, \quad j = \frac{b+a}{2\pi}$$

This enables us to immerse any TDHF solution into a system with constant fermion density (spacelike chiral spiral) or current density (timelike chiral spiral)