Wall Crossing

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India & KIAS

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Plan

1. Introduction

2. Wall crossing formulæ

3. ‘Derivation’

Consider an $\mathcal{N}=2$ supersymmetric string theory or gauge theory in D=4 at a generic point in the moduli space

Gauge group $\mathbf{U(1)^N}$.

In a gauge theory N = number of U(1) vector multiplets

In supergravity N = number of vector multiplets +1

A given state is characterized by N electric charges $\mathbf{q_I}$ and N magnetic charges $\mathbf{p^I}$ ($\mathbf{1} \le \mathbf{I} \le \mathbf{N}$).

$\gamma \equiv (\mathbf{q_I}, \mathbf{p^I})$ takes values over some lattice Γ.

$$\langle\gamma,\gamma'\rangle \equiv (\mathbf{q_I}\mathbf{p}'^{\mathbf{I}} - \mathbf{p^I}\mathbf{q}'_{\mathbf{I}})$$

Let t denote a point in the moduli space.

The BPS mass formula takes the form

$$\mathbf{m}(\gamma,\mathbf{t}) = |\mathbf{Z}(\gamma,\mathbf{t})|, \qquad \mathbf{Z}(\gamma,\mathbf{t}) = \mathbf{q_I}\,\mathbf{f^I}(\mathbf{t}) - \mathbf{p^I}\,\mathbf{g_I}(\mathbf{t})$$

for some known complex functions $\mathbf{f^I}(\mathbf{t})$, $\mathbf{g_I}(\mathbf{t})$.

We denote by $\Omega(\gamma,\mathbf{t})$ the index of BPS states carrying charge γ at the point t:

$$\Omega(\gamma,\mathbf{t}) = \mathbf{Tr}'_\gamma((-\mathbf{1})^{\mathbf{F}}) = \mathbf{Tr}'_\gamma((-\mathbf{1})^{\mathbf{2J_3}})$$

Tr′: removes the trace over the fermion zero modes associated with broken supersymmetries.

$\Omega(\gamma,\mathbf{t})$ is independent of t except for jumps across walls of marginal stability.

Let $\gamma_1, \gamma_2 \in \Gamma$ and take a codimension 1 subspace of the moduli space on which

$$\arg \mathbf{Z}(\gamma_1, \mathbf{t}) = \arg \mathbf{Z}(\gamma_2, \mathbf{t})$$

On this subspace,

$$|\mathbf{Z}(\gamma_1 + \gamma_2, \mathbf{t})| = |\mathbf{Z}(\gamma_1, \mathbf{t})| + |\mathbf{Z}(\gamma_2, \mathbf{t})|$$

Thus a state of charge $(\gamma_1 + \gamma_2)$ is marginally unstable against decay into a pair of states carrying charges γ_1 and γ_2.

More generally a state carrying charge $(\mathbf{M}\gamma_1 + \mathbf{N}\gamma_2)$ becomes marginally unstable.

$\Rightarrow \Omega(\mathbf{M}\gamma_1 + \mathbf{N}\gamma_2, \mathbf{t})$ **could jump as t crosses this wall.**

From now on we shall drop the argument t from Ω.

Goal: Compute the change in $\Omega(\mathbf{M}\gamma_1 + \mathbf{N}\gamma_2)$ across the wall.

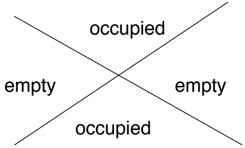
Note: Only states carrying charges in the 2D sublattice spanned by γ_1 and γ_2 are relevant near this wall.

Assumption: After taking appropriate linear combinations, it is possible to choose the vectors γ_1 and γ_2 such that:

Near the wall, BPS states carrying charges $(M\gamma_1 + N\gamma_2)$ exist only for $M, N \geq 0$ or $M, N \leq 0$.

Andriyash, Denef, Jafferis, Moore

$\Rightarrow$ **Define** $\tilde{\Gamma} \equiv \{m\gamma_1 + n\gamma_2 : m, n \geq 0, (m, n) \neq (0, 0)\}$



Consider two chambers in the moduli space separated by the wall of marginal stability.

$$\mathbf{c}^+ : \quad \gamma_{12}\, \mathbf{Im}\left(\mathbf{Z}^*_{\gamma_1}\, \mathbf{Z}_{\gamma_2}\right) > 0, \qquad \gamma_{12} \equiv \langle \gamma_1, \gamma_2 \rangle$$

$$\mathbf{c}^- : \quad \gamma_{12}\, \mathbf{Im}\left(\mathbf{Z}^*_{\gamma_1}\, \mathbf{Z}_{\gamma_2}\right) < 0$$

$\Omega^\pm(\gamma)$**: index in these two chambers**

We want to calculate Ω^- in terms of Ω^+ or vice versa.

