Group Theory and Hydrodynamics

Formalism, Gauge Fields & Anomalies

V. P. Nair

City College of the CUNY

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INTRODUCTION

- Lagrange's approach
  - Newton's equations for N point-particles $\rightarrow$ coarse graining using a smooth density function $\rightarrow$ fluid dynamics
- Point particle $\equiv$ a unitary irreducible representation (UIR) of the Poincaré group
- Classical action which upon quantization gives a UIR of a group $=$ A co-adjoint orbit action

> Can we construct fluid dynamics as
>
> Co-adjoint orbit action $\rightarrow$ coarse graining $\rightarrow$ fluid dynamics ?

- Advantages:
  - A single formalism where symmetries are foundational
  - Gauge fields $\rightarrow$ Abelian and nonabelian Magnetohydrodynamics
  - Spin, magnetic moment effects
  - Gravity easily included (Mathisson-Papapetrou equation)
  - Anomalous symmetries (chiral magnetic effect, chiral vorticity effect, etc.)

Motion of Particles with Internal Degrees of Freedom

- Consider nonrelativistic particles with internal "color" degrees of freedom. (Could be spin).
- Start with $SU(2)$ color, denote the color charge as $Q^a(t)$. Equations of motion are (Wong)

$$\dot{Q}^a - f^{abc} A_i^c \, \dot{x}_i \, Q^b = 0$$

$$\dot{p}_i - F_{ij}^a \, \dot{x}_j \, Q^a = 0$$

- These can be obtained from the action

$$S = \int \left[\frac{1}{2} m \, \dot{x}^2 + A_i^a Q^a \dot{x}_i \right] - i \, n \int \mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g}), \qquad g \in SU(2)$$

with $Q^a = \frac{n}{2} \mathrm{Tr}(g \, \sigma_3 \, g^{-1} \sigma^a)$.

- The crucial $\mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g})$ term was introduced, in this and related contexts, by Balachandran & *collaborators* in the 70s and early 80s. (Also related to Borel-Weil-Bott theory and to work by Kostant, Souriau, Kirillov + ...)

- Under $g \to g\,\exp(i\frac{\sigma_3}{2}\varphi)$ it changes by $\Delta S = n\,\Delta\varphi$. For closed paths in $SU(2)$, $\Delta\varphi = 2\pi \times \text{integer} \Rightarrow n \in \mathbb{Z}$.
- The co-adjoint term leads to something like a monopole field on $S^2 = SU(2)/U(1)$, which is the phase space for the "color" degrees of freedom.
- Quantizing $g \Longrightarrow$ one unitary irreducible representation of $SU(2)$, with $j\text{-value} = n/2$, leading to standard description of color by matrices.
- For group G, we have $rank(G) \equiv r$ mutually commuting generators in **G**, $rank(G)$ coefficients, and $g \in G$. The action is given by

$$S = -i \sum_{s=1}^{r} w_s \int d\tau\, \mathrm{Tr}(q_s\, g^{-1}\dot{g})$$

q_s are diagonal generators of G, and $\{w_s\}$ must be the weights for a UIR of G. Upon quantization, this gives the corresponding UIR.

- For relativistic point-particles, we must use this action with $G =$ Poincaré group
- We consider Poincaré group = contraction of de Sitter group; this makes some traces easier to define.
- For de Sitter algebra, use standard Dirac γ-matrices with

$$J_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \qquad P_\mu = \frac{\gamma_\mu}{r_0}, \qquad \text{Poincaré} = r_0 \to \infty \text{ limit}$$

- A general element is given by

$$g = \exp\left(i\gamma_\alpha\, x^\alpha / r_0\right)\, \Lambda, \qquad \Lambda = B(p)\, R$$

$$B(p) = \frac{1}{\sqrt{2m(p_0+m)}} \begin{bmatrix} p_0 + m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0 + m \end{bmatrix}$$

Λ is an element of the Lorentz group, R is a pure spatial rotation generated by J_{12}, J_{23}, J_{31}, and $m = \sqrt{p^2}$.

