Partition functions of $\mathcal{N} = 4$ Yang-Mills and applications

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Outline

1. Partition functions of topologically twisted $\mathcal{N} = 4$ $U(r)$ Yang-Mills theory on a four-manifold
   - Motivation: S-duality in gauge theory
   - Harder-Narasimhan recursion
   - Explicit expressions and modularity
2. Application
   - A 5d/2d/4d correspondence

Based on:
1109.4861
1211.0513 w. Haghighat & Vandoren

4-dim. Yang-Mills theory

Connection and field strength:

- Field strength: $F = dA + A \wedge A \in \Omega^2(S, \mathfrak{g})$
- Action: $S[A] = \frac{1}{g^2} \int_S \mathrm{Tr}\, F \wedge *F + \frac{i\theta}{8\pi^2} \mathrm{Tr}\, F \wedge F$

Electric and magnetic charge ($\mathfrak{g} = u(1)$):

$$Q_{\mathrm{e}} = \frac{1}{4\pi} \int_{S^2} *F, \qquad Q_{\mathrm{m}} = \frac{1}{4\pi} \int_{S^2} F$$

Solitonic monopoles ($\mathfrak{g} = su(2)$):

F, $\phi \Rightarrow \tilde{F}_{u(1)}$ with monopole charge

Mass: $M \geq |a||\tilde{Q}_{\mathrm{e}} + \tau\, \tilde{Q}_{\mathrm{m}}|, \qquad \text{with } \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

't Hooft (1974), Polyakov (1976)

S-duality

Electric-magnetic duality:

Montonen, Olive (1977)

$$S : \begin{cases} (F, *F) \to (*F, -F) \\ \tau \to -1/\tau \\ |a| \to |\tau a| \end{cases}$$

Translations:

$$\frac{1}{8\pi^2} \int \mathrm{Tr}\, F \wedge F \in \mathbb{Z}$$

$\Rightarrow \quad T : \tau \to \tau + 1$ is a symmetry of the theory

$S + T$ generate $SL_2(\mathbb{Z})$ S-duality group

Is the path integral invariant under $SL_2(\mathbb{Z})$?

$\Rightarrow$ Topologically twisted $\mathcal{N}=4$ Yang-Mills on S:

Vafa, Witten (1994)

$$h_r(\tau) = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}X\, e^{-S[A,\psi,X]}$$

S-duality $\Rightarrow$ $h_r(\tau)$ transforms as a modular form:

$$h_r\left(\frac{a\tau+b}{c\tau+d}\right) \sim (c\tau+d)^{-\chi(S)/2}\, h_r(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Path integral $\longrightarrow$ generating function

Path integral localizes on BPS-solutions:

- minimize $S[A]$, e.g. instantons: $F_{\mathrm{inst}} = - * F_{\mathrm{inst}}$
- instanton number: $\kappa(F) = -\frac{1}{8\pi^2}\int_S \mathrm{Tr}\, F \wedge F \in \mathbb{Z}$
- $e^{-S[A]} = q^\kappa$ with $q = e^{2\pi i \tau}$

$$\Rightarrow h_r(\tau) = \sum_\kappa \Omega(\kappa)\, q^\kappa + R(\tau, \bar{\tau})$$

- $\Omega(\kappa)$: Euler number of inst. moduli space $\sim$ BPS-invariant
- $R(\tau, \bar{\tau})$: arises due to reducible connections:

$$\text{e.g. } r = 2: \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix}$$

Verification of EM-duality for $U(r = 1, 2)$ YM and complex surfaces

Vafa, Witten (1994)

From YM theory to vector bundles

Computation of $h_r(\tau)$ for rational surfaces is possible using methods for complex surfaces Gottsche, Nakajima, Yoshioka,...

