Gauge Theories and Dessins d'Enfants

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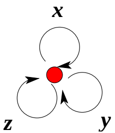
SUSY gauge theories with product gauge groups

- Brane World: D3-brane $\perp$ 6D space $\mathcal{M}$
  - world-volume: 3+1D with SUSY Yang-Mills and product gauge group; QUIVER THEORY
  - TRANSVERSE: local (affine, singular) Calabi-Yau 3-fold (cone over Sasaki-Einstein 5-manifold);
  - D3 probes the geometry of $\mathcal{M}$
- from Alg Geo perspective, AdS/CFT is dialogue between quiver gauge theory and affine CY variety:
  - $\mathcal{M} \longrightarrow$ Gauge Theory: Geometrical Engineering
  - Gauge Theory $\longrightarrow \mathcal{M}$: Forward Algorithm

Two most famous SCFTs in 4-D

- $\mathcal{N} = 4$ $U(N)$ Yang-Mills
  - 3 adjoint fields X, Y, Z with superpotential
    $W = \text{Tr}(X[Y,Z]) = \text{Tr}(XYZ - XZY)$
  - Original AdS/CFT: N D3-branes transverse to flat $\mathbb{R}^6$

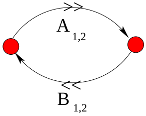


NODE = Gauge Group

ARROW = Bi−fundamental (Adj) Field

  - QUIVER = Finite graph (label = rk(gauge factor)) + relations from SUSY
    - Matter Content: Nodes + arrows
    - Relations (F-Terms): $D_i W = 0 \rightsquigarrow [X,Y] = [Y,Z] = [X,Z] = 0$
- # gauge factors $= N_g = 1$; # fields $= N_f = 3$; # terms in $W = N_w = 2$

- Klebanov-Witten $\mathcal{N}=1$ "conifold" Theory
    - $SU(N)\times SU(N)$ gauge theory with 4 bi-fundamental fields



QUIVER

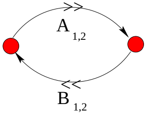
	$SU(N)$	$SU(N)$
$A_{i=1,2}$	$\square$	$\overline{\square}$
$B_{j=1,2}$	$\overline{\square}$	$\square$

$$W = \mathrm{Tr}(\epsilon_{il}\epsilon_{jk}A_iB_jA_lB_k)$$

    - string-theory realization: N D3-branes transverse to the conifold singularity: $\{uv=wz\}\subset\mathbb{C}^4$

- # gauge factors $= N_g = 2$; # fields $= N_f = 4$; # terms in $W = N_w = 2$
- $\boxed{\text{Observatio Curiosa: } N_g - N_f + N_w = 0}$
- true for almost all known cases in AdS_5/CFT_4

- Klebanov-Witten $\mathcal{N} = 1$ "conifold" Theory
  - $SU(N) \times SU(N)$ gauge theory with 4 bi-fundamental fields



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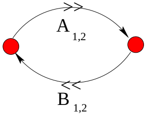

QUIVER

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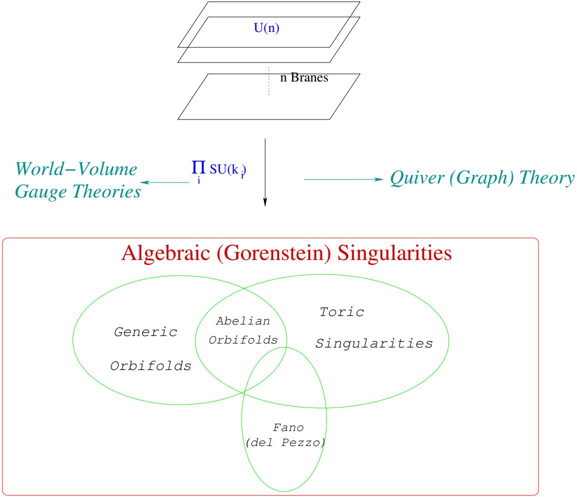
Brane probes: Paradigm

Stack of n parallel branes $\perp$ a Calabi-Yau (singularity) $\mathcal{M}$:



Orbifolds:

$\mathbb{C}^3/(\Gamma \subset SU(3))$

Toric:

e.g., conifold, $Y^{p,q}$, $L^{p,q}$ ...

