SL(2,R) duality symmetric action with sources

Hyunsoo Min

Department of Physics, University of Seoul

June 2014

Outline

1. Eltecro-magnetic duality in Maxwell system
2. Zwanziger action with two potentials
3. SL(2,R) duality in Maxwell-anyon-dilaton system
4. Born-Infeld theory with SL(2,R) duality

- based on the work with Choonkyu Lee
  Annals of physics, 339 (2013) 328-343 (arXiv:1306.5520)

Electromagnetic duality in Maxwell equations

Maxwell equation without electric currents

$$\nabla \cdot \vec{E} = 0, \qquad \nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\partial_t \vec{B}, \qquad \nabla \times \vec{B} = \partial_t \vec{E}$$

or, in a relativistic form

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial^*_\mu F^{\mu\nu} = 0$$

Their form is invariant under a SO(2) transformation

$$\begin{pmatrix} E' \\ B' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}$$

However the Maxwell action $S = 1/2 \int d^4x \, (E^2 - B^2)$ does not share the invariance. There are two approaches to construct an invariant action

- Zwanziger(1968): Introduce two vector potential A_μ and B_μ.
- Deser and Teitelboim(1976): Use physical degree of freedom $\mathbf{A}^T$ later it becomes Schwarz and Sen (1994)

Classical Electrodynamics

Classical electrodynamics is described by the action

$$S = \int d^4x \left[\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu_{(e)} A_\mu \right]$$

Then the equation of motion can be found by taking variations of F and A.

$$\partial_\nu F^{\mu\nu} = J^\mu_{(e)} \qquad \text{(Gaus law, Ampere law)}$$

$$\partial^*_\nu F^{\mu\nu} = 0 \qquad \text{(Faraday law)}$$

with ${}^*F_{\mu\nu} = 1/2 \epsilon_{\mu\nu\lambda\tau} F^{\lambda\tau}$ and the Lorentz force is

$$\mathcal{F}^\mu = \int d^3x \left[F^{\mu\nu}(\vec{x}, t) J_{(e)\nu}(\vec{x}, t) \right]$$

The currents are

$$J^\mu_{(e)}(x) = \sum q_i \int \delta(x - z_i(s_i)) \frac{dz_i^\mu}{ds_i} ds_i$$

in the case of point particles and

$$J^\mu_{(e)}(x) = \sum q_i \bar{\psi}_i \gamma^\mu \psi_i$$

in the case of Dirac theory.

Electromagnetic duality in Maxwell equations

Maxwell equation with the magnetic current (as well as the usual electric current):

$$\partial_\nu F^{\mu\nu} = J^\mu_{(e)}$$
$$\partial_\nu\, {}^*F^{\mu\nu} = J^\mu_{(m)}$$

the related Lorentz 4-force

$$\mathcal{F}^\mu = \int d^3x \left[F^{\mu\nu}(\vec{x},t)J_{(e)\nu}(\vec{x},t) + {}^*F^{\mu\nu}(\vec{x},t)J_{(m)\nu}(\vec{x},t)\right]$$

These are invariant under the SO(2) duality rotation

$$\{F'^{\mu\nu} = \cos\alpha\, F^{\mu\nu} + \sin\alpha\ {}^*F^{\mu\nu}\}$$
$$\begin{pmatrix} J'^\mu_{(e)} \\ J'^\mu_{(m)} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} J^\mu_{(e)} \\ J^\mu_{(m)} \end{pmatrix}.$$

Another form of the first

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}.$$

Remarks

Some Remarks

- Lagrangian is not invariant
- $F_{\mu\nu} \neq \partial_\mu A_\nu - \partial_\nu A_\mu$
- We need two vector potential A_μ and B_μ $\rightarrow$ Zwangziger action
- Dirac-Schwinger quantization rule: $q_1 g_2 - q_2 g_1 = 4\pi \times \text{integer}$

Possible generalization

- Possible extension of ‘axion’ interaction term $\theta {}^*F_{\mu\nu} F^{\mu\nu}$ $\rightarrow$ Axion electrodynamics of Wilczek.
- Enlargement of symmetry: from SO(2) to SL(2,R)
  after inclusion of dilaton as well as axion to the system

Our aim is to construct a local action for a dilaton-axion-electrodynamics with the SL(2,R) symmetry.