- The action is given by

$$S = i\, m\, r_0^2 \int d\tau \,\mathrm{Tr}\left(\frac{\gamma_0}{r_0} g^{-1} \frac{dg}{d\tau} \right) + i\, \frac{n}{2}\, \mathrm{Tr}(J_{12} g^{-1}\, dg) \;-\; \mathcal{H}$$

Using $B\gamma_0 B^{-1} = \gamma^\alpha\, p_\alpha / m$ and taking $r_0 \to \infty$, we find, for the Poincaré group,

$$S = -\int d\tau\, p_\mu\, \dot{x}^\mu + i\, \frac{n}{4} \int d\tau\, \mathrm{Tr}(\Sigma_3\, \Lambda^{-1}\, \dot{\Lambda}) \;-\; \mathcal{H}, \qquad \Sigma_3 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}$$

- $\mathcal{H}$ generates τ-evolution, so we should set it to zero as a constraint on quantum states. This leads to the wave equation.
- The addition of the term $e\, A_\mu\, \dot{x}^\mu$ leads to relativistic charged point-particle dynamics, with magnetic moment ($g = 2$) and spin-orbit coupling.

GENERALIZING TO FLUIDS

- Consider the point-particle *à la* WONG again. Take a collection of particles indexed by λ.

$$S = -in \int dt \, \mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g}) \quad \rightarrow \quad S = -i \int dt \sum_{\lambda} n_\lambda \mathrm{Tr}(\sigma_3 \, g_\lambda^{-1} \dot{g}_\lambda)$$

- We can take the continuum limit by $\lambda \to \vec{x}$, $\sum_\lambda \to \int d^3x/v$, $n_\lambda / v \to \rho(x)$.

- This leads to

$$S = -i \int d^4x \, \rho \, \mathrm{Tr}(\sigma_3 \, g^{-1} \dot{g})$$

  where $g = g(\vec{x}, t)$.

- This suggest the relativistic form

$$S = -i \int j^\mu \, \mathrm{Tr}(\sigma_3 \, g^{-1} \partial_\mu g)$$

- The difficulty for Poincaré is about what replaces $\dot{x}^\mu$. Only 3 of the 4 components are independent; also positions of particles are not well defined in the fluid version.

- Ordinary fluid dynamics can be described by a Poisson bracket system

$$\begin{aligned} [\rho(x), \rho(y)] &= 0 \\ [v_i(x), \rho(y)] &= \partial_{xi}\delta^{(3)}(x-y) \\ [v_i(x), v_j(y)] &= -\frac{\omega_{ij}}{\rho}\,\delta^{(3)}(x-y) \end{aligned}$$

$\omega_{ij} = (\partial_i v_j - \partial_j v_i)$.

$$H = \int d^3x \left[\frac{1}{2}\rho\, v^2 \;+\; V(\rho)\right]$$

- We get the usual equations of fluid motion with pressure $p = \rho \frac{\partial V}{\partial \rho} - V$.
- The PBs can be summarized as

$$[F, G] = \int \left[\frac{\delta F}{\delta \rho}\partial_i \left(\frac{\delta G}{\delta v_i}\right) - \frac{\delta G}{\delta \rho}\partial_i \left(\frac{\delta F}{\delta v_i}\right) - \frac{\omega_{ij}}{\rho}\frac{\delta F}{\delta v_i}\frac{\delta G}{\delta v_j}\right]$$

for any two functions $F,\ G$.

- The helicity C is given by

$$C = \frac{1}{8\pi} \int \epsilon^{ijk}\, v_i\, \partial_j v_k \;= \text{CS term for } v_i$$

- The helicity Poisson-commutes with all local observables, $[F, C] = 0$ for all F

  $\Longrightarrow C$ is superselected.

- Usually if $[\xi^a, \xi^b] = K^{ab}$, the Lagrangian is of the form $\mathcal{A}_b\, \dot{\xi}^b$, where $\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a = K^{-1}_{ab}$.

  Here K is not invertible, $\delta C/\delta v_i$ is a zero mode.

  This is the difficulty in writing down a Lagrangian.

- The solution is also clear: We must fix the value of C and seek a parametrization for the velocity which keeps the same value of C.