Fano (del Pezzo):

e.g., $dP_{0,\ldots,8}$

Gauge Theory Moduli Space & CY Geometry

$$S = \int d^4x \, [\int d^4\theta \, \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \, \mathrm{Tr}\, \mathcal{W}_\alpha \mathcal{W}^\alpha + \int d^2\theta \, W(\Phi) + \text{h.c.} \right)]$$

W = superpotential $\qquad V(\phi_i, \bar{\phi}_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{4} (\sum_i q_i |\phi_i|^2)^2$

- VACUUM $\sim \boxed{V(\phi_i, \bar{\phi}_i) = 0} \Rightarrow \begin{cases} \frac{\partial W}{\partial \phi_i} = 0 & \text{F-TERMS} \\ \sum_i q_i |\phi_i|^2 = 0 & \text{D-TERMS} \end{cases}$
- **$\mathcal{M}$:= vacuum moduli space = space of solutions to F and D-flatness = some complex variety**
- If $\mathcal{M}$ CY3, can realize in string theory as D3-brane probing $\mathcal{M}$
- for N-branes, get $Sym^N \mathcal{M} = \mathcal{M}^N / \Sigma_N$

Hilbert Series: a Fundamental Generating Function

- $\mathcal{M}$ ideal in graded polynomial ring $\sim$ affine cone over (weighted) projective variety: $I = \bigoplus_k I_k \rightsquigarrow f(t) = g_1(t) = \sum\limits_n a_n t^n$ = Hilbert Series of $\mathcal{M} \sim$ generating function of Chiral ring
- Single to Multi-Trace: pure combinatorics Plethystic Exponential (Littlewood)

$$\boxed{g_\infty(t) = PE[f(t)] = \exp\left(\sum_{k=1}^{\infty} \frac{f(t^k)-f(0)}{k}\right) = \prod_{n=1}^{\infty} (1-t^n)^{-a_n}}$$

Bosonic oscillator partition function

- Inverse: Plethystic Logarithm (Analytic!)

$$\boxed{f(t) = PE^{-1}(g(t)) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t^k))}$$

$\mu(k)$ is the Möbius function $\mu(k) = \begin{cases} 0 & k \text{ has one or more repeated prime factors} \\ 1 & k = 1 \\ (-1)^n & k \text{ is a product of } n \text{ distinct primes} \end{cases}$

The Plethystic Programme

- Geometry $\leftarrow$ Gauge Theory $\rightarrow$ Combinatorics

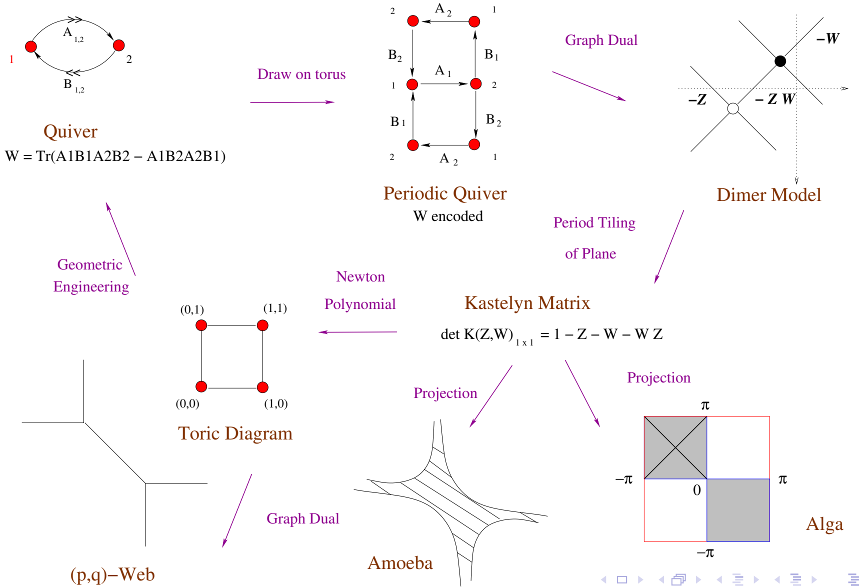
$$\boxed{\begin{array}{c} f_1(t) \\ \text{Syzygies} \end{array}} \xleftarrow{PE^{-1}} \boxed{\begin{array}{c} \text{Fundamental gen. func.} \\ \text{Hilbert Series} \\ f(t) = f_\infty(t) = g_1(t) \\ \text{Single–Trace} \end{array}} \xrightarrow{PE} \boxed{\begin{array}{c} g(t) = g_\infty(t) \\ \text{Multi–Trace} \end{array}}$$