We will use a shorthand notations
$(A \wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu$, $\quad {}^*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\tau} F^{\lambda\tau}$, $\quad (n \cdot F)_\mu = n^\nu F_{\nu\mu}$ and and
$(\partial \wedge A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Zwanziger action 1968

Two vector potential for the field strength F

$$F = n\wedge[n\cdot(\partial\wedge A)] - {}^*\{n\wedge[n\cdot(\partial\wedge B)]\}$$

n^μ is a fixed, spacelike, normalized four vector with the condition $n^\mu n_\mu = 1$. which is related with the direction of string attached to the monopole.
Action for the electrodynamics:

$$S = S_Z(A,B) + \int d^4x[J^\mu_{(e)}A_\mu + J^\mu_{(m)}B_\mu]$$

Zwanziger action: Zwanziger Phys.Rev.D3(1971)880

$$S_Z(A,B) = -\frac{1}{2}\int d^4x\Big\{[n\cdot(\partial\wedge A)]\cdot[n\cdot {}^*(\partial\wedge B)]$$
$$- [n\cdot(\partial\wedge B)]\cdot[n\cdot {}^*(\partial\wedge A)] + [n\cdot(\partial\wedge A)]^2 + [n\cdot(\partial\wedge B)]^2\Big\}.$$

Currents for point particles

$$J^\mu_{(e)} = \sum_j q_j \int ds_j \frac{dz^\mu_j(s_j)}{ds_j}\delta^4(x - z_j(s_j)),$$
$$J^\mu_{(m)} = \sum_j g_j \int ds_j \frac{dz^\mu_j(s_j)}{ds_j}\delta^4(x - z_j(s_j))$$

Currents in terms of Dirac fields

$$J^\mu_{(e)} = \sum_j q_j\overline{\psi}_j(x)\gamma^\mu\psi_j(x), \quad J^\mu_{(m)} = \sum_j g_j\overline{\psi}_j(x)\gamma^\mu\psi_j(x),$$

Properties of Zwanziger action

- $n \cdot F = n \cdot (\partial \wedge A)$ and $n \cdot^* F = n \cdot (\partial \wedge B)$
- SO(2)-duality invariance

$$\begin{pmatrix} A'^{\mu} \\ B'^{\mu} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} A^{\mu} \\ B^{\mu} \end{pmatrix}$$

- Euler-Lagrange equation from the action reproduces the Maxwell equation correctly
- Particle equation of motion (Dirac Veto)

$$m\frac{d}{ds}\left[\frac{u^{\mu}}{\sqrt{u^2}}\right] = [e(\partial \wedge A)^{\mu\nu} + g(\partial \wedge B)^{\mu\nu}]u_{\nu}$$

$$= (eF \cdot u + g\,{}^*F \cdot u) + \underbrace{eu_{\nu}(n \cdot \partial)^{-1}\,{}^*(n \wedge J_{(m)})^{\mu\nu} - gu_{\nu}(n \cdot \partial)^{-1}\,{}^*(n \wedge J_{(e)})^{\mu\nu}}$$

- The string is described by the Green function $(n{\cdot}\partial)^{-1}(x, x') = \frac{1}{2}\{\theta[n{\cdot}(x - x')] - \theta[-n{\cdot}(x - x')]\}\delta_n^3(x - x')$
- Lorentz invariance is not manifest, but

$$n \wedge \frac{\partial}{\partial n}\mathcal{L} = \text{terms vanishing outside of string}$$

- In quantum mechanics, rounding a string produces a phase factor which becomes invisible by the Dirac quantization rule.

Extended system

Maxwell equations with dilaton and axion fields (ϕ , a): (Gaillard and Zumino, NP B193(1981))

$$\partial_\nu {}^*G^{\mu\nu} = J^\mu_{(m)},$$
$$\partial_\nu H^{\mu\nu} = J^\mu_{(e)}$$
$$(H^{\mu\nu}(x) \equiv e^{-\phi(x)} G^{\mu\nu}(x) - a(x)^* G^{\mu\nu}(x))$$

the Lorentz force law

$$\mathcal{F}^\mu = \int d^3x \left[G^{\mu\nu}(\vec{x},t) J_{(e)\nu}(\vec{x},t) + {}^*H^{\mu\nu}(\vec{x},t) J_{(m)\nu}(\vec{x},t) \right].$$

SL(2,R) Duality transformation

$$\begin{pmatrix} G'^{\mu\nu} \\ {}^*H'^{\mu\nu} \end{pmatrix} = \begin{pmatrix} s & r \\ q & p \end{pmatrix} \begin{pmatrix} G^{\mu\nu} \\ {}^*H^{\mu\nu} \end{pmatrix}$$

$$\tau' = \frac{p\tau + q}{r\tau + s}, \qquad (\tau(x) \equiv a(x) + i\, e^{-\phi(x)})$$

$$\begin{pmatrix} J'^\mu_{(e)} \\ J'^\mu_{(m)} \end{pmatrix} = \begin{pmatrix} p & -q \\ -r & s \end{pmatrix} \begin{pmatrix} J^\mu_{(e)} \\ J^\mu_{(m)} \end{pmatrix}$$

p, q, r, s are real numbers satisfying the condition $ps - qr = 1$,

Vector potentials

Q1: What is the vector potential?
Q2: What is the invariant action?