- Such a parametrization exists. It is the so-called Clebsch parametrization,

$$v_i = \partial_i \theta \;+\; \alpha\, \partial_i \beta$$

  θ, α, β are arbitrary functions.

- For v_i parametrized in terms of well-defined θ, α, β,

$$C = \int (\text{total derivative}) = 0$$

- A suitable action which gives the PBs is now (C.C. LIN)

$$S = \int \rho\, \dot{\theta} + \rho\, \alpha\, \dot{\beta} - \int \left[\frac{1}{2} \rho\, v^2 - V \right]$$

- We can also write this as

$$S = \int J^\mu \, (\partial_\mu \theta + \alpha\, \partial_\mu \beta) \; - \; \int \left[J^0 \; - \; \frac{J^i J^i}{2\, \rho} + V \right]$$

$J^0 = \rho$; elimination of the auxiliary J^i leads to the previous version. $\int J^0$ is a constant.

- The relativistic generalization is

$$S = \int J^\mu \, (\partial_\mu \theta + \alpha\, \partial_\mu \beta) \; - \; \int f(n)$$

$f(n) = n + V(n), \qquad n^2 = J^2 = (J^0)^2 - J^i J^i.$

- The lesson from this is to treat
  - Translational part of action $\rightarrow$ Clebsch parametrization
  - Rest of the action in terms of the co-adjoint orbit version
- The general action is thus

$$S = \int d^4x \left[j^\mu (\partial_\mu \theta + \alpha \partial_\mu \beta) - \frac{i}{4} j^\mu_{(s)} \mathrm{Tr}(\Sigma_3 \Lambda^{-1} \partial_\mu \Lambda) + i \sum_a j^\mu_{(a)} \mathrm{Tr}(q_a\, g^{-1} D_\mu\, g) - f(\{n\}) \right] + S(A)$$

- Generally, we must have different currents $j^\mu, j^\mu_{(s)}, j^\mu_{(a)}$ for mass flow, spin flow and the transport of other quantum numbers.
- Coupling to gauge fields follow from covariant derivatives on the group elements

Group-theoretic Description of Fluids (*cont'd.*)

- $f(\{n\})$ depends on all invariant combinations of the currents and characterize the nature of the fluid, $n = \sqrt{j^\mu j_\mu}$, $n_a = \sqrt{j^\mu_{(a)} j_{\mu\,(a)}}$, etc.
- The group-valued fields are related to flow velocities and currents and given by the equations of motion,

$$
\begin{aligned}
\frac{1}{n}\frac{\partial f}{\partial n}\, j_\mu &= \partial_\mu \theta + \alpha\, \partial_\mu \beta \\
\frac{1}{n_a}\frac{\partial f}{\partial n_a}\, j_{\mu\,(a)} &= i\,\mathrm{Tr}\,(q_a\, g^{-1} D_\mu\, g), \qquad \text{etc.}
\end{aligned}
$$

Remark: The Clebsch parametrization can also be written in a "group" form,

$$
-i\,\mathrm{Tr}(\sigma_3\, g^{-1} dg) = d\theta + \alpha\, d\beta
$$

where $g \in SU(1,1)$,

$$
g = \frac{1}{\sqrt{1 - \bar{u}u}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad \alpha = \frac{2\,\bar{u}u}{(1 - \bar{u}u)}, \quad \beta = -\frac{i}{2}\log(u/\bar{u})
$$

- We will discuss 3 examples
  - $SU(2)$ internal symmetry (Nonabelian Magnetohydrodynamics)
  - Magnetohydrodynamics including spin, magnetic moment and spin-orbit effects
  - Spin and coupling to gravity
- We will also discuss generalization to include anomalies

- Consider the action (BISTROVIC, JACKIW, LI, NAIR, PI)

$$S = \int J^\mu \,(\partial_\mu\theta + \alpha\,\partial_\mu\beta) - i\int j^\mu\,\mathrm{Tr}(\sigma_3\, g^{-1}D_\mu g) - \int f(n) + S_{YM}$$

$$\begin{aligned} D_\mu g &= \partial_\mu g + A_\mu\, g & \qquad A_\mu = -i\, t^a A^a_\mu, \quad t^a = \tfrac{1}{2}\sigma^a \\ J^\mu &= n_m\, U^\mu, & U^2 = 1 \\ j^\mu &= n\, u^\mu, & u^2 = 1 \end{aligned}$$

- We also include a background field which couples to the color charge.

