

The factorization of the 3d index

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(POSTECH)

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Based on:
C. Hwang, B. Han, D. Kim, J. Kim, P. Kim,
"Factorization of the 3d exponential index",
arXiv:1301.4502

Contents

- Motivation
- Factorization
- Application to 3d Seiberg-like dualities
- Relation to the topological string amplitude
- Concluding remarks

Motivation

Factorization of sphere partition functions:

- Lee and Park for the squashed 3-sphere (Preprint 1301)
- Lee and Park for the Engquist-Shewokh-Chen-Yang (ECSY) theory (Hwang, Kim, Park, Lee 1302)

$$Z_{\text{Sphere}} = \sum_{\text{Diagrams}} \dots$$

↑ the matching trick
3d exponential index

Factorization

N=2 U(1) Chern-Simons-matter theories

• U(1) gauge theory with N_f fundamental and N_m meson chiral multiplets \mathcal{Q}, \mathcal{M}

• $U(1) \times U(1)$ gauge theory with N_f fundamental and N_m meson chiral multiplets \mathcal{Q}, \mathcal{M}

• N_f fundamental and N_m meson chiral multiplets

Application to 3d Seiberg-like dualities

The Abelian duality for N=2 U(1) gauge theories:

- U(1) gauge theory with N_f fundamental and N_m meson chiral multiplets \mathcal{Q}, \mathcal{M}
- $U(1) \times U(1)$ gauge theory with N_f fundamental and N_m meson chiral multiplets \mathcal{Q}, \mathcal{M}
- N_f fundamental, N_m meson, N_v vector and N_a adjoint chiral multiplets

The duality example U(1) theory with $N_f=2, N_m=1, N_v=1, N_a=1$

$$Z_{\text{U(1)}} = \sum_{\text{Diagrams}} \dots$$

Relation to the topological string amplitude

N=2 theory on CP^2 (3d):

- MS theory with U(1) gauge theory with $N_f=2, N_m=1, N_v=1, N_a=1$ (3d exponential index)
- One meson multiplet \mathcal{M} can be the BPS state spectrum of 3d BPS state counting in the M-theory (Hwang, Kim, Park, Kim, Kim, Lee 1302)

The simplest example 3d U(1) CS4 theory with one charged chiral

$$Z_{\text{3d}} = \sum_{\text{Diagrams}} \dots$$

Concluding remarks

- Direct observation of the factorization using the Higgs branch localization
- Analytic proof of the index agreement
- New information from the index-matching (i.e. N=2 Seiberg-like duality [Kim, Kim, Kim, Lee 1302])
- Explicit realization of 3d defects for general cases

Geometric interpretation [Bers, Dimock, Pasquier 12]

• Nonabelian PF as the squashed 3-sphere (Lee 1302)

The superconformal index in 3d

$$Z_{\text{SCIF}} = \sum_{\text{Diagrams}} \dots$$

The colored index

$$Z_{\text{Colored}} = \sum_{\text{Diagrams}} \dots$$

• A physical property of the colored partition function

$$Z_{\text{Colored}} = \sum_{\text{Diagrams}} \dots$$

The colored duality

• An explicit proof of the index agreement for non-abelian gauge theories (Hwang, Kim, Kim, Lee 1302)

• Generalization

$$Z_{\text{Colored}} = \sum_{\text{Diagrams}} \dots$$

• Example of U(1) gauge theory with $N_f=2, N_m=1, N_v=1, N_a=1$

$$Z_{\text{Colored}} = \sum_{\text{Diagrams}} \dots$$

N=2 U(1) theory in 3d

• U(1) CS4 theory with one charged chiral (Lee 1302)

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The sphere partition function of U(1) gauge theories [Kim, Kim, Lee 1302]

$$Z_{\text{Sphere}} = \sum_{\text{Diagrams}} \dots$$

• A physical property of the sphere partition function

$$Z_{\text{Sphere}} = \sum_{\text{Diagrams}} \dots$$

Another example

• U(1) CS4 theory with one charged chiral (Lee 1302)

• U(1) CS4 theory with one charged chiral (Lee 1302)

In general

- 3d index (Hwang, Kim, Kim, Lee 1302)
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The factorization of the 3d index

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C. Hwang, Hee-Cheol Kim, Jaemo Park,
"Factorization of the 3d superconformal index",
[arXiv:1211.6023]

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Motivation

Factorization of sphere partition functions:

3d case: abelian PF on the squashed 3-sphere [Pasquetti '11]

$$Z_{S_b^3} = \sum_{i=1}^N Z_{cl}^i Z_{1-loop}^i Z_V^i \times \bar{Z}_{1-loop}^i \bar{Z}_V^i$$

2d case: PF on the 2-sphere [Benini, Cremonesi '12] [Doroud, Gomis, Floch, Lee '12]

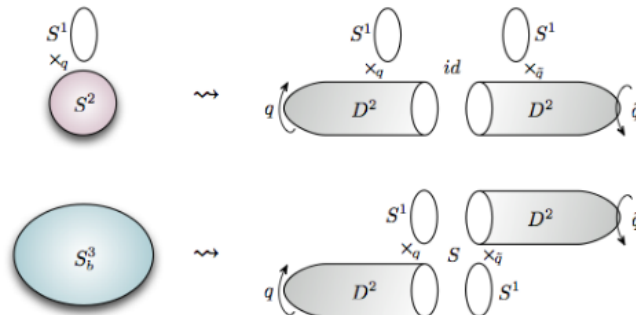
$$Z_{U(N)}(w, \bar{w}) = \sum_{\vec{l} \in C(N, N_f)} e^{i(w+\bar{w}) \sum_{r=1}^N \tau_{l_r}} Z_{1-loop}^{(\vec{l})} Z_v^{(\vec{l})}(e^{-w}) Z_{av}^{(\vec{l})}(e^{-\bar{w}})$$



the vanishing circle limit

3d superconformal index

Geometric interpretation [Beem, Dimofte, Pasquetti '12]



Heegaard splitting to Melvin cigars [BDP 12]

The partition function:

$$Z = \langle 0_q | 0_{\bar{q}} \rangle = \sum_{\alpha} \langle 0_q | \alpha \rangle \langle \alpha | 0_{\bar{q}} \rangle$$

$$q = e^{2\pi i \tau}$$

$$\tau \rightarrow \tilde{\tau} = -id \cdot \tau = -\tau, \quad \tau \rightarrow \tilde{\tau} = -S \cdot \tau = \frac{1}{\tau}$$

- Nonabelian PF on the squashed 3-sphere [Taki '13]

The superconformal index in 3d

$$I(x, t) = \text{Tr}(-1)^F \exp(-\beta' \{Q, S\}) x^{\epsilon + j_3} \prod_a t_a^{F_a}$$

Factorization

N=2 U(N) Chern-Simons-matter theories

$$I(x, t, w, \kappa) = \sum_{m \in \mathbb{Z}^N / S_N} \oint \prod_j \frac{dz_j}{2\pi i z_j} \frac{1}{|\mathcal{W}_m|} e^{-S_{cl}(\alpha, a, m)} Z_{gauge}(x, z, m) \prod_{\Phi} Z_{\Phi}(x, t, z, m)$$

the CS contribution:

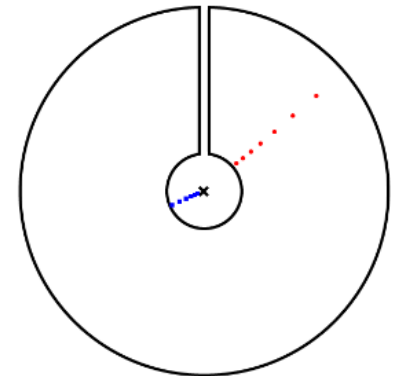
$$e^{-S_{cl}(\alpha, a, m)} = w^{\sum_j m_j} \prod_{j=1}^N (-z_j)^{-\kappa m_j},$$

the vector multiplet:

$$Z_{gauge}(x, z, m) = \prod_{\substack{i,j=1 \\ (i \neq j)}}^N x^{-|m_i - m_j|/2} (1 - z_i z_j^{-1} x^{|m_i - m_j|}),$$