First note that

$$\partial^*_\nu (G - (n\cdot\partial)^{-1}\, {}^*(n\wedge J_{(m)}))^{\mu\nu} = 0$$

Which implies that

$$G^{\mu\nu} = (\partial{\wedge}A)^{\mu\nu} + (n{\cdot}\partial)^{-1}\, {}^*(n{\wedge}J_{(m)})^{\mu\nu}.$$

By the same token

$$H^{\mu\nu} \equiv e^{-\phi} G^{\mu\nu} - a\, {}^*G^{\mu\nu} = -{}^*(\partial{\wedge}B)^{\mu\nu} - (n{\cdot}\partial)^{-1} (n{\wedge}J_{(e)})^{\mu\nu}.$$

It follows that

$$n{\cdot}G = n{\cdot}(\partial{\wedge}A),$$
$$n{\cdot}\, {}^*G = e^{\phi}[n{\cdot}(\partial{\wedge}B) - a\, n{\cdot}(\partial{\wedge}A)].$$

There exists an identity for a rank two tensor Q (Zwanziger)

$$Q = n{\wedge}(n{\cdot}Q) - {}^*[n{\wedge}(n{\cdot}\, {}^*Q)],$$

Hence we may conclude

$$G = n{\wedge}[n{\cdot}(\partial{\wedge}A)] - e^{\phi}\, {}^*\{n{\wedge}[n{\cdot}(\partial{\wedge}B) - a\, n{\cdot}(\partial{\wedge}A)]\}.$$

Duality transformation of vector potentials

And then

$$^*G = e^{\phi}\, n\wedge[n\cdot(\partial\wedge B) - a\, n\cdot(\partial\wedge A)] + {}^*\{n\wedge[n\cdot(\partial\wedge A)]\}$$

$$H = e^{\phi}(a^2 + e^{-2\phi})n\wedge[n\cdot(\partial\wedge A)] - {}^*\{n\wedge[n\cdot(\partial\wedge B)]\} - a\, e^{\phi}\, n\wedge[n\cdot(\partial\wedge B)].$$

SL(2,R) duality transformation rules for the vector potentials:

$$\begin{pmatrix} A'^{\mu} \\ B'^{\mu} \end{pmatrix} = \begin{pmatrix} s & r \\ q & p \end{pmatrix} \begin{pmatrix} A^{\mu} \\ B^{\mu} \end{pmatrix}$$

Remember

$$\tau' = \frac{p\tau + q}{r\tau + s}, \qquad (\tau(x) \equiv a(x) + i\, e^{-\phi(x)})$$

One can check the SL(2,R) transformation rules for the field strengths $*G$ and H

Duality symmetric action: non-local

We start from a non-local action (Schwinger-Yan type)

It is a first-order and we take B and ${}^*G$ as independent variables

$$S = \int d^4x \left[\frac{1}{4} e^{-\phi}\, {}^*G^{\mu\nu}\, {}^*G_{\mu\nu} - \frac{1}{2}\, {}^*G^{\mu\nu} (\partial_\mu B_\nu - \partial_\nu B_\mu) + \frac{1}{4} a\, {}^*G^{\mu\nu}\, G_{\mu\nu} \right] + \int d^4x \left[J^\mu_{(e)} A_\mu(G) + J^\mu_{(m)} B_\mu \right],$$

$A(G)$ denotes the nonlocal function of G

$$A_\mu = (n{\cdot}\partial)^{-1} (n{\cdot}G)_\mu .$$

Note: variation of B and ${}^*G$ yields

$$\begin{aligned} \partial_\nu\, {}^*G^{\mu\nu} &= J^\mu_{(m)}, \\ e^{-\phi}\, {}^*G^{\mu\nu} + aG^{\mu\nu} &= \partial^\mu B^\nu - \partial^\nu B^\mu - (n{\cdot}\partial)^{-1}\, {}^*(n \wedge J_{(e)})^{\mu\nu}, \end{aligned}$$

Now we may take A and B as independent variables instead of B and ${}^*G$ and transformation rules are already found.