We shall introduce a refined 'index':

$$\mathbf{I}(\gamma;\mathbf{y}) \equiv \mathbf{Tr}'_\gamma(-\mathbf{y})^{2\mathbf{J}_3}$$

This index is not protected in general but we can get the protected index $\Omega(\gamma)$ by setting y=1 at the end.

We also define 'rational invariants'

$$\widetilde{\mathbf{I}}(\gamma;\mathbf{y}) \equiv \sum_{\mathbf{m}|\gamma} \frac{1}{\mathbf{m}} \frac{\mathbf{y}-\mathbf{y}^{-1}}{\mathbf{y}^{\mathbf{m}}-\mathbf{y}^{-\mathbf{m}}}\, \mathbf{I}\left(\frac{\gamma}{\mathbf{m}};\mathbf{y}^{\mathbf{m}}\right)$$

I can be computed from $\widetilde{\mathbf{I}}$ and vice versa but wall crossing formulæ are easy to state in terms of $\widetilde{\mathbf{I}}$.

$\widetilde{\mathbf{I}}^{\pm}$ $\equiv$ the rational invariants in chambers $\mathbf{c}^{\pm}$

Kontsevich-Soibelman (KS) formula

Introduce an abstract algebra with elements e_γ labelled by the charge vector γ:

$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle\gamma,\gamma'\rangle} \frac{y^{\langle\gamma,\gamma'\rangle} - y^{-\langle\gamma,\gamma'\rangle}}{y - y^{-1}} \, e_{\gamma+\gamma'}$$

Define

$$V_\gamma^\pm = \exp\left[\sum_{n=1}^{\infty} \widetilde{I}^\pm(n\gamma; y)\, e_{n\gamma}\right]$$

Then KS wall crossing formula is:

$$\prod_{\substack{M\ge 0, N\ge 0, \gcd(M,N)=1 \\ M/N\uparrow}} V^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\ge 0, N\ge 0, \gcd(M,N)=1 \\ M/N\downarrow}} V^-_{M\gamma_1+N\gamma_2}$$

$$V_\gamma^\pm = \exp\left[\sum_{n=1}^{\infty} \widetilde{I}^\pm(n\gamma; y)\, e_{n\gamma}\right]$$

$$\prod_{\substack{M\geq 0, N\geq 0, \gcd(M,N)=1 \\ M/N\uparrow}} V^+_{M\gamma_1+N\gamma_2} = \prod_{\substack{M\geq 0, N\geq 0, \gcd(M,N)=1 \\ M/N\downarrow}} V^-_{M\gamma_1+N\gamma_2}$$

$M/N\uparrow$: the products are ordered from left to right in the order of increasing M/N, beginning with (M,N)=(0,1) at the extreme left.

$M/N\downarrow$: order is reversed

Using B-C-H formula to rearrange the product of the left hand side we can compute $\widetilde{I}^-$ in terms of $\widetilde{I}^+$.

An alternative wall crossing formula (MPS):

Manschot, Pioline, A.S.

$$\widetilde{\mathbf{I}}^{-}(\gamma;\mathbf{y}) = \widetilde{\mathbf{I}}^{+}(\gamma;\mathbf{y}) + \sum_{\mathbf{n}=2}^{\infty}\frac{\mathbf{1}}{\mathbf{n}!}\sum_{\{\alpha_{\mathbf{i}}\}}\mathbf{g}(\alpha_{\mathbf{1}},\cdots\alpha_{\mathbf{n}};\mathbf{y})\widetilde{\mathbf{I}}^{+}(\alpha_{\mathbf{1}};\mathbf{y})\cdots\widetilde{\mathbf{I}}^{+}(\alpha_{\mathbf{n}};\mathbf{y})$$

$\sum_{\{\alpha_{\mathbf{i}}\}}$: **sum over all ordered set of charge vectors** $\alpha_{\mathbf{i}} = \mathbf{m_i}\gamma_{\mathbf{1}} + \mathbf{n_i}\gamma_{\mathbf{2}}$ **with** $\mathbf{m_i} \geq \mathbf{0}$, $\mathbf{n_i} \geq \mathbf{0}$ **and satisfying**

$$\sum_{\mathbf{i}=1}^{\mathbf{n}}\alpha_{\mathbf{i}} = \gamma$$

$\mathbf{g}(\alpha_{\mathbf{1}},\cdots\alpha_{\mathbf{n}};\mathbf{y})$: **a function to be specified shortly.**

$g(\alpha_1, \cdots \alpha_n; y)$ is independent of the ordering of the α_i's, and hence can be specified for any convenient choice of the order.

We shall order the α_i's such that

$$\alpha_{ij} \equiv \langle \alpha_i, \alpha_j \rangle > 0 \quad \text{for } i < j$$

Then

$$\begin{aligned} g(\alpha_1, \cdots, \alpha_n; y) &= (-1)^{\sum_{i<j} \alpha_{ij} + n - 1} (y - y^{-1})^{1-n} \\ &\quad \sum_\sigma N(\sigma)\, y^{\sum_{i<j} \alpha_{\sigma(i)\sigma(j)}} \end{aligned}$$

$\sum_\sigma$: sum over all permutations σ of $1, 2, \cdots n$.

$N(\sigma)$: will be specified shortly

$$
\begin{aligned}
\mathbf{N}(\sigma) &= (-1)^{\sum_{k=1}^{n-1} H(\sigma(k)-\sigma(k+1))} \\
&\prod_{\substack{k=1 \\ \sigma(k+1)<\sigma(k)}}^{n} H\left(\left\langle \alpha_1 + \cdots \alpha_n, \sum_{i=1}^{k} \alpha_{\sigma(i)} \right\rangle\right) \\
&\prod_{\substack{k=1 \\ \sigma(k+1)>\sigma(k)}}^{n} H\left(\left\langle \sum_{i=1}^{k} \alpha_{\sigma(i)}, \alpha_1 + \cdots \alpha_n \right\rangle\right)
\end{aligned}
$$

$$
H(x) \equiv \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{cases} .
$$

Note: H(x) is ill-defined at x=0.

Prescription for computing g$(\alpha_1, \cdots \alpha_n)$:

1. Deform all the α_i's by small random linear combinations of γ_1, γ_2 such that all the arguments appearing inside H are non-zero.

2. Compute N(σ) and hence g.

3. Take the deformed α_i's back to the original α_i's in the final expression for g.

$\Rightarrow$ MPS wall crossing formula.

Mozgovoy and Reineke has now given a slightly different prescription for computing g which does not require this deformation.

The equivalence of the MPS and KS wall crossing formula have now been proved.

A.S.

In the rest of this talk I shall outline the ‘derivation’ of the MPS wall crossing formula.

Manschot, Pioline, A.S.

Kim, Park, Wang, Yi

Supergravity picture: Denef; Denef, Moore

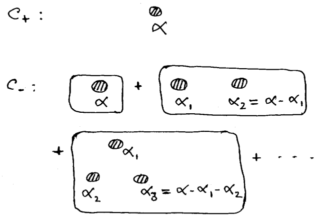
BPS states: Single centered black holes or multi-centered bound states of single centered black holes.

In $\mathbf{c}^+$ the only configurations which contribute to $\mathbf{I}^+(\gamma; \mathbf{y})$ are single units (<u>molecules</u>) which remain immortal across the wall of marginal stability.

A molecule can contain a single BH of charge $\gamma \in \tilde{\Gamma}$ or bound states of multiple BH, each with charge lying outside $\tilde{\Gamma}$, but total charge $\gamma \in \tilde{\Gamma}$.

As we approach the wall the size of the molecule remains finite.

In c^- the index $I^-(\gamma; y)$ gets contribution from single molecules and also bound states of these molecules.



As we approach the wall from c^- the intermolecular separations go to ∞ and only single molecule states remain on the other side.

$\Rightarrow$ I^+ describes the index for single molecules.

I^- describes the index for single molecules + molecular bound states.

If the α_i's are all distinct, then

$$\mathbf{I}^-(\alpha;\mathbf{y}) = \mathbf{I}^+(\alpha;\mathbf{y}) + \sum_{n\ge 2}\frac{1}{n!}\sum_{\substack{\{\alpha_1,\cdots\alpha_n\}\\ \alpha_1+\cdots\alpha_n=\alpha}} \mathbf{g}(\alpha_1,\cdots\alpha_n;\mathbf{y})\,\mathbf{I}^+(\alpha_1;\mathbf{y})\cdots\mathbf{I}^+(\alpha_n;\mathbf{y})$$

$\mathbf{g}(\alpha_1,\cdots\alpha_n;\mathbf{y})$: Index of supersymmetric bound states of n distinguishable centers with charges $\alpha_1,\cdots\alpha_n$, ignoring the internal contribution to the index from each center.

If some α_i's are identical, we need to take into account the effect of symmetrization for identical particles.

Example: Two identical bosonic centers, each with degeneracy Ω, will produce $\Omega(\Omega+1)/2$ states.

Results:

1. For all α_i's distinct, quantization of multi-centered black hole quantum mechanics leads to the formula for g given before.

2. We find that the effect of having identical particles is to replace $I^{\pm}$ by $\widetilde{I}^{\pm}$ in the formula.

$\Rightarrow$ a physical derivation of the MPS wall crossing formula.