N_f fundamental and \tilde{N}_f antifundamental chiral multiplets:

$$\begin{aligned} & \prod_{\Phi} Z_{\Phi}(x, t, \tilde{t}, \tau, z, m) \\ &= x^{(1-\Delta_{\Phi})(N_f + \tilde{N}_f) \sum |m_j|/2} \left[\prod_{j=1}^N (-z_j)^{-(N_f - \tilde{N}_f) |m_j|/2} \right] \tau^{-(N_f + \tilde{N}_f) \sum |m_j|/2} \\ & \prod_{j=1}^N \prod_{k=0}^{\infty} \left(\prod_{a=1}^{N_f} \frac{1 - z_j^{-1} t_a^{-1} \tau^{-1} x^{|m_j| + 2 - \Delta_{\Phi} + 2k}}{1 - z_j t_a \tau x^{|m_j| + \Delta_{\Phi} + 2k}} \right) \left(\prod_{a=1}^{\tilde{N}_f} \frac{1 - z_j \tilde{t}_a^{-1} \tau^{-1} x^{|m_j| + 2 - \Delta_{\Phi} + 2k}}{1 - z_j^{-1} \tilde{t}_a \tau x^{|m_j| + \Delta_{\Phi} + 2k}} \right) \end{aligned}$$



The factorized index:

$$I(x, t, \tilde{t}, \tau, w, \kappa) = \frac{1}{N!(N_f - N)!} \sum_{\sigma} I_{1-loop}(x, \sigma(t), \tilde{t}, \tau) \times \sum_{\vec{n}=0}^{\infty} Z_{vort}^{\vec{n}}(x, \sigma(t), \tilde{t}, \tau; w, \kappa) \times \sum_{\vec{n}=0}^{\infty} Z_{vort}^{\vec{n}}(x, \sigma(t), \tilde{t}, \tau; w^{-1}, -\kappa)$$

t, \tilde{t} : flavor symmetry, τ : axial symmetry, w : topological symmetry

$$I_{1-loop}(x, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau) = \left(\prod_{\substack{i,j=1 \\ (i \neq j)}}^N 2 \sinh \frac{iM_i - iM_j}{2} \right) \left[\prod_{j=1}^N \prod_{k=0}^{\infty} \left(\frac{\prod_{a=1(\neq j)}^{N_f} 1 - t_j t_a^{-1} x^{2+2k}}{\prod_{a=1}^{\tilde{N}_f} 1 - t_j \tilde{t}_a \tau^2 x^{2k}} \right) \left(\frac{\prod_{a=1}^{\tilde{N}_f} 1 - t_j^{-1} \tilde{t}_a^{-1} \tau^{-2} x^{2+2k}}{\prod_{a=1(\neq j)}^{N_f} 1 - t_j^{-1} t_a x^{2k}} \right) \right]$$

$$Z_{vort}^{\vec{n}}(x = e^{-\gamma}, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau = e^{i\mu}; w, \kappa) = (-1)^{-[\kappa + (N_f - \tilde{N}_f)/2]} \sum n_j e^{i\kappa \sum_j (M_j n_j + \mu n_j + i\gamma n_j^2)} \times (-w)^{\sum n_j} \left[\prod_{j=1}^N \prod_{k=0}^{n_j-1} \frac{\prod_{a=1}^{\tilde{N}_f} 2 \sinh \frac{-i\tilde{M}_a - iM_j - 2i\mu + 2\gamma k}{2}}{\left(\prod_{i=1}^N 2 \sinh \frac{iM_i - iM_j + 2\gamma(k - n_i)}{2} \right) \left(\prod_{a=N+1}^{N_f} 2 \sinh \frac{iM_a - iM_j + 2\gamma(1+k)}{2} \right)} \right]$$

γ : chemical potential for $r+2j$, M, \tilde{M}, μ : mass parameters, w : vortex fugacity

cf. the vortex partition function of 3d N=4 gauge theories [Kim, Kim, Kim, Lee '12]:

$$I_{(k_1, k_2, \dots, k_N)} = \prod_{i=1}^N \prod_{s=1}^{k_i} \left[\prod_{j=1}^N \frac{\sinh \frac{E_{ij} - 2i(\gamma - \gamma')}{2}}{\sinh \frac{E_{ij}}{2}} \prod_{p=N+1}^{N_f} \frac{\sinh \frac{E'_{ip} - 2i(\gamma + \gamma')(R + \tilde{R}) + 2i(\gamma - \gamma')}{2}}{\sinh \frac{E'_{ip} - 2i(\gamma + \gamma')(R + \tilde{R})}{2}} \right]$$

$$E_{ij} = \mu_i - \mu_j + 4i\gamma(k_j - s + 1), \quad E'_{ij} = \mu_i - \mu_j - 4i\gamma(s - 1)$$

$$\gamma_{ours} = 2i\gamma, \quad iM_j + i\mu = \mu_j + 2\gamma, \quad iM_a + i\mu = \mu_a, \quad i\tilde{M}_b + i\mu = -\mu_b$$