Duality symmetric action: local

Noting

$$\frac{1}{4}e^{-\phi}\,{}^*G^{\mu\nu}\,{}^*G_{\mu\nu} - \frac{1}{2}\,{}^*G^{\mu\nu}(\partial_\mu B_\nu - \partial_\nu B_\mu) + \frac{1}{4}a\,{}^*G^{\mu\nu}\,G_{\mu\nu}$$
$$= \frac{1}{4}\,{}^*G^{\mu\nu}\,{}^*H_{\mu\nu} - \frac{1}{2}\,{}^*G^{\mu\nu}(\partial^\mu B^\nu - \partial^\nu B^\mu),$$

We find

$$S = -\frac{1}{2}\int d^4x\Big\{[n\cdot(\partial\wedge A)]\cdot[n\cdot{}^*(\partial\wedge B)] - [n\cdot(\partial\wedge B)]\cdot[n\cdot{}^*(\partial\wedge A)]$$
$$+ e^{\phi}(a^2 + e^{-2\phi})[n\cdot(\partial\wedge A)]^2 + e^{\phi}[n\cdot(\partial\wedge B)]^2$$
$$- 2ae^{\phi}[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)]\Big\} + \int d^4x\left[J^\mu_{(e)}A_\mu + J^\mu_{(m)}B_\mu\right].$$

(Zwanziger action is recovered setting $\phi = a = 0$)

Using the complex scalar τ,

$$S = \frac{1}{2}\int d^4x\Big\{\mathrm{Im}\left[e^{\phi}(n\cdot[\tau\partial\wedge A - \partial\wedge B])\cdot(n\cdot{}^*[\bar\tau\partial\wedge A - \partial\wedge B])\right]$$
$$- \mathrm{Re}\left[e^{\phi}(n\cdot[\tau\partial\wedge A - \partial\wedge B])\cdot(n\cdot[\bar\tau\partial\wedge A - \partial\wedge B])\right]\Big\}$$
$$+ \int d^4x\left[J^\mu_{(e)}A_\mu + J^\mu_{(m)}B_\mu\right].$$

Properties

- We have found a local action involving electromagnetic vector potential A and B, and dilaton ϕ, and axion a.
- It is invariant under SL(2,R) duality transformation
- It reproduces the right field equations.
- In free space, where $J_{(e)} = J_{(m)} = 0$

$$\partial \wedge B = e^{-\phi}\ {}^*(\partial \wedge A) + a\,(\partial \wedge A).$$

  There are only one independent vector potential.
- Quantization will change the symmetry from SL(2,R) to SL(2,Z). The four parameters p, q, r, and s should be integers with the condition $ps - qr = 1$ to be compatible with the Dirac-Schwinger condition $e_1 g_2 - e_2 g_1 = 2\pi \times \text{integer}$.

Constant-valued scalars ϕ and a

When both of scalars ϕ and a have constant values, we can remove a using the duality transformation

$$A'^{\mu} = e^{-\phi/2} A^{\mu}, \quad B'^{\mu} = e^{\phi/2}(-aA^{\mu} + B^{\mu})$$

Choice of SL(2,R) parameters : $p = e^{\phi/2}$, $q = -ae^{\phi/2}$, $r = 0$ and $s = e^{-\phi/2}$
$\tau' = \frac{p\tau+q}{r\tau+s} = i$, this means $a' = 0$ and $\phi' = 0$.
We get the old Zwanziger action:

$$S = -\frac{1}{2}\int d^4x \Big\{[n\cdot(\partial\wedge A')]\cdot[n\cdot {}^*(\partial\wedge B')] - [n\cdot(\partial\wedge B')]\cdot[n\cdot {}^*(\partial\wedge A')] + [n\cdot(\partial\wedge A')]^2 + [n\cdot(\partial\wedge B')]^2\Big\} + \int d^4x \left[e^{\phi/2}(J^{\mu}_{(e)} + aJ^{\mu}_{(m)})A'_{\mu} + e^{-\phi/2}J^{\mu}_{(m)}B'_{\mu}\right]$$

The charges are changed

$$q' = e^{\phi/2}(q + ag), \qquad g' = e^{-\phi/2}g$$

Recall the Witten's formula

Born-Infeld theory

- Born-Infeld electrodynamics is a non-linear field theory with the duality symmetry. (BI Proc. Roy. Soc. A144(1934); Schrödinger A150(1935))
- Gibbons and Rashed extend the system to include dilaton and axion (GR PL B365(1996); Nucl. Phys. B454(1995))

$$\mathcal{L}_{\rm GR} = 1 - \sqrt{1 + \frac{1}{2} e^{-\phi} G^2 - \frac{1}{16} e^{-2\phi} (G^* G)^2} + \frac{1}{4} a (G^* G)$$