- The current which couples to A^a_μ is given by

$$J^{a\mu} = \mathrm{Tr}(\sigma_3\, g^{-1} t^a g)\, j^\mu = Q^a\, j^\mu, \qquad Q^a = \mathrm{Tr}(\sigma_3\, g^{-1} t^a g)$$

  This is the Eckart form for currents.

- The equations of motion are

$$\begin{aligned} \partial_\mu j^\mu &= 0 \\ (D_\mu J^\mu)^a &= 0 \\ n\, u^\mu \partial_\mu (u_\nu f') - n\, \partial_\nu f' &= \mathrm{Tr}(J^\mu F_{\mu\nu}) \qquad (\text{“Euler equation”}) \end{aligned}$$

- The first two equations lead to the fluid generalization of the Wong equations

$$u^\mu (D_\mu Q)^a = (D_0 Q)^a + \vec{u} \cdot (\vec{D} Q)^a = 0$$

- We also have

$$\partial_\mu T^{\mu\nu} = \mathrm{Tr}\,(J^\mu F_{\mu\nu})$$

$T^{\mu\nu}$ has the perfect fluid form.

- The nonabelian charge density $\rho = \rho^a t^a$ (which is the time-component of $J^{a\mu}$) transforms, under gauge transformations, as

$$\rho \to \rho' = h^{-1}\rho\, h, \qquad\qquad h \in SU(2)$$

- We can diagonalize ρ at each point by an $(\vec{x}, t)$-dependent transformation, $\rho_{diag} = \rho_0 \sigma_3$. Then $\rho = g\,\rho_{diag}\,g^{-1}$, or

$$\rho^a = \rho_0\,\mathrm{Tr}(g\,\sigma_3\,g^{-1}\,t^a) = j^0\,\mathrm{Tr}(g\,\sigma_3\,g^{-1}\,t^a)$$

- g is the transformation which diagonalizes the charge density at each point. The eigenvalues are gauge-invariant and are represented by n. Their flow is given by u^μ. Under a gauge transformation, $g \to h^{-1}\,g$.

- There are two (related) charge densities, j^0 and the nonabelian charge density $\rho^a = J^{a\,0}$.
- The basic (new) Poisson brackets are

$$
\begin{aligned}
\{j^0(\vec{x}), j^0(\vec{y})\} &= 0 \\
\{j^0(\vec{x}),\ g(\vec{y})\} &= -i\, g(\vec{x}) \left(\frac{\sigma_3}{2}\right) \delta(x-y) \\
\{\rho^a(\vec{x}), \rho^b(\vec{y})\} &= f^{abc} \rho^c(\vec{x})\, \delta(x-y) \\
\{\rho^a(\vec{x}),\ g(\vec{y})\} &= -i \left(\frac{\sigma_a}{2}\right) g(\vec{x})\, \delta(x-y)
\end{aligned}
$$

- Another interesting observation is that, since $\Pi_3[SU(N)] = \mathbb{Z}$, there are skyrmion-type solitons in any nonabelian magnetohydrodynamics. (DAI, NAIR)
- Remark: These equations of motion and charge algebra have some points of overlap with the work of GIBBONS, HOLM, KUPERSHMIDT

- Consider a special case where mass transport and charge transport are described by the same flow velocity.
- This applies when we have one species of particles with the same charge.
- Further, for dilute systems, if we neglect the possibility of spin-singlets forming (and moving independently), we can take spin flow velocity $\approx$ charge flow velocity
- The action for this case is (KARABALI, NAIR)

$$S = S(A) + \int d^4x \left[j^\mu \left(\partial_\mu \theta + \alpha \partial_\mu \beta + e A_\mu\right) - \frac{i}{4} j^\mu \,\mathrm{Tr}(\Sigma_3\, \Lambda^{-1} \partial_\mu \Lambda) - f(n, \sigma) \right]$$

$\Lambda = B\, R$ contains the same velocity u^μ as in $j^\mu = n\, u^\mu$.

