The factorziation of the 3d index

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(POSTECH)

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Pre Strings Workshop
(KIAS)

based on
C. Hwang, Hee-Cheol Kim, Jaemo Park,
"Factorization of the 3d superconformal index",
[arXiv:1211.6023]

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- Direct observation of the factorization using the Higgs branch localization
- Analytic proof of the index agreement
- New information from the index matching
  (cf. N=4 Seiberg-like duality [Kim, Kim, Kim, Lee '12])
- Explicit realization of 3d defects for general cases

Prezi

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- Relation to the topological string amplitude
- Concluding remarks

Motivation

Factorization of sphere partition functions:

3d case: abelian PF on the squashed 3-sphere [Pasquetti '11]

$$Z_{S_b^3} = \sum_{i=1}^{N} Z_{cl}^i Z_{1-loop}^i Z_V^i \times \bar{Z}_{1-loop}^i \bar{Z}_V^i$$

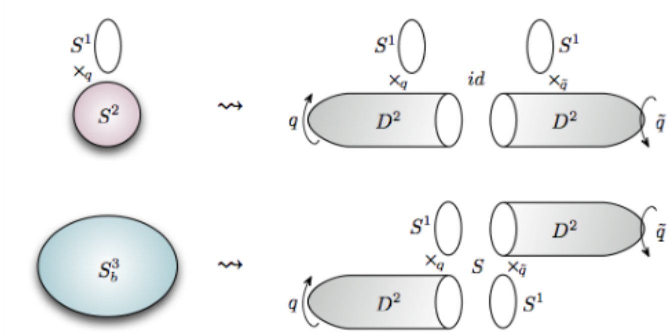
2d case: PF on the 2-sphere [Benini, Cremonesi '12] [Doroud, Gomis, Floch, Lee '12]

$$Z_{U(N)}(w, \overline{w}) = \sum_{\vec{l} \in C(N,N_f)} e^{i(w+\overline{w}) \sum_{r=1}^{N} \tau_{l_r}} \; Z_{\text{1-loop}}^{(\vec{l})} \; Z_{\text{v}}^{(\vec{l})}(e^{-w}) \; Z_{\text{av}}^{(\vec{l})}(e^{-\overline{w}})$$

↑ the vanishing circle limit

3d superconformal index

Geometric interpretation [Beem, Dimofte, Pasquetti '12]



Heegaard splitting to Melvin cigars [BDP 12]

The partition function:

$$Z = \langle 0_q | 0_{\tilde{q}} \rangle = \sum_\alpha \langle 0_q | \alpha \rangle \langle \alpha | 0_{\tilde{q}} \rangle$$

$$q = e^{2\pi i \tau}$$

$$\tau \to \tilde{\tau} = -id \cdot \tau = -\tau, \qquad \tau \to \tilde{\tau} = -S \cdot \tau = \frac{1}{\tau}$$

- Nonabelian PF on the squashed 3-sphere [Taki '13]

The superconformal index in 3d

$$I(x,t) = \mathrm{Tr}(-1)^F \exp(-\beta'\{Q,S\}) x^{\epsilon + j_3} \prod_a t_a^{F_a}$$

Factorization

N=2 U(N) Chern-Simons-matter theories

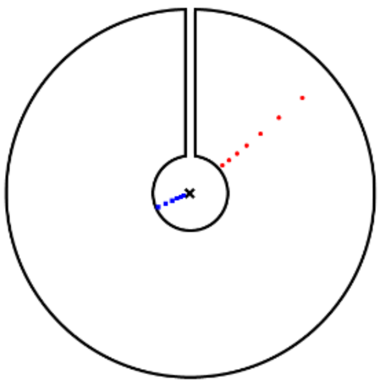
$$I(x,t,w,\kappa) = \sum_{m\in\mathbb{Z}^N/S_N} \oint \prod_j \frac{dz_j}{2\pi i z_j} \frac{1}{|\mathcal{W}_m|} e^{-S_{cl}(\alpha,a,m)} Z_{gauge}(x,z,m) \prod_\Phi Z_\Phi(x,t,z,m)$$

the CS contribution: $$e^{-S_{cl}(\alpha,a,m)} = w^{\sum_j m_j} \prod_{j=1}^{N} (-z_j)^{-\kappa m_j},$$

the vector multiplet: $$Z_{gauge}(x,z,m) = \prod_{\substack{i,j=1\\(i\neq j)}}^{N} x^{-|m_i-m_j|/2} \left(1 - z_i z_j^{-1} x^{|m_i-m_j|}\right),$$