- A functional property of the vortex partition function:

$$Z_{vort}^{\vec{n}}(x, t, \tilde{t}, \tau; w, \kappa) = Z_{vort}^{\vec{n}}(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1}; w, -\kappa)$$

$$I_{1-loop} = Z_{1-loop}(x, t, \tilde{t}, \tau) Z_{1-loop}(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1})$$

$$Z_{1-loop}(x, t = e^{iM}, \tilde{t} = e^{i\tilde{M}}, \tau) = \left(\prod_{i < j}^N 2 \sinh \frac{iM_i - iM_j}{2} \right) \left(\prod_{j=1}^N \prod_{k=0}^{\infty} \frac{\prod_{a=1(\neq j)}^{N_f} 1 - t_j t_a^{-1} x^{2+2k}}{\prod_{a=1}^{\tilde{N}_f} 1 - t_j \tilde{t}_a \tau^2 x^{2k}} \right)$$

$$I(x, t, \tilde{t}, \tau, w, \kappa) = I(x^{-1}, t^{-1}, \tilde{t}^{-1}, \tau^{-1}, w^{-1}, \kappa)$$

Application to 3d Seiberg-like dualities

The Aharony duality for $N=2$ $U(N)$ gauge theories:

- $U(N)$ gauge theory with N_f fundamental and antifundamental chiral multiplets Q, \tilde{Q}
- $U(N_f - N)$ gauge theory with N_f fundamental and antifundamental chiral multiplets q, \tilde{q} + N_f^2 singlets M_{ab} , 2 singlets V_+, V_- , and the superpotential:

$$W = Mq\tilde{q} + V_+v_- + V_-v_+$$

The simplest example: $U(1)$ theory with 1 flavor \leftrightarrow the XYZ model

$$I^{N=N_f=1} = \prod_{l=0}^{\infty} \frac{1 - \tau^{-2}x^{2l+2}}{1 - \tau^2x^{2l}} \times Z_{vortex}^{N=N_f=1} \times Z_{anti}^{N=N_f=1}$$



$$I^{M, V_{\pm}} = \text{PE}[f_M] \times \text{PE}[f_+] \times \text{PE}[f_-]$$

$$\text{PE}[f(\cdot)] = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} f(\cdot^n) \right]$$

$$f_M = \frac{\tau^2 - \tau^{-2}x^2}{1 - x^2},$$

$$f_+ + f_- = \frac{\tau^{-1}x - \tau x}{1 - x^2} (w + w^{-1}) = f_{V_+} + f_{V_-}$$

An obtained identity:

$$\sum_{n=0}^{\infty} (-w)^n \prod_{k=1}^n \frac{\tau^{-1} x^{-(k-1)} - \tau x^{k-1}}{x^{-(k-1-n)} - x^{k-1-n}} = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n} w^n \frac{(\tau^{-n} - \tau^n) x^n}{1 - x^{2n}} \right]$$

cf. a analytic proof of the index agreement for some mirror pairs [Krattenthaler, Spiridonov, Vartanov '11]

- General cases

$$\begin{aligned} I_1^{\{b_j\}} &= \tilde{I}_1^{\{b_j\}^c} \times \text{PE}[f_M], \\ Z_V^{\{b_j\}} &= \tilde{Z}_V^{\{b_j\}^c} \times \text{PE}[f_+], \\ Z_A^{\{b_j\}} &= \tilde{Z}_A^{\{b_j\}^c} \times \text{PE}[f_-] \end{aligned}$$

1-loop \leftrightarrow 1-loop + singlets M
 vortex \leftrightarrow vortex + scalar of V+ + spinor of V-
 antivortex \leftrightarrow antivortex + spinor of V+ + scalar of V-

$$\begin{aligned} I_1^{\{b_j\}} &\equiv I_{1\text{-loop}}(x, \{t_{b_1}, \dots, t_{b_N}, \dots\}, \tilde{t}, \tau), \\ Z_V^{\{b_j\}} &\equiv \sum_{\vec{n}}^{\infty} Z_{\text{vort}}^{\vec{n}}(x, \{t_{b_1}, \dots, t_{b_N}, \dots\}, \tilde{t}, \tau; w), \\ Z_A^{\{b_j\}} &\equiv \sum_{\vec{n}}^{\infty} Z_{\text{vort}}^{\vec{n}}(x, \{t_{b_1}, \dots, t_{b_N}, \dots\}, \tilde{t}, \tau; w^{-1}) \end{aligned}$$