- F depends on n and $\sigma = S^{\mu\nu}\, F_{\mu\nu}$, where $S^{\mu\nu}$ is the spin density,

$$S^{\mu\nu} = \frac{1}{2}\,\mathrm{Tr}\,(\Sigma_3\, \Lambda^{-1}\, J^{\mu\nu}\, \Lambda), \qquad J^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

- Having the same flow velocity can be satisfied by

$$\frac{2}{n}\frac{\partial f}{\partial n}\frac{\partial f}{\partial \sigma} = e$$

This is the fluid analog of the requirement of $g = 2$ for point-particles.

- The equations of motion are the Maxwell equations +

$$\begin{aligned}
u^\alpha \partial_\alpha (f' \, u_\nu) - \partial_\nu f' &= e\Big[u^\lambda \, F_{\lambda\nu} - \frac{4}{s^2 f'} \, \partial_\nu S^{\lambda\beta} (SFS - FSS)_{\lambda\beta} - \cdots\Big] \\
u^\alpha \partial_\alpha S_{\mu\nu} &= \frac{1}{f'}\left[S_\mu^{\ \lambda}(eF_{\lambda\nu} + G_{\lambda\nu}) - S_\nu^{\ \lambda}(eF_{\lambda\mu} + G_{\lambda\mu})\right] \\
&\quad - \frac{4e}{s^2 f'^2}(u_\mu S_\nu^{\ \lambda} - u_\nu S_\mu^{\ \lambda})\partial_\lambda S^{\rho\beta}(SFS - FSS)_{\rho\beta} + \cdots \\
G_{\lambda\nu} &= u_\lambda \, \partial_\nu f' - u_\nu \, \partial_\lambda f' \\
(S\,F\,S - F\,S\,S)_{\lambda\beta} &= S_\lambda^{\ \rho} \, F_{\rho\tau} \, S^\tau_{\ \beta} - F_\lambda^{\ \rho} \, S_{\rho\tau} \, S^\tau_{\ \beta}
\end{aligned}$$

- Spin density is subject to precession effects due to pressure gradient terms as well as due to the external field.

- Consider spin transport in the fluid in a gravitational background.
- Specializing the general fluid action to the Lorentz group (to describe spin), with covariant derivatives appropriate to a general curved manifold,

$$S[e, \omega, j, \Lambda] \quad = \quad \int \det e \left[-\frac{i}{2} j^\mu \, \mathrm{Tr}(\Sigma_3 \, \Lambda^{-1} D_\mu(\bar{\omega}(e)) \, \Lambda) - f(n) \right] - \frac{1}{32\,\pi\, G} \, \epsilon_{abcd} \int e^a \wedge e^b \wedge R^{cd}(\omega)$$

- Here we use $\bar{\omega}(e)$ to avoid generating torsion via equations of motion,

$$\bar{\omega}^{ab}_\mu = (e^{-1})^{\nu a} \partial_{[\mu} e^b_{\nu]} - (e^{-1})^{\nu b} \partial_{[\mu} e^a_{\nu]} - (e^{-1})^{\rho a} \, (e^{-1})^{\sigma b} \partial_{[\rho} e^c_{\sigma]} \, e_{\mu c}.$$

- The energy-momentum tensor is now

$$T_{\mu\nu} = T^{(f)}_{\mu\nu} \, + \, 2\,\nabla_\alpha (j_\mu Q^\alpha_{\ \nu} + j_\nu Q^\alpha_{\ \mu}).$$