Equation of motion

$$\partial_\nu\, {}^*G^{\mu\nu} = J^\mu_{(m)},$$
$$\partial_\nu H^{\mu\nu} = J^\mu_{(e)},$$

with

$$H^{\mu\nu} = \frac{e^{-\phi} G^{\mu\nu} - \frac{1}{4} e^{-2\phi} (G^* G)\, {}^*G^{\mu\nu}}{\sqrt{1 - \frac{1}{2} e^{-\phi} ({}^*G)^2 - \frac{1}{16} e^{-2\phi} (G^* G)^2}} - a\, {}^*G^{\mu\nu}$$

Now we know the drill.

- The first eq. determines $G = (\partial \wedge A) +$ magnetic current dependent term.
- The 2nd eq. determines $H =^* (\partial \wedge B) +$ electric current dependent term.

- We can identify $n \cdot G$, $n \cdot^* G$ (But the relation between H and G is quite complicated.)
- Find expressions for non-linear terms $(GG)^2$, $(G^* G)^2$

After some algebras, we found

$$
\begin{aligned}
G =& n\wedge[n\cdot(\partial\wedge A)] - \frac{1}{\sqrt{\mathcal{M}}}\left(e^{\phi} + [n\cdot(\partial\wedge A)]^2\right) {}^*\{n\wedge[n\cdot(\partial\wedge B)]\} \\
&+ \frac{1}{\sqrt{\mathcal{M}}}\left(ae^{\phi} + [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)]\right) {}^*\{n\wedge[n\cdot(\partial\wedge A)]\}, \\
H =& -\ {}^*\{n\wedge[n\cdot(\partial\wedge B)]\} - \frac{1}{\sqrt{\mathcal{M}}}\left(e^{\phi} a + [n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)]\right) n\wedge[n\cdot(\partial\wedge B)] \\
&+ \frac{1}{\sqrt{\mathcal{M}}}\left(e^{-\phi} + a^2 e^{\phi} + [n\cdot(\partial\wedge B)]^2\right) n\wedge[n\cdot(\partial\wedge A)],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M} =& 1 + e^{-\phi}(1 + e^{2\phi}a^2)[n\cdot(\partial\wedge A)]^2 + e^{\phi}[n\cdot(\partial\wedge B)]^2 - 2e^{\phi}a[n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)] \\
&+ [n\cdot(\partial\wedge A)]^2[n\cdot(\partial\wedge B)]^2 - \left([n\cdot(\partial\wedge A)]\cdot[n\cdot(\partial\wedge B)]\right)^2
\end{aligned}
$$

SL(2,R) invariant BI action

To find the BI action in terms of A and B, let us start from the Schwinger-Yan type action

$$S = \int d^4x \left[1 - \sqrt{X(G)} - \frac{1}{2} {}^*G^{\mu\nu}(\partial_\mu B_\nu - \partial_\nu B_\mu) + \frac{1}{4} a \, {}^*G^{\mu\nu} G_{\mu\nu} \right] + \int d^4x \left[J^\mu_{(e)} A_\mu(G) + J^\mu_{(m)} B_\mu \right],$$

where

$$X \equiv 1 - \frac{1}{2} e^{-\phi} ({}^*G)^2 - \frac{1}{16} e^{-2\phi} (G^*G)^2$$

Independent variables are: B_μ and ${}^*G_{\mu\nu}$. $A(G)$ and X are specified by these.

Following the steps same with the previous analyses, we found

$$S = \int d^4x \Big[1 - \frac{1}{2}(n \cdot [\partial \wedge A]) \cdot (n \cdot {}^*[\partial \wedge B]) + \frac{1}{2}(n \cdot [\partial \wedge B]) \cdot (n \cdot {}^*[\partial \wedge A]) - \sqrt{\mathcal{M}} + J^\mu_{(e)} A_\mu + J^\mu_{(m)} B_\mu \Big]$$

Conclusion

- We constructed local actions with two vector potentials, dilaton and axion for
  - linear Maxwell system
  - non-linear Born-Infeld system
- SL(2,R) symmetry is manifest
- It is n-dependent but the physics is n-independent
- Lorentz symmetry is hidden.
- We can study quantum effects

- There exit other formulations–Schwartz and Sen, Deser,...
- Coupling to gravity is interesting
- Other spin case (especially spin-2) is interesting
- Quantum properties should be interesting
- Application to condensed matter (topological insulator)
- Application to ADS theory (multi-faces Janus system)