N_f fundamental and $\tilde{N}_f$ antifundamental chiral multiplets:

$$\begin{aligned}&\prod_\Phi Z_\Phi\left(x,t,\tilde{t},\tau,z,m\right)\\ =&x^{(1-\Delta_\Phi)(N_f+\tilde{N}_f)\sum|m_j|/2}\left[\prod_{j=1}^{N}(-z_j)^{-(N_f-\tilde{N}_f)|m_j|/2}\right]\tau^{-(N_f+\tilde{N}_f)\sum|m_j|/2}\\ &\prod_{j=1}^{N}\prod_{k=0}^{\infty}\left(\prod_{a=1}^{N_f}\frac{1-z_j^{-1}t_a^{-1}\tau^{-1}x^{|m_j|+2-\Delta_\Phi+2k}}{1-z_j t_a \tau x^{|m_j|+\Delta_\Phi+2k}}\right)\left(\prod_{a=1}^{\tilde{N}_f}\frac{1-z_j\tilde{t}_a^{-1}\tau^{-1}x^{|m_j|+2-\Delta_\Phi+2k}}{1-z_j^{-1}\tilde{t}_a\tau x^{|m_j|+\Delta_\Phi+2k}}\right)\end{aligned}$$



The factorized index:

$$I(x,t,\tilde{t},\tau,w,\kappa)$$
$$=\frac{1}{N!(N_f-N)!}\sum_{\sigma} I_{1-loop}(x,\sigma(t),\tilde{t},\tau)\times\sum_{\vec{n}=0}^{\infty} Z^{\vec{n}}_{vort}(x,\sigma(t),\tilde{t},\tau;w,\kappa)\times\sum_{\vec{\bar{n}}=0}^{\infty} Z^{\vec{\bar{n}}}_{vort}(x,\sigma(t),\tilde{t},\tau;w^{-1},-\kappa)$$

$t,\tilde{t}$: flavor symmetry, τ : axial symmetry, w : topological symmetry

$$I_{1-loop}(x,t=e^{iM},\tilde{t}=e^{i\tilde{M}},\tau)$$
$$=\left(\prod_{\substack{i,j=1\\(i\neq j)}}^{N} 2\sinh\frac{iM_i-iM_j}{2}\right)\left[\prod_{j=1}^{N}\prod_{k=0}^{\infty}\left(\frac{\prod_{a=1(\neq j)}^{N_f}1-t_jt_a^{-1}x^{2+2k}}{\prod_{a=1}^{\tilde{N}_f}1-t_j\tilde{t}_a\tau^2x^{2k}}\right)\left(\frac{\prod_{a=1}^{\tilde{N}_f}1-t_j^{-1}\tilde{t}_a^{-1}\tau^{-2}x^{2+2k}}{\prod_{a=1(\neq j)}^{N_f}1-t_j^{-1}t_ax^{2k}}\right)\right]$$

$$Z^{\vec{n}}_{vort}\left(x=e^{-\gamma},t=e^{iM},\tilde{t}=e^{i\tilde{M}},\tau=e^{i\mu};w,\kappa\right)$$
$$=(-1)^{-[\kappa+(N_f-\tilde{N}_f)/2]\sum n_j}e^{i\kappa\sum_j(M_jn_j+\mu n_j+i\gamma n_j^2)}$$
$$\times(-w)^{\sum n_j}\left[\prod_{j=1}^{N}\prod_{k=0}^{n_j-1}\frac{\prod_{a=1}^{\tilde{N}_f}2\sinh\frac{-i\tilde{M}_a-iM_j-2i\mu+2\gamma k}{2}}{\left(\prod_{i=1}^{N}2\sinh\frac{iM_i-iM_j+2\gamma(k-n_i)}{2}\right)\left(\prod_{a=N+1}^{N_f}2\sinh\frac{iM_a-iM_j+2\gamma(1+k)}{2}\right)}\right]$$