- e.g. Conifold: $4t - t^2 \longleftarrow f(t) = \frac{1+t}{(1-t)^3} \longrightarrow \prod_{n=1}^{\infty} (1 - t^n)^{-(n+1)^2}$
- $\boxed{\textbf{Applicable to all gauge theories}}$, not just D-brane probes: Lagrangian $\Rightarrow$ F,D-Flat $\Rightarrow$ Vacuum Moduli Space $\mathcal{M} \Rightarrow f_1 = \text{Syzygy}(\mathcal{M}) \Rightarrow$ $f = PE[f_1] \Rightarrow g = PE^2[f_1]$

Gauge Theories with $\mathcal{M}$ Affine Toric CY3

- By far the largest class known and studied
- World-volume physics and Geometry of $\mathcal{M}$ $\sim$ Combinatorical Data $\sim$ (integer cones σ in $\mathbb{Z}^r$-lattice); (CY3 $\rightsquigarrow$ planar toric diag)
- Explicit Ricci-flat metric known for infinite families $Y^{p,q}$, L^{abc} (inc. conifold); [Candelas-de la Ossa, Cvectic, Gauntlett, Hanany, Pope, Sparks, Waldram ...]
- $N_g - N_f + N_w = 0$ is Euler relation for a torus!
  - N_W is even with each field appearing exactly twice with opposite sign
  - Toric (binomial) ideal: F-terms are of form "monomial = monomial"
  - associate black/white with +/- terms in W
- $\boxed{\text{Toric Quiver theory} = \text{Bi-partite periodic planar Graph}}$ brane-tiling or dimer model; Get a beautiful web of inter-connections
- *Cf. Reviews of Kennaway and of Yamazaki*

D-branes, Tilings and Dimers



Permutation Triple & Riemann Existence

- Label edges in dimer (SCFT fields) as $\{1, 2, \ldots, d\}$ and consider cycle notations in Σ_d (in the *same* orientation)

$$\text{Valency Data} \Rightarrow \begin{cases} \sigma_{Black} & := (i_1\ i_2\ \ldots i_k)_{B_1} \ldots (i_1\ i_2\ \ldots i_k)_{B_n} \ldots \\ \sigma_{White} & := (j_1\ j_2\ \ldots j_k)_{W_1} \ldots (i_1\ i_2\ \ldots i_k)_{W_n} \ldots \end{cases}$$

Permuation Triple: Define $\sigma_0 = \sigma_{Black}$, $\sigma_1 = \sigma_{White}$, $\sigma_0 \sigma_1 \sigma_\infty = \mathbb{I}_{\Sigma_d}$

recognize as $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\})$

- **Thm** [Riemann Existence]: Meromorphic functions (rational maps) on Riemann surface S are in 1:1 with permutation rep of π_1 of $\mathbb{P}^1$ minus ramification points.
- COR: rational maps from dimer on T^2 to $\mathbb{P}^1 \simeq$ permutation triples.

Grothendieck's Dessin d'Enfant

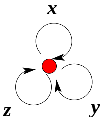
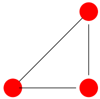
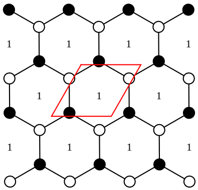
- Ramification data: $\left\{ \begin{array}{l} r_0(1), r_0(2), \ldots, r_0(B) \\ r_1(1), r_1(2), \ldots, r_1(W) \\ r_\infty(1), r_\infty(2), \ldots, r_\infty(I) \end{array} \right\}$ plus connectivity $\simeq$ dimer on torus $\simeq$ quiver gauge theory with toric $\mathcal{M}$ $\simeq$ permutation triple

  $\sigma_0 = (\ldots)_{r_0(1)}(\ldots)_{r_0(2)} \ldots (\ldots)r_0(B), \quad \sigma_1 = (\ldots)_{r_1(1)}(\ldots)_{r_1(2)} \ldots (\ldots)r_1(W)$