Donaldson-Uhlenbeck-Yau theorem:

$$\text{ASD connections} \Leftrightarrow \begin{cases} \text{- stable vector bundles} \\ \text{- reducible connections} \end{cases}$$

Characterized by their Chern classes $ch_i \in H^{2i}(S,\mathbb{Z})$:

$$ch_0 = r, \qquad ch_1 = \frac{i}{2\pi}\mathrm{Tr}\,F, \qquad ch_2 = \frac{1}{8\pi^2}\mathrm{Tr}\,F\wedge F$$

Gieseker stability

Choose polarization (Kähler modulus) $J \in C(S)$, and define the polynomial:

$$p_J(F) := \frac{c_1(F) \cdot J}{r(F)} + \frac{1}{r(F)} \left(\frac{c_1(F)^2 - K_S \cdot c_1(F)}{2} - c_2(F) \right)$$

Definition:

A bundle F is Gieseker stable if for every subbundle $F' \subsetneq F$, $p_J(F', n) \prec p_J(F, n)$ (semi-stable $\Rightarrow p_J(F', n) \preceq p_J(F, n)$)

Agrees with stability based on central charge in large volume limit: $\lim_{J \to \infty} Z(F, B + iJ)$

Douglas (2000), Bridgeland (2002)

BPS-invariants

Definition:

$$\Omega(\Gamma, w; J) := \frac{w^{-\dim_{\mathbb{C}} \mathcal{M}_J(\Gamma)}}{w - w^{-1}} \, p(\mathcal{M}_J(\Gamma), w),$$

where $p(X, s) = \sum_{i=0}^{2\dim_{\mathbb{C}}(X)} b_i s^i$ with $b_i = \dim H^2(X, \mathbb{Z})$
and $\mathcal{M}(\Gamma)$ is the moduli space of *semi-stable* vector bundles.

Generating function

Generating function:

$$h_{r,c_1}(z,\tau;S,J) := \sum_{c_2} \bar{\Omega}(\Gamma, w; J)\, q^{r\Delta - \frac{r\chi(S)}{24}}$$

- discriminant: $\Delta = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2)$
- modular parameter: $\tau \longrightarrow \frac{a\tau+b}{c\tau+d}, \quad q = e^{2\pi i\tau}$
- elliptic parameter: $z \longrightarrow \frac{z}{c\tau+d}, \quad w = e^{2\pi i z}$

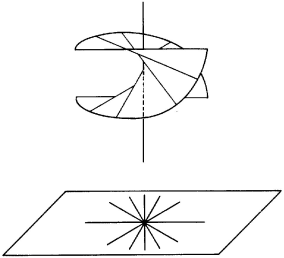
Also define:

$$f_{r,c_1}(z,\tau;S,J) := \frac{h_{r,c_1}(z,\tau;S,J)}{h_{1,0}(z,\tau;S,J)^r}$$

Isomorphism of $\mathcal{M}_J(\Gamma)$: $\quad c_1(E) \to c_1(E) + \mathbf{k}, \quad \mathbf{k} \in H^2(S, r\mathbb{Z}), \quad \Delta \quad \text{invariant}$

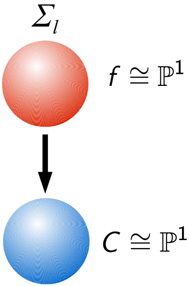
Rational surfaces

Blow-up Σ_1: two points of view:



Blow-up: $\phi : \Sigma_1 \to \mathbb{P}^2$

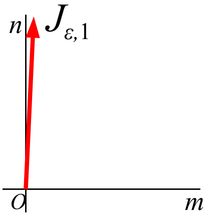
Bott & Tu



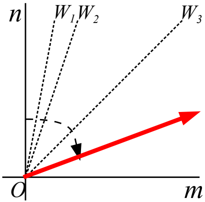
Rational ruled surface
$\pi : \Sigma_1 \to \mathbb{P}^1$

- $H_2(\Sigma_1, \mathbb{Z})$: generators C, f such that $\phi^*(C + f) = H$
- Intersection numbers: $C^2 = -1$, $f^2 = 0$, $C \cdot f = 1$
- Ample cone: $\mathcal{C}(\Sigma_1) = \{m(C + f) + nf;\ m, n > 0\}$

Outline of computations



1. Determine invariants for suitable polarization



2. Wall-crossing

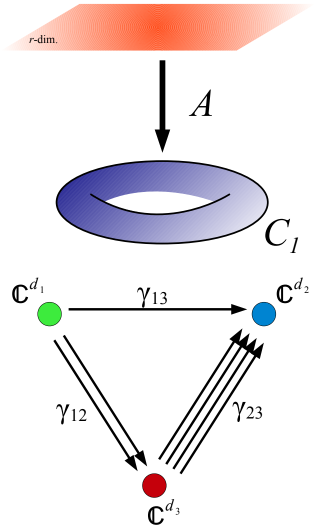
$$\phi : \tilde{S} \longrightarrow S$$

$$H^{\mu}_{r,\phi^* c_1 - kC_1}(z,\tau;\tilde{S},\phi^* J) = B_{r,k}(z,\tau)\, H^{\mu}_{r,c_1}(z,\tau;S,J)$$

3. Blow-up/down

Harder-Narasimhan recursion I

Crucial technique for computation of BPS invariants:



Vector bundles
on Riemann surfaces:

Harder, Narasimhan (1975),

Atiyah, Bott (1982)

Quivers:

Reineke (2003)

App. to BH's

Denef (2002), JM, Pioline, Sen (2010-12),. . .

Harder-Narasimhan recursion II

Computation of $\Omega(\Gamma, w)$ proceeds in 2 steps:

1. Determine the cohomology of the space of all bundles $B\mathscr{G}(\Gamma)$:

$$h_r(z;C) = -w^{r^2(1-g)} \frac{(1+w^{2r-1})^{2g}}{1-w^{2r}} \prod_{\ell=1}^{r-1} \frac{(1+w^{2\ell-1})^{2g}}{(1-w^{2\ell})^2}$$

   Harder, Narasimhan (1975), Atiyah, Bott (1982)

2. Subtract the unstable bundles

Harder-Narasimhan recursion III

Definition:
A Harder-Narasimhan filtration of a vector bundle F is a filtration $0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = F$ such that the quotients $E_i = F_i/F_{i-1}$ are semi-stable and satisfy $d_i/r_i > d_{i+1}/r_{i+1}$ for all i.

$\Rightarrow$ If $\ell \geq 2$ then F is unstable.

Enumerate vector bundles with a given HN filtration:

$$w^{-\sum_{i<j}(r_i d_j - r_j d_i)} \prod_{i=1}^{\ell} \Omega(\Gamma_i, w)$$

$\Rightarrow$ Leads to a recursion in Γ for $\gcd(r,d) = 1$, e.g. for $\Gamma = (2,1)$:

$$\Omega(\Gamma, w) = h_2(z;C) + \frac{w^2}{1-w^4} h_1(z;C)^2$$

$$g = 2 \Rightarrow \qquad = \frac{1}{w - w^{-1}} \left(w^{-5} + 4w^{-4} + 7w^{-3} + 12w^{-2} + 24w^{-1} + 32 + \ldots \right)$$

Recursion solved by Zagier (1996), Reineke (2003)

Suitable polarization

$c_1 \cdot f \neq 0 \mod r$: $\Rightarrow \Omega(\Gamma, w; J_{\varepsilon,1}) = 0$

$c_1 \cdot f = 0 \mod r$:

Conjecture: JM (2011)
The contribution to $h_{r,c_1}(z,\tau;\Sigma_\ell, J)$ with $c_1 \cdot f = 0 \mod r$ from the set of vector bundles whose restriction to the fibre is semi-stable is given by:

$$H_r(z,\tau;f) := \frac{i\,(-1)^{r-1}\,\eta(\tau)^{2r-3}}{\theta_1(2z,\tau)^2\,\theta_1(4z,\tau)^2 \dots \theta_1((2r-2)z,\tau)^2\,\theta_1(2rz,\tau)},$$

with $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, $\quad \theta_1(z,\tau) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} (-1)^r q^{\frac{r^2}{2}} w^r$

Proofs: $r=1$: Gottsche (1990), $r=2$: Yoshioka (1995), general r: Mozgovoy (2013)

Explicit expressions for Σ_1

Rank 1:

$$h_{1,c_1}(z,\tau;S) = \frac{i}{\theta_1(z,\tau)\,\eta(\tau)^{b_2(S)-1}}$$

Gottsche (1990)

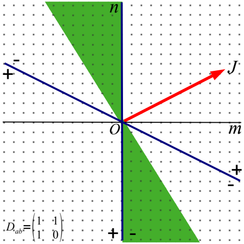
Rank 2:

$$f_{2,\beta C-\alpha f}(z,\tau;\Sigma_1,J_{m,n}) = \delta_{\beta,0}\left(\frac{w^{2\alpha}}{1-w^4} - \frac{1}{2}\delta_{\alpha,0} + \frac{i\eta(\tau)^3}{\theta_1(4z,\tau)}\right) \qquad 1 < |w| < |q^{-\frac{1}{4}}|$$

$$+ \sum_{\substack{(a,b)=-(\alpha,\beta) \mod 2\\ b\neq 0}} \tfrac{1}{2}\left(\operatorname{sgn}(b) - \operatorname{sgn}(bn-am)\right) w^{-b+2a}\, q^{\frac{1}{4}b^2+\frac{1}{2}ab}$$

Yoshioka (1995/6), Gottsche, Zagier (1996)

Indefinite theta function:



Rank 3: Σ_1

Explicit expression:

$$
\begin{aligned}
f_{3,-C-f}(z,\tau;\Sigma_1,J_{m,n}) \quad = \quad & \frac{1}{4}\sum_{\substack{(a,b)=(-2,2) \mod 3 \\ (c,d)=(a,b) \mod 2,\; d\neq 0}} \\
& \times\left(\operatorname{sgn}(b)-\operatorname{sgn}(bn-am)\right)\left(\operatorname{sgn}(d)-\operatorname{sgn}(d|a|-c|b|)\right) \\
& \times(w^{-b+2a}-w^{b-2a})w^{-d+2c}\,q^{\frac{1}{12}b^2+\frac{1}{6}ab+\frac{1}{4}d^2+\frac{1}{2}cd}+\ldots
\end{aligned}
$$

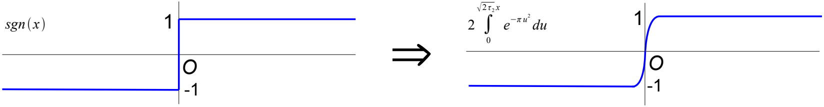
- signature of lattice is (2,2)
- quadratic condition on the lattice points

Modularity

Rank 1: $h_{1,c_1}(z,\tau;S)$ has expected transformation properties ✓

Rank 2: $h_{2,c_1}(z,\tau;\Sigma_1,J_{m,n})$ is not modular

However, replace:



$\Rightarrow \widehat{f}_{2,c_1}$ transforms as a modular form

Zwegers (2000), Ramanujan ($\sim$ 1915)

- Approaches $\mathrm{sgn}(x)$ for $\tau_2 \to \infty$
- Main physical importance: modularity & continuous in J
  Related to continuity of hypermultiplet metric, Gaiotto, Moore, Neitzke (2008), Alexandrov, JM, Pioline (2012)
- Modular properties of $f_{3,c_1}(z,\tau;\mathbb{P}^2)$ under investigation
  Bringmann, JM, Zwegers
- How to derive from path integral?

Geometric Engineering I

Limit of string theory on non-compact CY: $\mathcal{O}(-K_{\Sigma_1}) \to \Sigma_1$
$\Rightarrow \quad \mathcal{N} = 2$, $SU(2)$ gauge theory in 4 dimensions Katz, Klemm, Vafa (1997)

Field theory particles:

Particles	D-brane
W-boson	D2 on f
Monopole	D4 on Σ_1
(Flavor charge	D2 on blow-up C_i)

Dyon spectra $\Leftrightarrow$ $\mathcal{N} = 4$ Yang-Mills partition functions

Charges:
D4's: $P = r$
D2's: $Q = c_1 + rK_S/2$
D0's: $Q_0 = -r\Delta + \frac{1}{2r}Q^2 + r\chi(S)/24$

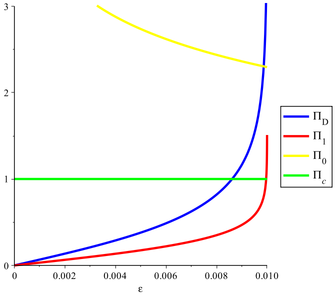
Geometric Engineering II

Field theory particles are the "light" D-branes.

Magnitude of periods: large volume limit → field theory limit:

Limit of Kähler classes:

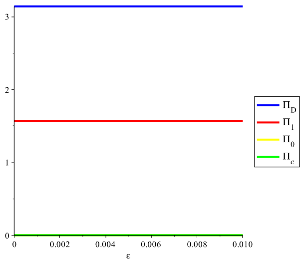
$$e^{2\pi i(B_C+iJ_C)} = \lim_{\varepsilon\to 0} \varepsilon^4 e^{4c_0}; \quad e^{2\pi i(B_f+iJ_f)} = \lim_{\varepsilon\to 0} \frac{1}{4}\left(1+\frac{2\pi^2\varepsilon^2 u}{M_0^2}\right)$$



Chuang, Diaconescu, JM, Moore, Soibelman (2012)

Geometric Engineering III

No walls:



Arguments of periods

For $r \geq 2$, D0-brane charge > 0 for $\Omega(\Gamma; J) \neq 0$

Dyon spectrum:
$\Omega((1, \frac{1}{2}n_e f, 0); J) = 1 \Rightarrow (n_m, n_e) = (\pm 1, 2n), n \in \mathbb{Z}$

Dimofte, Gukov, Soibelman (2009); Chuang *et al.* (2012)

Geometric Engineering IV

In agreement with:

- Vacuum moduli space

  Seiberg, Witten (1994/5), Bilal, Ferrari (1996)

- Computation of index of Dirac operator on monopole moduli space:

$$\begin{array}{ccc} O(r) \times SO(2N_f) & & \\ \downarrow & & \\ \mathcal{M}_r & = & \mathbb{R}^3 \times \frac{S^1 \times \widetilde{\mathcal{M}}_r}{\mathbb{Z}_r} \end{array} .$$

  Sethi, Stern, Zaslow (1995), Gauntlett, Harvey (1996)

Ground states of quantum mechanics on $\mathcal{M}_r$:

- 4r boson-fermion pairs (X^m, ψ^m)
- $2rN_f$ fermions χ^A

A 5d/2d/4d correspondence I

Five dimensional gauge theory on $\mathbb{R}^3 \otimes S^1_{\mathrm{t}} \otimes S^1_{\mathrm{M}}$:

BPS objects:

- Magnetic string: $\frac{T}{\sqrt{2}} = r\left(\frac{\phi}{2g_5^2} + \frac{\kappa}{4}\phi^2\right)$
- Electric charge: $Z_e = \phi$
- Instanton: $Z_I = \frac{1}{2g_5^2} + \frac{\kappa}{2}\phi$
- Momentum around S^1_{M}: Q_0
- Hypermultiplet mass: m_i, $i = 1, \ldots, N_f$

with $\kappa = 2(8 - N_f)$

Seiberg (1996), Morrison, Seiberg (1996)

Engineered by M-theory on $\mathcal{O}(K_{\Sigma_1}) \to \Sigma_1$:

$$\Rightarrow \quad \begin{cases} J = \frac{f}{4g_5^2} - \phi\, K_{\mathbb{B}_{N_f}} - \sum_{i,j}^{N_f} m_i C^{-1}_{D_n,ij} d_j \\ k = \frac{1}{2} n_e f - n_I\, K_{\mathbb{B}_{N_f}} - \frac{1}{2}\sum_i^{N_f} n_{f,i} d_j \end{cases}$$

A 5d/2d/4d correspondence II

Worldvolume theory on magnetic string $\Rightarrow$ $(0,4)$ σ-model with target space $\widetilde{\mathcal{M}}$:

- tension of the string: $\tilde{T} = \frac{T}{2\sqrt{2}}$
- radius of S^1: $\frac{2}{\phi}$
- momentum p: n_e
- winding w: n_I
- oscillator level: Q_0

CFT elliptic genus:

$$\mathcal{Z}_{\mathrm{CFT},r}(\tau,y) = \mathrm{Tr}_{\mathrm{R}}\left[\tfrac{1}{2}F^2(-1)^F q^{L_0-\frac{c_L}{24}}\bar{q}^{\bar{L}_0-\frac{c_R}{24}}e^{2\pi i y_i J_0^i}\right]$$

- $L_0 - \frac{c_L}{24} = \frac{\left(p\frac{\phi}{2}+w\frac{\tilde{T}}{\phi}\right)^2}{2\tilde{T}} - Q_0$
- BPS condition $\Rightarrow$ $\bar{L}_0 - \frac{c_R}{24} = \frac{\left(p\frac{\phi}{2}+w\frac{\tilde{T}}{\phi}\right)^2}{2\tilde{T}}$
- spectral flow $\Rightarrow$ $\mathcal{Z}_{\mathrm{CFT},r}(\tau,y) = \sum_\mu h_{r,\mu}^{\mathrm{CFT}}(\tau)\,\Theta_{r,\mu}(\tau,y)$

A 5d/2d/4d correspondence III

Proposal:

$$\mathcal{Z}_{\mathrm{CFT},r}(\tau, y) = \mathcal{Z}_{\mathrm{YM},r}(\tau, y; J)$$

Haghighat, JM, Vandoren (2012)

Prediction of elliptic genera of hyper-Kähler monopole moduli spaces!

A 5d/2d/4d correspondence IV

Proven for $SU(2)$ gauge theory with N_f flavors and $r=1$

(0,4) σ-model:

- Condition on gauge theory charges: $(-1)^{n_e} = (-1)^H \Rightarrow$ lift to orbifold CFT
- $\Theta_{D_{N_f}} \Rightarrow$ flavor symmetry $SO(2N_f)$

$\mathcal{N}=4$ Yang-Mills:

- $H^2(S,\mathbb{Z})$: $\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & -1 & & \\ & & & \dots & \\ & & & & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 8-n & 2 & \\ 2 & 0 & \\ & & -\mathcal{Q}_{D_n} \end{pmatrix}$
- gluing vectors $\mathbf{g}_i$

A 5d/2d/4d correspondence V

For $r \geq 2$ various additional intriguing aspects arise:

- Dependence on $J \Leftrightarrow$ renormalization group flow ?
- What does the $(0, 4)$ elliptic genus compute in terms of the target space?

Conclusions

Partition functions of $\mathcal{N} = 4$, $U(r)$ Yang-Mills theory on rational surfaces can be determined explicitly

Applications:

- Geometric engineering:
  - dyon spectra
  - a 5d/2d/4d correspondence
- Elliptic Calabi-Yau manifolds 1207.1795 w. Klemm & Wotschke
- D3-instantons 1207.1109 w. Alexandrov & Pioline
- Localization
- Quivers
- ...