where $T^{(f)}_{\mu\nu}$ has the perfect fluid form,

$$T^{(f)}_{\mu\nu} = n f' \, u_\mu u_\nu - g_{\mu\nu}(n f' - f)$$

where $Q^{\alpha\beta}$ is the spin density,

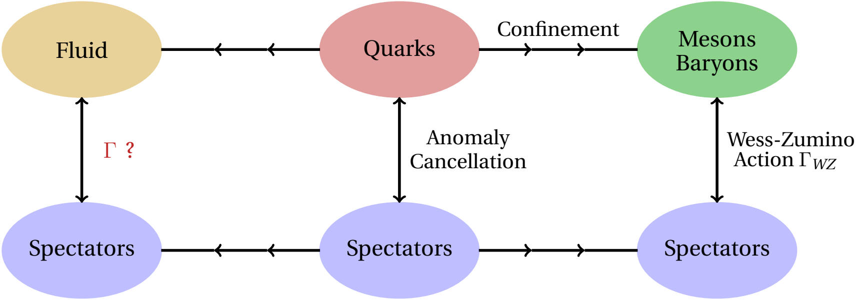
$$Q^{\alpha\beta} = (e^{-1})^\alpha_{\ a} \, (e^{-1})^\beta_{\ b} \; \frac{1}{2} \mathrm{Tr} \left(\Sigma_3 \, \Lambda^{-1} J^{ab} \, \Lambda \right)$$

- The conservation law may be written as

$$\nabla_\mu \, T^{(f)\mu\nu} - 2\,(R_{\alpha\beta})^\nu_{\ \lambda} \, j^\lambda \, Q^{\alpha\beta} = 0$$

- The second term is the fluid version of the spin-curvature coupling occurring in the Mathisson-Papapetrou equation.

- ’t Hooft argument for the Wess-Zumino action for anomalies



- A similar argument for the fluid phase suggests an effective action for anomalies in terms of fluid variables. What is this action?

- Since we have formulated fluid dynamics using group variables, this is easy. We can use the same Γ_{WZ} but using fluid group element instead of meson fields.

- The suggestion is (NAIR, RAY, ROY)

$$
\begin{aligned}
S \;=\; & -i \int \Big[j_3^\mu \, \mathrm{Tr}\left(\frac{\lambda_3}{2}\, g_L^{-1} D_\mu\, g_L \right) + j_8^\mu \, \mathrm{Tr}\left(\frac{\lambda_8}{2}\, g_L^{-1} D_\mu\, g_L \right) + k_3^\mu \, \mathrm{Tr}\left(\frac{\lambda_3}{2}\, g_R^{-1} D_\mu\, g_R \right) \\
& + k_8^\mu \, \mathrm{Tr}\left(\frac{\lambda_8}{2}\, g_R^{-1} D_\mu\, g_R \right) + j_0^\mu \, \mathrm{Tr}\left(g_L^{-1} D_\mu\, g_L \right) + k_0^\mu \, \mathrm{Tr}\left(g_R^{-1} D_\mu\, g_R \right) \Big] \\
& - f(n_3, n_8, n_0, m_3, m_8, m_0) \;+\; S_{YM}(A) \;+\; \Gamma_{WZ}(A_L, A_R, g_L\, g_R^\dagger)
\end{aligned}
$$

- $\Gamma_{WZ}(A_L, A_R, g_L\, g_R^\dagger)$ is the standard Wess-Zumino term $\Gamma_{WZ}(A_L, A_R, U)$ with $U \Longrightarrow g_L\, g_R^\dagger$.

- There are other ways to incorporate anomalies (SON & SUROWKA; SADOFYEV & ISACHENKOV; ABANOV *et al*; BASAR, DUNNE, KHARZEEV; + *many others*); an approach somewhat similar to ours is by SHU LIN.

- In full it is given by (WITTEN; KAYMAKCALAN, RAJEEV, SCHECHTER; + ...)