- Belyǐ Map: rational map $\beta : \Sigma \longrightarrow \mathbb{P}^1$ ramified only at $(0, 1, \infty)$
  - **Theorem [Belyǐ]:** β exists $\Leftrightarrow$ Σ can be defined over $\overline{\mathbb{Q}}$
  - (β, Σ) Belyǐ Pair
  - Dessin d'Enfants $= \beta^{-1}([0,1] \in \mathbb{P}^1)$ is a *bi-partite graph* on Σ: label $\beta^{-1}(0)$ black and $\beta^{-1}(1)$ white, then $\beta^{-1}(\infty)$ lives one per face
  - Grothendieck: !!!
- Typical existence theorem

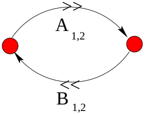
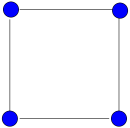
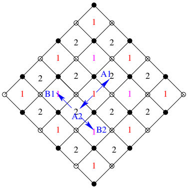
AdS/CFT and Dessins

- Matter content and interaction of SUSY gauge theory with toric moduli space is specified by Belyǐ pair
- Our most familiar example of $\mathcal{N}=4$ super-Yang-Mills:

Theory	Toric Diag	Belyǐ Pair	Dessin on T^2 (dimer)
 $W = \mathrm{Tr}(X[Y,Z])$	 $\mathcal{M} \simeq \mathbb{C}^3$	$y^2 = x^3 + 1$ $\beta(x,y) = \frac{y+1}{2}$ $\sigma_0 = \sigma_1 = \sigma_\infty$ $= (123)$	

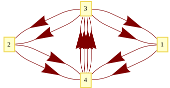
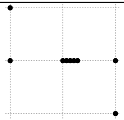
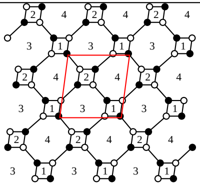
- Rmk: Absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts *faithfully* over the dessin, even on subsets like dessin on $\mathbb{P}^1$ or T^2... Relation to Langlands???

- Conifold

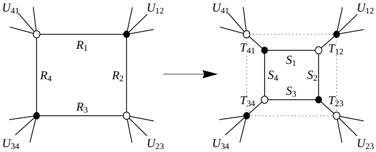
Theory	Toric Diag
 $W = \mathrm{Tr}(\epsilon_{il}\epsilon_{jk}A_iB_jA_lB_k)$	 $\mathcal{M} \simeq \{uv - wz = 0\} \subset \mathbb{C}^4$
Belyĭ Pair	**Dessin on T^2 (dimer)**
$y^2 = x(x-1)(x-\frac{1}{2})$ $\beta(x,y) = \frac{x^2}{2x-1}$ $\sigma_0 = (1234), \sigma_1 = (1234),$ $\sigma_\infty = (12)(34)$	

Plethora of Non-Trivial Examples

e.g., Cone over $F_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (zeroth Hirzebruch surface); *In Piam Memoriam ...*

Theory	Toric Diag
 $W = X^1_{14}X^4_{43}X^2_{31} + X^2_{14}X^2_{43}X^1_{31} + X^1_{24}X^1_{43}X^1_{32} + X^2_{24}X^3_{43}X^2_{32} - X^1_{43}X^1_{31}X^1_{14} - X^2_{43}X^2_{32}X^1_{24} - X^3_{43}X^2_{31}X^2_{14} - X^4_{43}X^1_{32}X^2_{24}$	 $\mathcal{M} \simeq$Hirzebruch 0
Belyĭ Pair	**Dessin on T^2 (dimer)**
$y^2 = x - x^3$ $\beta(x,y) = \frac{i\left(x^2 - \sqrt[3]{-1}\right)^3}{3\sqrt{3}x^2(x^2-1)}$	

- Can capture many interesting physical phenomena, e.g. Seiberg Duality



so-called "Urban Renewal"

- Klein-j of τ-parametre of (isoradial) dimer is Seiberg-invariant
- SUSY R-charges: a-maximization or by volume minimization of SE
  - R-charges and normalized volume of dual geometry are *algebraic numbers*
  - Degree of transcendence over $\mathbb{Q}$ is Seiberg-invariant
- Many open puzzles: e.g. matching τ of isoradial dimer with that of T^2 in mirror T^3-fibration and that of dessin

Grothendieck, 1984

This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my centre of interest in mathematics, which suddenly found itself strongly focussed. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact.

This is surely because of the very familiar, non-technical nature of the objects considered, of which any child's drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke.