From $4d$ dualities to $3d$ dualities

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IAS

Based on work with:

*O. Aharony, N. Seiberg, and B. Willett (1305.3924, 1306.****)*

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IR dualities

- IR dualities: different UV descriptions flowing in the IR to the same fixed point.

- 4d IR (Seiberg) dualities:

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\begin{aligned}
SU(N_c)_{N_f} &\longleftrightarrow SU(N_f - N_c)_{N_f} + W_1 \\
SO(N_c)_{N_f} &\longleftrightarrow SO(N_f - N_c + 4)_{N_f} + W_2 \\
USp(2N_c)_{N_f} &\longleftrightarrow USp(2(N_f - N_c - 2))_{N_f} + W_3
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- 3d IR (Aharony,...) dualities:

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- If yes, how one explains the differences?

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Outline

- The road from 4d to 3d
- Examples
- Partition functions
- Summary

From 4d to to 3d by dimensional reduction

- Say we have an IR 4d duality between theories UV_4^A and UV_4^B.
- Both these theories flow to the same CFT_4 in the IR.
- Let us consider the theories $UV_3^{A,B}$ obtained by a dimensional reduction to $3d$ of UV_4^A.
- That is same matter, same gauge interactions, just one coordinate less.
- One finds that the two theories $UV_3^{A,B}$ in general do not flow to the same IR CFT in $3d$.
- Moduli spaces in $3d$ might not match
- $3d$ theory has actually more symmetry than the $4d$ parent one. Some classical symmetries are anomalous in $4d$ but are symmetries of the quantum theory in $3d$.
- This can be seen in the different partition functions on $S^2 \times S^1$ and on S^3(Dolan,Spiridonov,Vartanov;Niarchos). $3d$ partition functions of $UV_3^{A,B}$ refined with fugacities/real masses for symmetries anomalous in $4d$ do not match, whereas the unrefined ones do match.

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Why?

- Why the duality does not survive the dimensional reduction?
- 4d duality means that physics described by $UV_4^{A,B}$ when probed at energies $E << \Lambda,\ \widetilde{\Lambda},\ 1/R$ is the same.
- The strong coupling scale Λ is given by $\Lambda = \exp(-\frac{8\pi^2}{b\, g_4^2})$.
- After compactification $g_4^2 = 2\pi R\, g_3^2$, $\Lambda = \exp(-\frac{4\pi}{b\, R\, g_3^2})$.
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- However keeping g_3^2 and $\widetilde{g}_3^2$ fixed and taking $R \to 0$ also $\Lambda \to 0$.
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From 4d to effective 3d theory

- However, if we consider a 4d theory on a circle at energies $E << 1/R$ it's behavior is effectively 3-dimensional.
- There are though at least two caveats.
- 4d theories on a circle have scalars coming from $P\exp(i\oint A_3)$. These scalars can acquire VEVs and parametrize the Coulomb branch of the theory on the circle. By definition they are compact.
- 3d theories have scalars in their multiplet which also parametrize their Coulomb branches. But they are not compact.
- The effective 3d theory describing the 4d theory on a circle has non-perturbative superpotentials W which the naive 4d reduction does not have. *E.g.* in case the theory we put on the circle is SQCD with $SU(N)$ gauge group one has

$$W = \eta\, Y, \qquad \eta \sim \Lambda^b$$

  where Y is the coordinate on the Coulomb branch. (W breaks explicitly the symmetry which is anomalus in 4d)
- Thus the effective 3d theories with the superpotentials and the compact Coulomb branches have to be dual whereas the naively reduced ones do not.

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From 3d effective theory to 3d UV theory

- The final step is to find a 3d theory which will flow to the same effective one as the 4d theory on the circle, $UV_3^{A,B}(\eta)$
- One can find operators in $3d$ (built from monopoles) which flow to the Coulomb branch coordinate Y.
- In some cases the compact Coulomb branches are lifted by superpotentials and in others one can focus on particular points of the Coulomb branch so that the compactness will not be an issue.
- The procedure outlined here is completely generic.
- However, the actual details are very much case specific.
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Example: 3d SU duality

- Let us give an example of the procedure reducing $SU(N)$ dualities in 4d to 3d.
- 4d Seiberg duality:

$$A: \qquad SU(N)_{N_f}, \quad Q,\, \widetilde{Q}\,(R_Q = \frac{N_f - N}{N_f}), \qquad W = 0$$

$$\updownarrow$$

$$B: \qquad SU(N_f - N)_{N_f}, \quad q, \widetilde{q}\,(R_q = \frac{N}{N_f}), \quad M\,(R_M = 2R_Q)\,, \qquad W = Mq\widetilde{q}\,.$$

$SU(N)$: from 4d to 3d

- Putting theories A and B on the circle one obtains effective 3d theories with same matter content as in 4d but with the following superpotentials

$$A: \quad W = \eta\, Y, \qquad\qquad B: \quad W = Mq\widetilde{q} + \widetilde{\eta}\widetilde{Y}, \qquad \eta\, \widetilde{\eta} \sim 1\,.$$

- The superpotemtials can be written using monopole operators and they actually lift completely the compact Colomb branches.

- We can get rid of the η superpotential on one side of the duality, say A, by turning on real masses.

- Real mass in 3d is a VEV for σ inside a background vector field for a global symmetry.

- Consider the 3d duality of the previous slide with $N_f + 1$ flavors and turn on large real mass $\hat{m}$ for the last flavor.

- The theory with the real mass has a vacuum at the origin of the moduli space. In this vacuum the last flavor disappears from the IR physics. Moreover, since the η superpotential is charged under the symmetries of the heavy fields it also cannot appear in the IR. Thus we obtain precisely 3d $SU(N)$ SQCD on side A.

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SU duality in 3d

- On the dual side B all the quarks acquire real masses. The first N_f quarks get real masses $\hat{m}_1$ and the last real mass $\hat{m}_2$,

$$\hat{m}_1 = \frac{\hat{m}}{N_f - N + 1}\,, \qquad \hat{m}_2 = \frac{\hat{m}\,(N - N_f)}{N_f - N + 1}.$$

- In presence of the real masses theory B has a vacuum where $N_f - N$ σs acquire VEV $\hat{m}_1$ and one σ has VEV $\hat{m}_2$.

- Thus, the gauge group in this vacuum is broken to $SU(N_f - N) \times U(1)$.

- The duality after adding the real masses is thus

$$\begin{array}{ccc} SU(N)_{N_f}\ Q, \widetilde{Q}, & \leftrightarrow & SU(N_f - N)_{N_f} \times U(1),\ q,\ \widetilde{q},\ b,\ \widetilde{b},\ M,\ Y \\ W = 0 & & W = Mq\widetilde{q} + Yb\widetilde{b} + \widetilde{X}_+ + \widetilde{X}_- \,. \end{array}$$

- $Q^N \to q^{N_f - N}\, b, \quad \widetilde{Q}^N \to \widetilde{q}^{N_f - N}\, \widetilde{b}, \quad Q\widetilde{Q} \to M, \quad Y \to Y, \quad \ldots$

- A related duality can be obtained by assuming Aharony duality in 3d and gauging the $U(1)_J$ symmetry (Kapustin;ASRW;Park,Park)

More examples

- $4d$ $Usp(2N)_{N_f} \leftrightarrow Usp(2(N_f - N - 2))_{N_f}$ duality reduces to $3d$ in a similar way.
- Here, starting with $N_f + 1$ flavors and turning on real mass the vacuum at the origin of the Coulomb branch on side A maps to the origin of side B.
- The duality in $3d$ thus is: $Usp(2N)_{N_f} \leftrightarrow Usp(2(N_f - N - 1))_{N_f}$
- Reduction of $so(N)$ dualities is more involved.
- Putting $so(N)_{N_f} \leftrightarrow so(N_f - N + 4)_{N_f}$ duality on a circle not all of the compact Coulomb branches are lifted.
- The duality on the circle relates different points on the compact Coulomb branches on the two sides of the duality.
- In particular vacuum at the origin on side A maps to vacuum away of the origin on side B where the gauge group is broken, $SO(N_f - N + 4) \to SO(N_f - N + 2) \times SO(2)$.
- After analyzing the dynamics of the $SO(2)$ factor the duality in $3d$ is $SO(N)_{N_f} \leftrightarrow SO(N_f - N + 2)_{N_f}$.

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Partition functions

- Supersymmetric partition functions are a useful tool to indicate whether a putative duality is wrong or might be correct.
- One of the strongest checks of the 4d Seiberg dualities is the equality of the supersymmetric indices ($S^3 \times S^1$) (Dolan,Osborn; Spiridonov, Vartanov, ...)
- In 3d one can easily compute the 3d index ($S^2 \times S^1$) and the 3d partition function on S^3: these also have been shown to be equal for all the known dualities. (Benini,Closset,Cremonesi,Hwang,Kapustin,Kim,Krattenhaler, Park,Park,Spiridonov,Vartanov,Willett,Yaakov, ...)
- The 4d index can be directly related to the S^3 partition function of the theory 3d theory obtained by careful dimensional reduction. (Spiridonov,Vartanov;Gadde,Yan;Imamura)
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From 4d index to 3d partition functions on S^3

- One can think of the $4d$ index as a twisted partition function on $S^3 \times S^1$.
- Taking into account the twists coming from the fugacities this partition function can be thought of as a partition function on $S_b^3 \times \widetilde{S}^1$. (Imamura and Yokoyama)
- For a chiral field reducing on $\widetilde{S}^1$ one can write thus the $4d$ index as a product over S_b^3 partition functions of the KK modes,

$$\mathcal{I}_{(4d)}(p,\, q;\, u) \propto \prod_{n=-\infty}^{\infty} \mathcal{Z}_{(3d)}(\omega_1,\, \omega_2;\, m + \frac{n}{\tilde{r}})$$

- This product should be properly regularized and the 4d index appropriately normalized so that the above becomes an exact equality,

$$e^{\mathcal{I}_0}\, \Gamma(e^{2\pi i m \widetilde{r}}; e^{2\pi i \omega_1 \widetilde{r}},\, e^{2\pi i \omega_2 \widetilde{r}}) =$$
$$e^{-\Delta} \prod_{n=-\infty}^{\infty} e^{-\mathrm{sign}(n)\, \frac{\pi i}{2\omega_1\omega_2}\left((m+\frac{n}{\tilde{r}}-\omega)^2 - \frac{\omega_1^2+\omega_2^2}{12}\right)} \Gamma_h(m + \frac{n}{\tilde{r}}; \omega_1, \omega_2)\,.$$

(This equality is mathematically precisely the $SL(3,Z)$ property of elliptic Gamma functions.)

- Sending the radius $\widetilde{r}$ to zero only the zero mass KK mode survives and we get that the 4d index of a chiral reduces to the 3d S_b^3 partition function.

Partition function on S^3

- The 3d partition function on S^3 has the following form (here we have $U(N)$ gauge theory)

$$\mathcal{Z} = \frac{1}{N!}\int \prod_{\ell=1}^{N} \frac{d\sigma_\ell}{\sqrt{-\omega_1\omega_2}} e^{\frac{2\pi i\xi\sum_\ell \sigma_\ell}{\omega_1\omega_2}} e^{\frac{\pi i k\sum_\ell \sigma_\ell^2}{\omega_1\omega_2}} \prod_{i\neq j} \frac{1}{\Gamma_h(\sigma_i-\sigma_j;\omega_1,\omega_2)} \times$$
$$\prod_{i=1}^{N}\prod_{a=1}^{N_f} \Gamma_h(\omega R+\sigma_i+m_a+m_A;\omega_1,\omega_2)\,\Gamma_h(\omega R-\sigma_i+\tilde{m}_a+m_A;\omega_1,\omega_2)\,.$$

- The integral over σs is literally over the scalars in the vector multiplet. The parameters m_a, $\widetilde{m}_a$ and m_A are real masses for the global symmetries.

- The limit of taking large real masses might or might not commute with the integrals.

- One can argue that taking the limit of large real masses under the integrals while simultaneously shifting the values of σ corresponds to the different vacua of the theory.

$SU(N)$ example: before real masses

- The partition functions of theories with η superpotential are given by

$$
\begin{aligned}
\mathcal{Z}_A &= \frac{1}{N_c!}\int\prod_{j=1}^{N_c}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\delta\left(\sum_{j=1}^{N_c}\sigma_j\right)\frac{\prod_{j=1}^{N_c}\prod_{a=1}^{N_f}\Gamma_h(\sigma_j\pm m_a+\hat\mu_a\pm\frac{\beta}{N_c})}{\prod_{i<j}\Gamma_h(\pm(\sigma_i-\sigma_j))},\\
\mathcal{Z}_B &= \prod_{a,b}\Gamma_h(m_a+\hat\mu_a-m_b+\hat\mu_b)\,\frac{1}{(N_f-N_c)!}\times\\
&\int\prod_{\ell=1}^{N_f-N_c}\frac{d\sigma_\ell}{\sqrt{-\omega_1\omega_2}}\delta\left(\sum_{j=1}^{N_f-N_c}\sigma_j\right)\frac{\prod_{j=1}^{N_f-N_c}\prod_{a=1}^{N_f}\Gamma_h(\omega+\sigma_j\mp m_a-\hat\mu_a\pm\frac{\beta}{N_f-N_c})}{\prod_{i<j}\Gamma_h(\pm(\sigma_i-\sigma_j))}\,.
\end{aligned}
$$

- The constraint coming from the superpotential is

$$
\sum_{a=1}^{N_f}\hat\mu_a=\omega\,(N_f-N_c)\,.
$$

$SU(N)$ example: after real masses

- Taking now the limit of large real masses under the integrals carefully we obtain the following equality

$$\frac{1}{N_c!}\int\prod_{j=1}^{N_c}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\,\delta\left(\sum_{j=1}^{N_c}\sigma_j\right)\frac{\prod_{j=1}^{N_c}\prod_{a=1}^{N_f}\Gamma_h(\pm\sigma_j\pm m_a+\hat{\mu}_a\pm\frac{\beta}{N_c})}{\prod_{i<j}\Gamma_h(\pm(\sigma_i-\sigma_j))}=$$

$$\left(\prod_{a,b}^{N_f}\Gamma_h(m_a+\hat{\mu}_a-m_b+\hat{\mu}_b)\right)\Gamma_h(2\omega(N_f+1-N_c)-2\sum_{a=1}^{N_f}\hat{\mu}_a)\times$$

$$\frac{1}{(N_f-N_c)!}\int\prod_{j=1}^{N_f-N_c}\frac{d\sigma_j}{\sqrt{-\omega_1\omega_2}}\frac{\prod_{a=1}^{N_f}\Gamma_h(\omega\pm\sigma_j\mp m_a-\hat{\mu}_a\pm\frac{\beta}{N_f-N_c})}{\prod_{i<j}\Gamma_h(\pm(\sigma_i-\sigma_j))}\times$$

$$\Gamma_h(-\omega(N_f+1-N_c)\pm\sum_j\sigma_j+\sum_{a=1}^{N_f}\hat{\mu}_a)\,.$$

Summary

- Starting from any 4d dualities one can deduce a 3d duality
- From this 3d duality one can flow to many other ones using the 3d tool-kit.
- This way one can derive many known 3d dualities as well as many new ones. In particular explaining why 3d dualities are so similar to 4d ones.
- The fact that we find a consistent web of dualities in 3d can be seen as yet another check of the 4d dualities.
- The partition functions are a very useful tool to deduce, sometimes intricate, physics.

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