Relation to the topological string amplitude

M-theory on CY 3-folds

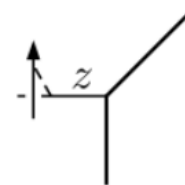
- M5-branes wrapping Lagrangian submanifolds of CY 3-folds -> 3 dimensional theories
- Open topological strings capture the BPS state degeneracy of M2-branes ending on the M5-branes. [Ooguri, Vafa '99] [Gukov, Schwarz, Vafa '04]

The simplest example: $k=1/2$ U(1) CSM theory with one charged chiral

$$I(x = e^{-\gamma}, w) = \sum_{n=0}^{\infty} \left[w^n x^{-\frac{n^2}{2}} \left(\prod_{k=1}^n 2 \sinh \gamma k \right)^{-1} \right] \times \sum_{\bar{n}=0}^{\infty} \left[(-w)^{-\bar{n}} x^{\frac{\bar{n}^2}{2}} \left(\prod_{k=1}^{\bar{n}} 2 \sinh \gamma k \right)^{-1} \right]$$

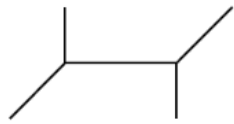
$$\downarrow \quad wx^{-\frac{1}{2}} = z, \quad x = q^{1/2}$$

$$Z_{\text{BPS}}^{\text{open}}(z; q) = \sum_{n=0}^{\infty} \frac{z^n}{(1-q) \cdots (1-q^n)} = Z_{\text{K-theory}}^{\text{vortex}}(z; q) \xrightarrow{\beta \rightarrow 0} Z_{\text{hom}}^{\text{vortex}} = \sum_{n=0}^{\infty} \frac{z^n}{n! \hbar^n}$$

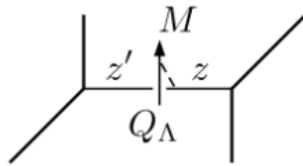


[Dimofte, Gukov, Hollands '10]

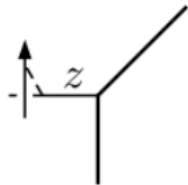
N=2 U(1) theory in 5d



↓ + a 3d defect



↓ the decoupling limit



$$Z_{\text{BPS}}^{\text{closed}} = \prod_{i,j=1}^{\infty} \left(1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q_\Lambda\right) \quad I^{5d} = \int \frac{dQ_\Lambda}{Q_\Lambda} |Z_{\text{BPS}}^{\text{closed}}|^2$$

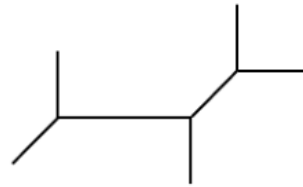
$$Z_{\text{BPS}}^{\text{brane}} = Z_{\text{BPS}}^{\text{closed}} Z_{\text{BPS}}^{\text{open}} = \frac{\prod_{i,j=1}^{\infty} \left(1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q_\Lambda\right)}{\prod_{i=1}^{\infty} \left(1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} z\right) \left(1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} Q_\Lambda z^{-1}\right)}$$

$$I^{5d+3d} = \int \frac{dQ_\Lambda}{Q_\Lambda} |Z_{\text{BPS}}^{\text{closed}}|^2 |Z_{\text{BPS}}^{\text{open}}|^2$$

$$Z_{\text{BPS}}^{\text{open}} = \prod_{i=1}^{\infty} \frac{1}{1 - q_1^{i-\frac{1}{2}} q_2^{\frac{1}{2}} z} \quad I^{3d} = |Z_{\text{BPS}}^{\text{open}}|^2$$

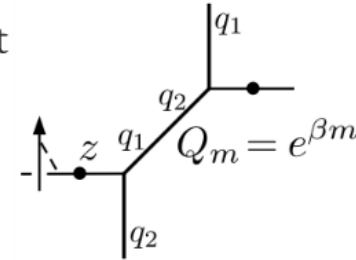
- 5d N=2 U(1) theory can have a 3d defect which in the decoupling limit is the 3d CSM theory of the level 1/2, with one charged chiral.