γ : chemical potential for r+2j, $M,\tilde{M},\mu$: mass parameters, w : vortex fugacity cf. the vortex partition function of 3d N=4 gauge theories [Kim, Kim, Kim, Lee '12]:

$$I_{(k_1,k_2,\cdots,k_N)} = \prod_{i=1}^{N}\prod_{s=1}^{k_i}\left[\prod_{j=1}^{N}\frac{\sinh\frac{E_{ij}-2i(\gamma-\gamma')}{2}}{\sinh\frac{E_{ij}}{2}}\prod_{p=N+1}^{N_f}\frac{\sinh\frac{E'_{ip}-2i(\gamma+\gamma')(R+\tilde{R})+2i(\gamma-\gamma')}{2}}{\sinh\frac{E'_{ip}-2i(\gamma+\gamma')(R+\tilde{R})}{2}}\right]$$

$$E_{ij} = \mu_i - \mu_j + 4i\gamma(k_j - s + 1), \quad E'_{ij} = \mu_i - \mu_j - 4i\gamma(s-1)$$

$$\gamma_{ours} = 2i\gamma, \quad iM_j + i\mu = \mu_j + 2\gamma, \quad iM_a + i\mu = \mu_a, \quad i\tilde{M}_b + i\mu = -\mu_b$$

- A functional property of the vortex partition function:

$$Z^{\vec{n}}_{vort}(x,t,\tilde{t},\tau;w,\kappa) = Z^{\vec{n}}_{vort}(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1};w,-\kappa)$$

$$I_{1-loop} = Z_{1-loop}\left(x,t,\tilde{t},\tau\right) Z_{1-loop}\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1}\right)$$

$$Z_{1-loop}(x,t=e^{iM},\tilde{t}=e^{i\tilde{M}},\tau)$$
$$=\left(\prod_{i<j}^{N}2\sinh\frac{iM_i-iM_j}{2}\right)\left(\prod_{j=1}^{N}\prod_{k=0}^{\infty}\frac{\prod_{a=1(\neq j)}^{N_f}1-t_j t_a^{-1}x^{2+2k}}{\prod_{a=1}^{\tilde{N}_f}1-t_j\tilde{t}_a\tau^2 x^{2k}}\right)$$

$$\boxed{I\left(x,t,\tilde{t},\tau,w,\kappa\right) = I\left(x^{-1},t^{-1},\tilde{t}^{-1},\tau^{-1},w^{-1},\kappa\right)}$$

Application to 3d Seiberg-like dualities

The Aharony duality for N=2 U(N) gauge theories:

- $U(N)$ gauge theory with N_f fundamental and antifundamental chiral multiplets $Q, \tilde{Q}$
- $U(N_f - N)$ gauge theory with N_f fundamental and antifundamental chiral multiplets $q, \tilde{q}$ + ${N_f}^2$ singlets M_{ab}, 2 singlets V_+, V_-, and the superpotential:

$$W = Mq\tilde{q} + V_+ v_- + V_- v_+$$

The simplest example: U(1) theory with 1 flavor <-> the XYZ model

$$I^{N=N_f=1} = \prod_{l=0}^{\infty} \frac{1-\tau^{-2}x^{2l+2}}{1-\tau^2 x^{2l}} \times Z_{vortex}^{N=N_f=1} \times Z_{anti}^{N=N_f=1}$$

$$I^{M,V_\pm} = \mathrm{PE}[f_M] \times \mathrm{PE}[f_+] \times \mathrm{PE}[f_-]$$

$$\mathrm{PE}[f(\cdot)] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(\cdot^n)\right]$$

$$f_M = \frac{\tau^2 - \tau^{-2}x^2}{1-x^2},$$

$$f_+ + f_- = \frac{\tau^{-1}x - \tau x}{1-x^2}(w + w^{-1}) = f_{V_+} + f_{V_-}$$

An obtained identity:

$$\sum_{n=0}^{\infty}(-w)^n \prod_{k=1}^{n} \frac{\tau^{-1}x^{-(k-1)} - \tau x^{k-1}}{x^{-(k-1-n)} - x^{k-1-n}} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{(\tau^{-n} - \tau^n)x^n}{1 - x^{2n}}\right]$$

cf. a analytic proof of the index agreement for some mirror pairs [Krattenthaler, Spiridonov, Vartanov '11]

- General cases

$$I_1^{\{b_j\}} = \tilde{I}_1^{\{b_j\}^c} \times \mathrm{PE}[f_M],$$
$$Z_V^{\{b_j\}} = \tilde{Z}_V^{\{b_j\}^c} \times \mathrm{PE}[f_+],$$
$$Z_A^{\{b_j\}} = \tilde{Z}_A^{\{b_j\}^c} \times \mathrm{PE}[f_-]$$

1-loop <-> 1-loop + singlets M

vortex <-> vortex + scalar of V+ + spinor of V-

antivortex <-> antivortex + spinor of V+ + scalar of V-

$$I_1^{\{b_j\}} \equiv I_{1-loop}\left(x, \{t_{b_1}, \cdots, t_{b_N}, \cdots\}, \tilde{t}, \tau\right),$$
$$Z_V^{\{b_j\}} \equiv \sum_{\vec{n}}^{\infty} Z_{vort}^{\vec{n}}\left(x, \{t_{b_1}, \cdots, t_{b_N}, \cdots\}, \tilde{t}, \tau; w\right),$$
$$Z_A^{\{b_j\}} \equiv \sum_{\vec{n}}^{\infty} Z_{vort}^{\vec{n}}\left(x, \{t_{b_1}, \cdots, t_{b_N}, \cdots\}, \tilde{t}, \tau; w^{-1}\right)$$

Relation to the topological string amplitude

M-theory on CY 3-folds

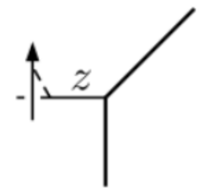
- M5-branes wrapping Lagrangian submanifolds of CY 3-folds -> 3 dimensional theories
- Open topological strings capture the BPS state degeneracy of M2-branes ending on the M5-branes. [Ooguri, Vafa '99] [Gukov, Schwarz, Vafa '04]

The simplest example: k=1/2 U(1) CSM theory with one charged chiral

$$I\left(x=e^{-\gamma},w\right) = \sum_{n=0}^{\infty}\left[w^n x^{-\frac{n^2}{2}}\left(\prod_{k=1}^{n} 2\sinh\gamma k\right)^{-1}\right]\times\sum_{\bar{n}=0}^{\infty}\left[(-w)^{-\bar{n}} x^{\frac{\bar{n}^2}{2}}\left(\prod_{k=1}^{\bar{n}} 2\sinh\gamma k\right)^{-1}\right]$$

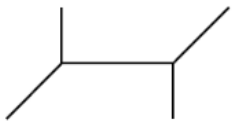
$$\downarrow \quad wx^{-\frac{1}{2}}=z,\quad x=q^{1/2}$$

$$Z_{\rm BPS}^{\rm open}(z;q)=\sum_{n=0}^{\infty}\frac{z^n}{(1-q)\cdots(1-q^n)}=Z^{\rm vortex}_{\text{K-theory}}(z;q)\xrightarrow{\beta\to 0}Z^{\rm vortex}_{\rm hom}=\sum_{n=0}^{\infty}\frac{z^n}{n!\hbar^n}$$



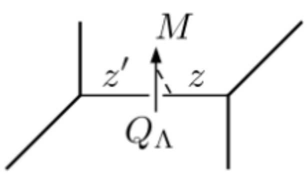
[Dimofte, Gukov, Hollands '10]

N=2 U(1) theory in 5d



$$Z^{\text{closed}}_{\text{BPS}} = \prod_{i,j=1}^{\infty} \left(1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q_\Lambda\right) \qquad I^{5d} = \int \frac{dQ_\Lambda}{Q_\Lambda} |Z^{\text{closed}}_{\text{BPS}}|^2$$

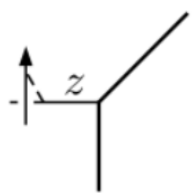
\+ a 3d defect



$$Z^{\text{brane}}_{\text{BPS}} = Z^{\text{closed}}_{\text{BPS}} Z^{\text{open}}_{\text{BPS}} = \frac{\prod_{i,j=1}^{\infty}\left(1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q_\Lambda\right)}{\prod_{i=1}^{\infty}\left(1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} z\right)\left(1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} Q_\Lambda z^{-1}\right)}$$

$$I^{5d+3d} = \int \frac{dQ_\Lambda}{Q_\Lambda} |Z^{\text{closed}}_{\text{BPS}}|^2 \, |Z^{\text{open}}_{\text{BPS}}|^2$$

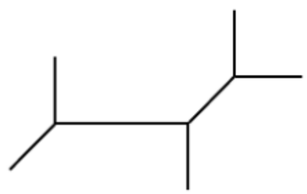
the decoupling limit



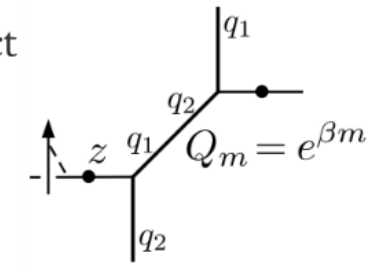
$$Z^{\text{open}}_{\text{BPS}} = \prod_{i=1}^{\infty} \frac{1}{1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} z} \qquad I^{3d} = |Z^{\text{open}}_{\text{BPS}}|^2$$

- 5d N=2 U(1) theory can have a 3d defect which in the decoupling limit is the 3d CSM theory of the level 1/2, with one charged chiral.

Another example:



the decoupled 3d defect



5d U(1) theory
with a fundamental chiral multiplet

3d U(1) theory with fundamental and
antifundamental chiral mutliplets

$$\begin{aligned} &Z_{\rm BPS}^{\rm closed} \\ &= \prod_{i,j}^{\infty} (1 - Q_m q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}}) \\ &\quad \times \sum_{\nu} (-Q_\Lambda)^{|\nu|} q_2^{\frac{||\nu||^2}{2}} q_1^{\frac{||\nu^t||^2}{2}} \tilde{Z}_\nu(q_1,q_2) \tilde{Z}_{\nu^t}(q_2,q_1) \\ &\qquad \times \prod_{i,j\in\nu} (1 - Q_m q_1^{i-\nu_j^t-\frac{1}{2}} q_2^{j-\frac{1}{2}}) \end{aligned}$$

$$\begin{aligned} &\tilde{Z}_\nu(q_1,q_2)\tilde{Z}_{\nu^t}(q_2,q_1) \\ &= \prod_{s\in\nu} \left(1 - q_1^{a_\nu(s)+1} q_2^{\ell_\nu(s)}\right)^{-1} \left(1 - q_2^{a_\nu(s)} q_1^{\ell_\nu(s)+1}\right)^{-1} \end{aligned}$$

$$Z_{\rm BPS}^{\rm open} = \sum_{m=0}^{\infty} \frac{\left(1 - q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} Q_m\right)\left(1 - q_1^{m-\frac{1}{2}} q_2^{-\frac{1}{2}} Q_m\right)}{(1-q_1)(1-q_1^m)} \left(q_2^{\frac{1}{2}} z\right)$$

$$q_1 = x^2, \quad q_2 = 1,$$
$$Q_m = \tau^2 x^{-1}, \quad z = w\tau x^{-1}$$

$$Z_V = \sum_{n=0}^{\infty} \left(w\tau x^{-1}\right)^n \prod_{k=1}^{n} \frac{1-\tau^2 x^{2(k-1)}}{1-x^{2k}}$$

In general

- 5d index [Iqbal, Vafa '12]:

$$I^{5d} = \int \frac{dU_i}{U_i} \left| Z_{top}^{closed}(q_1, q_2, U_i; z_j) \right|^2 = \int \frac{dU_i}{U_i} \left| Z_{top}^{pert} \right|^2 \left| Z_{top}^{np} \right|^2$$

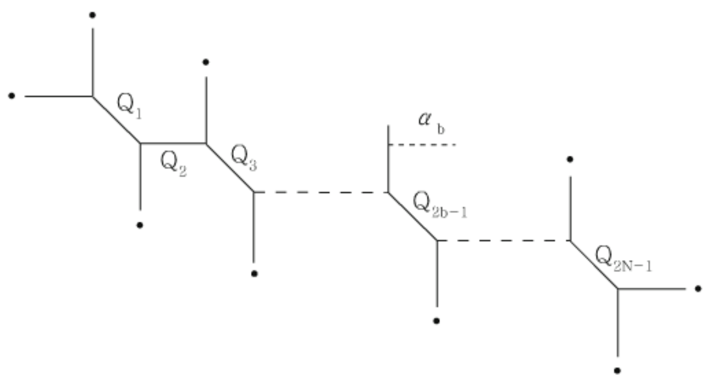
cf. 5d index from the localization [Kim, Kim, Lee '12]:

$$I(x,y,m,q) = \int [d\alpha] \ \mathrm{PE}\left[f_{mat}\left(x,y,e^{i\alpha},e^{im}\right) + f_{vec}\left(x,y,e^{i\alpha}\right)\right] \left| I^{inst}\left(x,y,e^{i\alpha},e^{im},q\right)\right|^2$$

- 5d theory + 3d defect index [Iqbal, Vafa '12]:

$$I^{5d+3d} = \int \frac{dU_i}{U_i}\frac{dV_j}{V_j} \left| Z_{top}^{closed}(q_1, q_2, U_i; z_j) \right|^2 \left| Z_{top}^{open}(q_1, q_2, U_i, V_j; z_k) \right|^2$$

- 3d vortex partition function = the topological open string amplitude [Pasquetti '11] [Taki '13]



Concluding remarks

- Direct observation of the factorization using the Higgs branch localization
- Analytic proof of the index agreement
- New information from the index matching
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- Explicit realization of 3d defects for general cases