$$
\begin{aligned}
\Gamma_{WZ} = & -\frac{iN}{240\pi^2}\int_D \mathrm{Tr}(dU\,U^{-1})^5 - \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[(A_L\,dA_L + dA_L\,A_L + A_L^3)\,dUU^{-1}] \\
& - \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[(A_R\,dA_R + dA_R\,A_R + A_R^3)\,U^{-1}dU] \\
& + \frac{iN}{96\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[A_L\,dUU^{-1}A_L\,dUU^{-1} - A_R\,U^{-1}dU\,A_R\,U^{-1}dU] \\
& + \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[A_L(dUU^{-1})^3 + A_R(U^{-1}dU)^3 + dA_L\,dU\,A_R\,U^{-1} - dA_R\,d(U^{-1})\,A_L\,U] \\
& + \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[A_R\,U^{-1}\,A_L\,U(U^{-1}dU)^2 - A_L\,U\,A_R\,U^{-1}(dUU^{-1})^2] \\
& - \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[(dA_R\,A_R + A_R\,dA_R)\,U^{-1}\,A_L\,U - (dA_L\,A_L + A_L\,dA_L)\,U\,A_R\,U^{-1}] \\
& - \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[A_L\,U\,A_R\,U^{-1}\,A_L\,dUU^{-1} + A_R\,U^{-1}\,A_L\,U\,A_R\,U^{-1}dU] \\
& - \frac{iN}{48\pi^2}\int_{\mathcal{M}} \mathrm{Tr}[A_R^3\;U^{-1}\,A_L\,U - A_L^3\;U\,A_R\,U^{-1} + \tfrac{1}{2}\,U\,A_R\,U^{-1}\,A_L\,U\,A_R\,U^{-1}\,A_L]
\end{aligned}
$$

with $U \Longrightarrow g_L\,g_R^\dagger$.

- This action gives the chiral magnetic effect (& other anomaly related effects) for all flavor gauge fields and chemical potentials (A_0 components become the chemical potentials μ.)
- The electromagnetic current, for example, is given by (*previous refs, also* CALLAN & WITTEN)

$$
\begin{aligned}
J^\mu \quad = \quad & J_3^\mu + \frac{e}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\Big[Q(\partial_\nu U\, U^{-1}\,\partial_\alpha U\, U^{-1}\,\partial_\beta U\, U^{-1} \\
& + U^{-1}\partial_\nu U\, U^{-1}\partial_\alpha U\, U^{-1}\partial_\beta U)\Big] \\
& + i\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu A_\alpha \mathrm{Tr}\Big[Q^2(\partial_\beta U\, U^{-1} + U^{-1}\partial_\beta U) + \frac{1}{2}(Q\partial_\beta U\, QU^{-1} \\
& - QUQ\partial_\beta U^{-1})\Big]
\end{aligned}
$$

- We can restrict to two flavors by choosing

$$
U = e^{i\theta}\begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix}
$$

- The current is now

$$J^\mu = J_3^\mu + \frac{e}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(\mathcal{I}_\nu\,\mathcal{I}_\alpha\,\mathcal{I}_\beta) + i\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\mathrm{Tr}\left[(\Sigma_{3L}+\Sigma_{3R})\,I_\beta\right] + J_\theta^\mu$$

$$J_\theta^\mu = -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\partial_\beta\theta\left[2+\frac{1}{4}\mathrm{Tr}\,(\Sigma_{3L}\,\Sigma_{3R}-1)\right]$$

$$\mathcal{I}_\beta = g_L^{-1}\partial_\beta g_L - g_R^{-1}\partial_\beta g_R, \qquad \Sigma_{3L} = g_L^{-1}\sigma_3 g_L,\ \Sigma_{3R} = g_R^{-1}\sigma_3 g_R.$$

- If we further restrict to $g_L = g_R$, we get

$$J_\theta^\mu = -\frac{e^2}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}(\partial_\nu A_\alpha)\,\partial_\beta\theta$$

$$J_i = -\frac{e^2}{4\,\pi^2}\,(\mu_L-\mu_R)\,B_i$$

  This reproduces the chiral magnetic effect which was originally calculated using Feynman diagrams (KHARZEEV, MCLERRAN, WARRINGA, FUKUSHIMA + ...).

- The full set of equations describe hydrodynamic transport of flavor charges.

- The anomaly term Γ_{WZ} also has terms proportional to Z_μ, so there is also an induced isospin current (CAPASSO, NAIR, TEKEL).
- The relevant term is

$$\Gamma_{WZ} = -\frac{Ne^2}{6\,\pi^2}(\cot 2\theta_W) \int \epsilon^{\mu\nu\alpha\beta} Z_\mu \partial_\nu A_\alpha \, \partial_\beta \theta$$

- This leads to

$$J^{Z\,\mu} = -\frac{e}{8\,\pi^2}(\cos 2\theta_W)\, \epsilon^{\mu\nu\alpha\beta} F_{\nu\alpha}\, \partial_\beta \theta$$
$$J^{3\,\mu} = \frac{e}{8\,\pi^2}(\mu_L - \mu_R)\, B^i$$

- In terms of pion fields, $J^{3\,\mu} \approx -\frac{1}{2} f_\pi \partial^\mu \Pi^0 + \cdots$. So we can interpret this as a pion field of gradient

$$\partial^i \Pi^0 = -\frac{e}{4\,\pi^2 f_\pi}(\mu_L - \mu_R)\, B^i$$

- This can manifest itself as an asymmetry in the neutral pion distribution.

- Generally, there is a contribution even when the background fields are zero.
- If we eliminate the group elements in favor of velocities, we get

$$\begin{aligned} J^\mu &= J_3^\mu + J_\theta^\mu + i\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\partial_\nu A_\alpha\,\mathrm{Tr}\left[(\Sigma_{3L}+\Sigma_{3R})\,I_\beta\right] \\ &\quad + \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu \mathrm{Tr}(g_L^{-1}\partial_\alpha g_L\;g_R^{-1}\partial_\beta g_R) \\ &\quad + \frac{e}{12\pi^2}\epsilon^{\mu\nu\alpha\beta}\left[\left(\frac{\partial f}{\partial n_3}\right)^2 u_{3L\,\nu}\,\partial_\alpha u_{3L\,\beta} - \left(\frac{\partial f}{\partial m_3}\right)^2 u_{3R\,\nu}\,\partial_\alpha u_{3R\,\beta}\right]. \end{aligned}$$

- A left-right asymmetry with nonzero vorticity can generate an electromagnetic current
- Finally, we know that the standard model can have mixed gauge-gravity anomalies in some restricted cases, due to the 6-form index density,

$$I_6 = \frac{i}{384\,\pi^3}\;(\mathrm{Tr}F)\;\mathrm{Tr}\,(R\wedge R)$$

- If up, down and strange are described by fluid variables, while charm is described as a Dirac field, the fluid part can show an anomaly.
- The hypercharge conservation law is changed to

$$\partial_\mu J^\mu = -i \frac{N_c}{768\,\pi^2} \frac{\epsilon^{\mu\nu\alpha\beta}}{\sqrt{-g}} \mathrm{Tr}(R_{\mu\nu}\, R_{\alpha\beta}).$$

- The corrected energy-momentum tensor is given by the previous result +

$$T^{\nu\sigma}\big]_{corr} = -i \frac{N_c}{192\pi^2} \frac{1}{\sqrt{-g}} \nabla_\lambda \left[\mathrm{Tr}_Y(\partial_\mu \theta)\, (R_{\alpha\beta})^{\lambda\sigma} \epsilon^{\mu\nu\alpha\beta} + (\nu \leftrightarrow \sigma) \right].$$

The remaining trace is over the hypercharge values. If we replace $\dot\theta$ by the chemical potentials, as can be done for the chiral magnetic effect,

$$\mathrm{Tr}_Y(\dot\theta) \to \frac{1}{2} \left[\frac{1}{3}(\mu_L^u + \mu_L^d + \mu_L^s) + \frac{2}{3}(\mu_R^d + \mu_R^s - 2\mu_R^u) \right].$$

- A number of new directions are suggested by this. Just to name a few:
  - Relating the Poincaré group version to the formulation of fluid dynamics in terms of the group of diffeomorphisms
  - The chiral vorticity effect and the mixed anomaly effects may be significant in certain gravitational backgrounds, such as cosmic strings.
  - Exploring soliton solutions. (Topological solitons in the fluid are possible in this formalism.)
  - Quantum ground states for fluids can be meaningful in some contexts, as in liquid Helium. How do they look in more general situations?

Thank you