Another example:



5d U(1) theory
with a fundamental chiral multiplet

the decoupled 3d defect



3d U(1) theory with fundamental and
antifundamental chiral multiplets

$$\begin{aligned}
 & Z_{\text{BPS}}^{\text{closed}} \\
 &= \prod_{i,j} (1 - Q_m q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}}) \\
 &\quad \times \sum_{\nu} (-Q_{\Lambda})^{|\nu|} q_2^{\frac{||\nu||^2}{2}} q_1^{\frac{||\nu^t||^2}{2}} \tilde{Z}_{\nu}(q_1, q_2) \tilde{Z}_{\nu^t}(q_2, q_1) \\
 &\quad \times \prod_{i,j \in \nu} (1 - Q_m q_1^{i-\nu_j^t-\frac{1}{2}} q_2^{j-\frac{1}{2}})
 \end{aligned}$$

$$\begin{aligned}
 & \tilde{Z}_{\nu}(q_1, q_2) \tilde{Z}_{\nu^t}(q_2, q_1) \\
 &= \prod_{s \in \nu} \left(1 - q_1^{a_{\nu}(s)+1} q_2^{\ell_{\nu}(s)}\right)^{-1} \left(1 - q_2^{a_{\nu}(s)} q_1^{\ell_{\nu}(s)+1}\right)^{-1}
 \end{aligned}$$

$$Z_{\text{BPS}}^{\text{open}} = \sum_{m=0}^{\infty} \frac{\left(1 - q_1^{\frac{1}{2}} q_2^{-\frac{1}{2}} Q_m\right) \left(1 - q_1^{m-\frac{1}{2}} q_2^{-\frac{1}{2}} Q_m\right)}{(1 - q_1) (1 - q_1^m)} \left(q_2^{\frac{1}{2}} z\right)$$

$q_1 = x^2, \quad q_2 = 1,$
 $Q_m = \tau^2 x^{-1}, \quad z = w\tau x^{-1}$

$$Z_V = \sum_{n=0}^{\infty} (w\tau x^{-1})^n \prod_{k=1}^n \frac{1 - \tau^2 x^{2(k-1)}}{1 - x^{2k}}$$

In general

- 5d index [Iqbal, Vafa '12]:

$$I^{5d} = \int \frac{dU_i}{U_i} |Z_{top}^{closed}(q_1, q_2, U_i; z_j)|^2 = \int \frac{dU_i}{U_i} |Z_{top}^{pert}|^2 |Z_{top}^{np}|^2$$

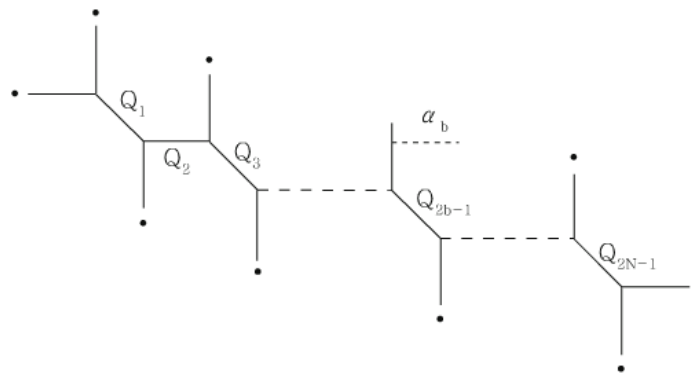
- cf. 5d index from the localization [Kim, Kim, Lee '12]:

$$I(x, y, m, q) = \int [d\alpha] \text{PE} [f_{mat}(x, y, e^{i\alpha}, e^{im}) + f_{vec}(x, y, e^{i\alpha})] |I^{inst}(x, y, e^{i\alpha}, e^{im}, q)|^2$$

- 5d theory + 3d defect index [Iqbal, Vafa '12]:

$$I^{5d+3d} = \int \frac{dU_i}{U_i} \frac{dV_j}{V_j} |Z_{top}^{closed}(q_1, q_2, U_i; z_j)|^2 |Z_{top}^{open}(q_1, q_2, U_i, V_j; z_k)|^2$$

- 3d vortex partition function = the topological open string amplitude [Pasquetti '11] [Taki '13]



Concluding remarks

- Direct observation of the factorization using the Higgs branch localization
- Analytic proof of the index agreement
- New information from the index matching
(cf. $N=4$ Seiberg-like duality [Kim, Kim, Kim, Lee '12])
- Explicit realization of 3d defects for general cases