Entropy Production in Continuous Stochastic Dynamics with Odd-Parity Variables

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The 6th KIAS Conference on Statistical Physics, KIAS, Seoul
8-11 July, 2014

with

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Integral Fluctuation Theorem (IFT)

IFT for

\[ \mathcal{R}[\mathbf{q}(t)] = \ln \frac{\mathcal{P}[\mathbf{q}(t)]}{\hat{\mathcal{P}}[\hat{\mathbf{q}}(t)]]} \]

- \( \mathcal{P}[\mathbf{q}(t)]: \) forward path probability density for a stochastic path \( \mathbf{q}(t) \) for \( 0 \leq t \leq \tau \)
- \( \hat{\mathcal{P}}[\hat{\mathbf{q}}(t)]: \) path probability for the transformed path \( \hat{\mathbf{q}}(t) \) with a specified time dependence.
- \( \mathbf{q}(t) \to \hat{\mathbf{q}}(t) \) has the Jacobian of unity.
- \( \langle \exp(-\mathcal{R}[\mathbf{q}]) \rangle = 1 \), follows from the normalization of \( \hat{\mathcal{P}} \).
- \( \langle \mathcal{R}[\mathbf{q}] \rangle \geq 0 \) from Jensen’s inequality.
Integral Fluctuation Theorem (IFT)

- IFT for

\[ \mathcal{R}[q(t)] = \ln \frac{\mathcal{P}[q(t)]}{\hat{\mathcal{P}}[\hat{q}(t)]} \]

- \( \mathcal{P}[q(t)]: \) forward path probability density for a stochastic path \( q(t) \) for \( 0 \leq t \leq \tau \)
- \( \hat{\mathcal{P}}[\hat{q}(t)]: \) path probability for the transformed path \( \hat{q}(t) \) with a specified time dependence.
- \( q(t) \to \hat{q}(t) \) has the Jacobian of unity.
- \( \langle \exp(-\mathcal{R}[q]) \rangle = 1 \), follows from the normalization of \( \hat{\mathcal{P}} \).
- \( \langle \mathcal{R}[q] \rangle \geq 0 \) from Jensen’s inequality

- Total entropy production:
  - \( \hat{q}(t) \to \epsilon q(\tau - t) \) [\( \epsilon = \pm 1 \) for even (odd) variables like position (momentum)]
  - \( \hat{\mathcal{P}}: \) reverse protocol
  - \( \mathcal{R} \to \Delta S_{\text{tot}} \)
\[ \Delta S_{\text{tot}} = \Delta S_{\text{excess}} + \Delta S_{\text{housekeeping}} \]

- \( \Delta S_{\text{excess}} \)
  - Hatano and Sasa (2001)
  - produced during transitions between stationary states
  - satisfies IFT on its own
  - transient component \( \sim \) nonadiabatic EP (Esposito and Van den Broeck (2010))
Steady State Thermodynamics

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  - Speck and Seifert (2005)
  - necessary to maintain noneq. steady state
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  - satisfies IFT on its own
  - adiabatic component

- \( \hat{P} \leftarrow \text{Dual (Adjoint) Dynamics} \)
In the presence of **odd** parity variables

- **Spinney and Ford (2012):**
  - $\Delta S_{\text{excess}}$ can be separated out & shown to satisfy IFT
  - $\Delta S_{\text{housekeeping}}$ does **not** satisfy IFT
  - $\Delta S_{\text{housekeeping}} = \Delta S_2 + \Delta S_3$
    - $\Delta S_2$ satisfies IFT; nontransient
    - $\Delta S_3$ does **not** satisfy IFT; parity asymmetry of SSD; transient
In the presence of odd parity variables

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    - $\Delta S_3$ does not satisfy IFT; parity asymmetry of SSD; transient

- Lee, Kwon and Park (2013):
  - Master eq. for discrete variables
  - $\Delta S_{\text{housekeeping}} = \Delta S_{bDB} + \Delta S_{as}$
    - $\Delta S_{bDB}$ satisfies IFT; measures directly DB breakage; nontransient
    - $\Delta S_{as}$ does not satisfy IFT; parity asymmetry of SSD; nontransient
Our Work

- Apply the splitting scheme of LKP (2013) to continuous variable cases
- Our findings:
  - $\Delta S_{bDB}$ for continuous variables is ill-defined & needs regularization
  - Splitting of $\Delta S_{\text{housekeeping}}$ is not unique
  - Can be done in many ways (described by a parameter $\sigma$)
  - Spinney & Ford $\leftrightarrow \sigma = 0$
Outline

1. Review: EPs in overdamped case (Even variables only)
2. Dual Dynamics
3. In the Presence of Odd Variables
   - Detailed Balance
   - Generalized Dual Dynamics
4. Summary
Overdamped Dynamics / Even Only Case

\[ \dot{x} = f(x, \lambda_t) + \xi(t), \quad \text{where} \quad \langle \xi(t)\xi(t') \rangle = 2D\delta(t - t') \]

- **Fokker-Planck Eq.**

\[ \partial_t \rho(x, t) = -\partial_x [f(x) - D\partial_x] \rho(x, t) = -\partial_x j(x) \]

with \( j(x) = f(x) \rho(x) - D\partial_x \rho(x) \).

- **Stationary State Dist. (SSD)** \([\rho_s(x) \equiv \exp[-\phi(x)]]\)

  ➤ **Current**

\[ j_s(x) = [f(x) + D\partial_x \phi(x)] \rho_s(x), \quad \partial_x j_s(x) = 0 \]

  ➤ **Mean Velocity**

\[ \nu_s(x) \equiv \frac{j_s(x)}{\rho_s(x)} = f(x) + D\partial_x \phi(x) \]
Overdamped / Entropy Productions (EP)

- Total EP: \( \hat{x}(t) = x(\tau - t), \hat{P} \)
  \[
  \Delta S_{\text{tot}} = \Delta S_{\text{env}} + \Delta S_{\text{sys}}
  \]
- System EP:
  \[
  \Delta S_{\text{sys}} = -\ln \rho(x(\tau), \tau) + \ln \rho(x(0), 0)
  \]
- Environment EP for time interval \( dt \):
  \[
  d \Delta S_{\text{env}} = \ln \frac{\Gamma[x', t + dt|x, t]}{\Gamma[x, t + dt|x', t]}
  = \frac{dt}{D} \dot{x}f(x)
  = -\frac{dt}{D} \left[ -\dot{x} + \xi \right]
  = -\frac{dt}{D} \dot{Q}
  \]
- Conditional probability: Onsager-Machlup with Stratonovich convention
  \[
  \Gamma[x', t + dt|x, t] = \frac{1}{\sqrt{4\pi D dt}} \exp \left[ -\frac{dt}{4D} \left\{ \dot{x} - f(x) \right\}^2 - \frac{dt}{2} \partial_x f(x) \right],
  \]
  where \( \dot{x} = (x' - x)/(dt) \).
Dual process

\[ \Gamma^*[x', t + dt|x, t] \equiv \Gamma[x, t + dt|x', t] \frac{\rho^S(x')}{\rho^S(x)}, \]

**Detailed Balance** holds if \( \Gamma^*[x', t + dt|x, t] = \Gamma[x', t + dt|x, t] \).

\[ \Gamma^*[x', t + dt|x, t] = \exp \left[ -\frac{dt}{4D} \left\{ \frac{d}{dt} + f(x) + 2D \partial_x \phi(x) \right\}^2 \right. \]

\[ \left. + \frac{dt}{2} \partial_x (f(x) + 2D \partial_x \phi(x)) \right], \]

\[ f \rightarrow -f - 2D \partial_x \phi \text{ and } j_S \rightarrow -j_S. \]
Overdamped / Dual Dynamics / Housekeeping EP

- Dual process

\[ \Gamma^*[x', t + dt|x, t] \equiv \Gamma[x, t + dt|x', t] \frac{\rho^s(x')}{\rho^s(x)} , \]

- Detailed Balance holds if \( \Gamma^*[x', t + dt|x, t] = \Gamma[x', t + dt|x, t] \).

\[ \Gamma^*[x', t + dt|x, t] = \frac{1}{\sqrt{4\pi D dt}} \exp \left[ - \frac{dt}{4D} \left\{ \dot{x} + f(x) + 2D \partial_x \phi(x) \right\}^2 
+ \frac{dt}{2} \partial_x (f(x) + 2D \partial_x \phi(x)) \right] \]

- \( f \to -f - 2D \partial_x \phi \) and \( j_s \to -j_s \)

- Housekeeping EP:

\[ d\Delta S_{\text{housekeeping}} \equiv \ln \frac{\Gamma[x', t + dt|x, t]}{\Gamma^*[x', t + dt|x, t]} \]

\[ = \frac{dt}{D} \dot{x} \nu_s(x) \]

(Speck and Seifert (2005))
Overdamped / Dual Dynamics / Excess EP

- **Dual + Time Reversal**
  
  $$
  d\Delta S_1 \equiv \ln \frac{\Gamma[x', t + dt|x, t]}{\Gamma^*[x, t + dt|x', t]} = -(dt)\dot{x}\partial_x\phi(x)
  $$

- **Excess EP:**
  
  $$
  \Delta S_{\text{excess}} = \Delta S_{\text{sys}} + \int d\Delta S_1.
  $$

- **If the initial and final distributions are also given by \( \rho_s \), then**
  
  $$
  \Delta S_{\text{excess}} = \int_0^\tau dt \; \lambda(t) \frac{\partial \phi(x; \lambda)}{\partial \lambda},
  $$

  (Hatano and Sasa (2001))
Overdamped / Dual Dynamics / Excess EP

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(Hatano and Sasa (2001))

- \( d\Delta S_1 + d\Delta S_{hk} = d\Delta S_{\text{env}} \rightarrow \Delta S_{\text{tot}} = \Delta S_{\text{exc}} + \Delta S_{hk} \)
Overdamped / Average EP Rate

Using

\[ \langle A(x')B(x) \rangle = \int dx' \int dx A(x') \Gamma[x', t + dt|x, t] B(x) \rho(x). \]

one can show that

\[ \frac{d}{dt} \langle \Delta S_{\text{tot}} \rangle = \frac{1}{D} \int dx \frac{|j(x)|^2}{\rho(x)} \geq 0 \]

\[ \frac{d}{dt} \langle \Delta S_{\text{housekeeping}} \rangle = \frac{1}{D} \int dx \left( \frac{j_s(x)}{\rho_s(x)} \right)^2 \rho(x) \geq 0 \]

\[ \frac{d}{dt} \langle \Delta S_{\text{excess}} \rangle = \frac{1}{D} \int dx \left( \frac{j(x)}{\rho(x)} - \frac{j_s(x)}{\rho_s(x)} \right)^2 \rho(x) \geq 0 \]

\[ = -\int dx (\partial_t \rho) \ln \frac{\rho(x)}{\rho_s(x)} \]
Odd-parity Variables
Odd Variables

\[ q = (x, p), \quad \epsilon q = (x, -p) \]

\[
\dot{x} = \frac{p}{m}, \\
\dot{p} = -G \frac{p}{m} + f(q; \lambda) + \xi(t),
\]

where \( G = \{G_{ij}\} \), \((Gp)_i = \sum_j G_{ij} p_j\), and

\[
\langle \xi(t) \rangle = 0, \quad \langle \xi_i(t)\xi_j(t') \rangle = 2D_{ij} \delta(t - t')
\]

Reversible and Irreversible Forces: \( f = f^{\text{rev}}(q) + f^{\text{ir}}(q) \)

\[
f^{\text{rev}}(q) = \frac{1}{2}(f(q) + f(\epsilon q)), \quad f^{\text{ir}}(q) = \frac{1}{2}(f(q) - f(\epsilon q)),
\]
\[ \Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{env}}, \]

where \( \Delta S_{\text{sys}} = -\ln \rho(q(\tau), \tau) + \ln \rho(q(0), 0) \)

- **Environment EP:**
  
  \[ \Delta S_{\text{env}} = \ln \frac{\Gamma[q(\tau)|q(0)]}{\Gamma^R[\epsilon q(0)|\epsilon q(\tau)]}, \quad d\Delta S_{\text{env}} = \ln \frac{\Gamma[q', t + dt|q, t]}{\Gamma[\epsilon q', t + dt|\epsilon q', t]}. \]
\[ \Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{env}}, \]

where \( \Delta S_{\text{sys}} = -\ln \rho(q(\tau), \tau) + \ln \rho(q(0), 0) \)

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  \Delta S_{\text{env}} = \ln \frac{\Gamma[q(\tau)|q(0)]}{\Gamma^R[\epsilon q(0)|\epsilon q(\tau)]}, \quad d\Delta S_{\text{env}} = \ln \frac{\Gamma[q', t + dt|q, t]}{\Gamma[\epsilon q, t + dt|\epsilon q', t]}.
  \]

- **Onsager-Machlup:**

  \[
  \Gamma[q', t + dt|q, t] = \frac{\delta(x' - x - dt(p/m))}{(4\pi dt)^{d/2} |\det(D)|^{1/2}}
  \]
  \[
  \times \exp \left[ -\frac{dt}{4} \left\{ \dot{p} + G \frac{p}{m} - f(q) \right\} \right. \]
  \[
  \left. -\frac{dt}{2} \partial_p \left\{ -G \frac{p}{m} + f(q) \right\} \right].
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\[ \Delta S_{\text{tot}} = \Delta S_{\text{sys}} + \Delta S_{\text{env}}, \]

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  \]

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  \times \exp \left[ -\frac{dt}{4} \left\{ \dot{p} + G \frac{p}{m} - f(q) \right\} D^{-1} \left\{ \dot{p} + G \frac{p}{m} - f(q) \right\} \right. \\
  \left. -\frac{dt}{2} \partial_p \left\{ - G \frac{p}{m} + f(q) \right\} \right],
  \]

- \[ d\Delta S_{\text{env}} = dt \left( -G \frac{p}{m} + f^{\text{ir}}(q) \right) D^{-1} \left( \dot{p} - f^{\text{rev}}(q) \right) - dt \partial_p f^{\text{rev}}(q). \]
Fokker-Planck (Kramer’s) Eq. \( \partial_t \rho(q, t) = -\partial_x j_x - \partial_p j_p \)

\[
\partial_t \rho(q, t) = H_K \rho(q, t)
\equiv - \left[ \partial_x \frac{p}{m} + \partial_p \left( -G \frac{p}{m} + f(q) - D \partial_p \right) \right] \rho(q, t).
\]

Currents \( j = j^{\text{rev}} + j^{\text{ir}} \).

\[
\begin{align*}
  j_x^{\text{rev}} &= \left( \frac{p}{m} \right) \rho(q), & j_x^{\text{ir}} &= 0, \\
  j_p^{\text{rev}} &= f^{\text{rev}}(q) \rho(q), & j_p^{\text{ir}} &= \left( -G \frac{p}{m} + f^{\text{ir}}(q) - D \partial_p \right) \rho(q).
\end{align*}
\]

Average rate of total EP:

\[
\frac{d}{dt} \langle \Delta S_{\text{tot}} \rangle = \int dq \frac{j_p^{\text{ir}}(q) D^{-1} j_p^{\text{ir}}(q)}{\rho(q)} \geq 0
\]

Note: D is positive-definite
**Excess EP / Odd Variables**

- **Dual Process**: (same as even only case)
  \[ \Gamma^*[\mathbf{q}', t + dt | \mathbf{q}, t] \equiv \Gamma[\mathbf{q}, t + dt | \mathbf{q}', t] \frac{\rho^s(\mathbf{q}')}{\rho^s(\mathbf{q})}, \]

- **dual + time reversal (without parity change)**
  \[ d\Delta S_1 \equiv \ln \frac{\Gamma[\mathbf{q}', t + dt | \mathbf{q}, t]}{\Gamma^*[\mathbf{q}, t + dt | \mathbf{q}', t]} = -dt \left( \dot{\mathbf{p}} \partial_p \phi(\mathbf{q}) + \dot{x} \partial_x \phi(\mathbf{q}) \right) \]

- **Excess EP**:
  \[ \Delta S_{\text{excess}} = \Delta S_{\text{sys}} + \int d\Delta S_1. \]

  - If the initial and final distributions are also given by \( \rho_s \), then
    \[ \Delta S_{\text{excess}} = \int^T_0 dt \; \dot{\lambda}(t) \partial_\lambda \phi \]

- **Average EP rate**:
  \[
  \frac{d}{dt} \langle \Delta S_{\text{excess}} \rangle = \int d\mathbf{q} \left\{ \frac{j_p(\mathbf{q})}{\rho(\mathbf{q})} - \frac{j_p^s(\mathbf{q})}{\rho^s(\mathbf{q})} \right\} D^{-1} \left\{ \frac{j_p(\mathbf{q})}{\rho(\mathbf{q})} - \frac{j_p^s(\mathbf{q})}{\rho^s(\mathbf{q})} \right\} \rho(\mathbf{q}),
  \]

  \[ = -\int d\mathbf{q} \left( \partial_t \rho(\mathbf{q}) \right) \ln \frac{\rho(\mathbf{q})}{\rho^s(\mathbf{q})}, \]
Detailed Balance (DB)

\[ \omega[q, q'] \rho_s(q') = \omega[\epsilon q', \epsilon q] \rho_s(\epsilon q), \]

with transition rate \( \omega \):

\[ \partial_t \rho(q) = \int dq' \omega[q, q'] \rho(q'), \quad \omega[q, q'] = H_K(q) \delta(q - q') \]

or

\[ \Gamma[q', t + dt|q, t] = \delta(q' - q) + (dt)\omega[q', q]. \]
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\[ \omega[q, q'] \rho_s(q') = \omega[\epsilon q', \epsilon q] \rho_s(\epsilon q), \]

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\[ \Gamma[q', t + dt|q, t] = \delta(q' - q) + (dt)\omega[q', q]. \]

**Consequences of DB:** (c.f. Risken)

\[ j^{s,ir}_p = 0, \quad \partial_x j^{s,rev}_x + \partial_p j^{s,rev}_p = 0, \quad \phi(q) = \phi(\epsilon q) \]

**Steady state currents** \( j = j^{s,rev} + j^{s,ir} \):

\[ j^{s,rev}_x = (p/m) \rho^s(q), \quad j^{s,ir}_x = 0, \]

and

\[ j^{s,rev}_p = f^{rev}(q) \rho^s(q), \quad j^{s,ir}_p = \left( -G \frac{p}{m} + f^{ir}(q) + D(\partial_p \phi(q)) \right) \rho^s(q). \]
Master Eq. $\dot{\rho}_x = \sum_y \omega_{x,y} \rho_y$

- DB Condition: $\omega_{x,y} \rho^s_y = \omega_{\epsilon y, \epsilon x} \rho^s_{\epsilon x}$ for $x \neq y$.
- $\omega_{x,x} \rho^s_x = \omega_{\epsilon x, \epsilon x} \rho^s_{\epsilon x}$ follows from $\omega_{x,x} = \sum_{y \neq x} \omega_{y,x}$
- $\rho^s_x = \rho^s_{\epsilon x}$ does not follow from DB
Master Eq. $\dot{\rho}_x = \sum_y \omega_{x,y} \rho_y$

- DB Condition: $\omega_{x,y} \rho_y^s = \omega_{\epsilon,y,\epsilon} \rho_{\epsilon_x}^s$ for $x \neq y$.
- $\omega_{x,x} \rho_x^s = \omega_{\epsilon,x,\epsilon} \rho_{\epsilon_x}^s$ follows from $\omega_{x,x} = \sum_{y \neq x} \omega_{y,x}$
- $\rho_{x}^s = \rho_{\epsilon_x}^s$ does not follow from DB

Dual process (Lee, Kwon & Park (2013)):

$$\omega^\dagger_{x,y} \equiv \omega_{\epsilon,y,\epsilon} \frac{\rho_{\epsilon_x}^s}{\rho_y^s}$$

$$\Gamma^\dagger_{x,y} \equiv \delta_{x,y} + (dt)\omega^\dagger_{x,y} = \Gamma_{\epsilon,y,\epsilon} \frac{\rho_{\epsilon_x}^s}{\rho_y^s} + \left[1 - \frac{\rho_{\epsilon_x}^s}{\rho_y^s}\right] \delta_{x,y}$$

- DB $\leftrightarrow \Gamma^\dagger_{x,y} = \Gamma_{x,y}$
- EP associated with breakage of DB:

$$\ln \frac{\Gamma_{x,y}}{\Gamma^\dagger_{x,y}}$$
Define $\uparrow$-process as

$$\Gamma^{\uparrow}[\mathbf{q}', t + dt | \mathbf{q}, t] \equiv \Gamma[\epsilon \mathbf{q}, t + dt | \epsilon \mathbf{q}', t] \frac{\rho_s(\epsilon \mathbf{q}')}{\rho_s(\mathbf{q})} + \delta(\mathbf{q} - \mathbf{q}') \left( 1 - \frac{\rho_s(\epsilon \mathbf{q})}{\rho_s(\mathbf{q})} \right)$$

And try to calculate

$$d \Delta S_{bDB} \equiv \ln \frac{\Gamma[\mathbf{q}', t + dt | \mathbf{q}, t]}{\Gamma^{\uparrow}[\mathbf{q}', t + dt | \mathbf{q}, t]}.$$
Result:

\[
\Gamma^\dagger[q', t + dt|q, t] = \frac{\delta(x' - x - dt(p/m)e^{\phi_A(q)})}{(4\pi e^{\phi_A} dt)^{d/2}|\text{det}(D)|^{1/2}} \exp \left[ -\frac{dt}{4e^{\phi_A}} h(q) D^{-1} h(q) \right. \\
\left. - \frac{dt}{2} \partial_p \cdot \left\{ e^{\phi_A} \left( G \frac{p}{m} + f(\epsilon q) - 2D\partial_p \phi(\epsilon q) - \frac{1}{2} D\partial_p \phi_A(q) \right) \right\} \right] - \frac{dt}{2} \partial_x \cdot \left\{ e^{\phi_A} \frac{p}{m} \right\},
\]

where

\[
h(q) \equiv \dot{p} - e^{\phi_A} \left( G \frac{p}{m} + f(\epsilon q) - 2D\partial_p \phi(\epsilon q) - D\partial_p \phi_A(q) \right),
\]

and

\[
\phi_A(q) \equiv \phi(q) - \phi(\epsilon q).
\]
Result:

\[ \Gamma^+[q', t+dt|q, t] = \frac{\delta(x' - x - dt(p/m)e^{\phi_A(q)})}{(4\pi e^{\phi_A} dt)^d/2 |\det(D)|^{1/2}} \exp \left[ -\frac{dt}{4e^{\phi_A}} h(q)D^{-1}h(q) \right] \]

\[ -\frac{dt}{2} \partial_p \cdot \left\{ e^{\phi_A} \left( \frac{p}{m} + f(\epsilon q) - 2D\partial_p \phi(\epsilon q) - \frac{1}{2}D\partial_p \phi_A(q) \right) \right\} - \frac{dt}{2} \partial_x \cdot \left\{ e^{\phi_A} \frac{p}{m} \right\}, \]

where

\[ h(q) \equiv \dot{p} - e^{\phi_A} \left( \frac{p}{m} + f(\epsilon q) - 2D\partial_p \phi(\epsilon q) - D\partial_p \phi_A(q) \right), \]

and

\[ \phi_A(q) \equiv \phi(q) - \phi(\epsilon q). \]

- \( d\Delta S_{bDB} \) is ill-defined in continuous stochastic dynamics
  - Two different delta function constraints for \( \dot{x} \) and \( p \)
  - \( O(1) \) terms in \( d\Delta S_{bDB} \)
\begin{itemize}
    \item \( \Gamma^\dagger \) corresponds to Langevin equations with a multiplicative noise
    \begin{align*}
        \dot{x} &= \left( \frac{p}{m} \right) e^{\phi_A(q)}, \\
        \dot{p} &= \left( G \frac{p}{m} + f(\epsilon q) - 2D \partial_p \phi(\epsilon q) - \frac{1}{2} D \partial_p \phi_A(q) \right) e^{\phi_A(q)} + e^{1/2 \phi_A(q)} \xi(t).
    \end{align*}
\end{itemize}
\( \Gamma^\dagger \) corresponds to Langevin equations with a *multiplicative* noise

\[
\begin{align*}
\dot{x} &= \left( \frac{p}{m} \right) e^{\phi_A(q)}, \\
\dot{p} &= \left( G \frac{p}{m} + f(\epsilon q) - 2D \partial_p \phi(\epsilon q) - \frac{1}{2} D \partial_p \phi_A(q) \right) e^{\phi_A(q)} + e^{\frac{1}{2} \phi_A(q)} \xi(t).
\end{align*}
\]

  - Regard the above eqs. as those for \( x(s) \) and \( p(s) \) (\( s = \text{new time variable} \))
  - Change of time variable: \( ds = dt \exp(-\phi_A) \) or

\[
s = s(t) = \int_0^t dt' \, e^{-\phi_A(q(t'))}.
\]
New eqs. of motion for $x(t)$ and $p(t)$:

\[
\frac{dx}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = G \frac{p}{m} + f(\epsilon q) - 2D \partial_p \phi(\epsilon q) - \frac{D}{2} \partial_p \phi_A(q) + \xi(t),
\]
Housekeeping EP / Regularization of $d\Delta S_{bDB}$

- New eqs. of motion for $x(t)$ and $p(t)$:

\[
\frac{dx}{dt} = \frac{p}{m},
\]

\[
\frac{dp}{dt} = G \frac{p}{m} + f(\epsilon q) - 2D \partial_p \phi(\epsilon q) - \frac{D}{2} \partial_p \phi_A(q) + \xi(t),
\]

- Corresponding transition prob. during $dt$: $\tilde{\Gamma}^+[q', t + dt|q, t]$
New eqs. of motion for $x(t)$ and $p(t)$:

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$$\frac{dp}{dt} = G\frac{p}{m} + f(\epsilon q) - 2D\partial_p \phi(\epsilon q) - \frac{D}{2} \partial_p \phi_A(q) + \xi(t),$$

Corresponding transition prob. during $dt$: $\tilde{\Gamma}^+[q', t + dt|q, t]$

Regularized $d\Delta S_{bDB}$:

$$d\Delta \tilde{S}_{bDB} \equiv \ln \frac{\Gamma[q', t + dt|q, t]}{\tilde{\Gamma}^+[q', t + dt|q, t]}.$$
Result:

\[ d \Delta \tilde{S}_{\text{bDB}} = -dt \left\{ \left( \dot{p} - f^{\text{rev}}(q) + D \partial_p \phi(\epsilon q) + \frac{D}{4} \partial_p \phi_A(q) \right) \right. \]

\[ \times D^{-1} \left( \frac{p}{m} - f^{\text{ir}}(q) - D \partial_p \phi(\epsilon q) - \frac{D}{4} \partial_p \phi_A(q) \right) \right\} \]

\[ + dt \partial_p \left( \frac{p}{m} - f^{\text{ir}}(q) - D \partial_p \phi(\epsilon q) - \frac{D}{4} \partial_p \phi_A(q) \right) \]

\( \Delta \tilde{S}_{\text{bDB}} \) satisfies the IFT
Housekeeping EP / Regularization of $d\Delta S_{bDB}$

- Result:

$$d\Delta \tilde{S}_{bDB} = -dt \left\{ \left( \dot{p} - f^{rev}(q) + D \partial_p \phi(\epsilon q) + \frac{D}{4} \partial_p \phi_A(q) \right) \right.$$ 

$$\times D^{-1} \left( \frac{G}{m} \frac{p}{m} - f^{ir}(q) - D \partial_p \phi(\epsilon q) - \frac{D}{4} \partial_p \phi_A(q) \right) \right\}$$ 

$$+ dt \left( \frac{G}{m} \frac{p}{m} - f^{ir}(q) - D \partial_p \phi(\epsilon q) - \frac{D}{4} \partial_p \phi_A(q) \right)$$

- $\Delta \tilde{S}_{bDB}$ satisfies the IFT
- Average EP rate:

$$\frac{d}{dt} \langle \Delta \tilde{S}_{bDB} \rangle = \int dq \left\{ \frac{j_{p}^{s,ir}(\epsilon q)}{\rho^s(\epsilon q)} - \frac{D}{4} \partial_p \phi_A(q) \right\} D^{-1} \left\{ \frac{j_{p}^{s,ir}(\epsilon q)}{\rho^s(\epsilon q)} - \frac{D}{4} \partial_p \phi_A(q) \right\} \rho(q),$$

- When DB is satisfied, $j_{p}^{s,ir} = 0$ and $\phi_A(q) = 0$
- Spinney and Ford: no $\phi_A$ terms

\[
\Gamma^*_\psi[q', t + dt|q, t] \equiv \Gamma[q, t + dt|q', t] \frac{\rho_\psi(q')}{\rho_\psi(q)} e^{-(dt)\pi(q)},
\]

for an arbitrary reference distribution \(\rho_\psi(q) \equiv \exp[-\psi(q)]\) where

\[
\pi(q) = \frac{1}{\rho_\psi(q)} H_K(q) \rho_\psi(q)
\]

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For the case where \( \rho_\psi(q) = \rho^s(q) \), \( H_K \rho^s = 0 \), thus \( \pi(q) = 0 \). Consequently \( \Gamma^*_\psi \Rightarrow \Gamma^* \).

\[ \Gamma^*_\psi[q', t + dt|q, t] \equiv \Gamma[q, t + dt|q', t] \frac{\rho_\psi(q')}{\rho_\psi(q)} e^{-(dt)\pi(q)}, \]

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For the case where \( \rho_\psi(q) = \rho^s(q), H_K \rho^s = 0 \), thus \( \pi(q) = 0 \). Consequently \( \Gamma^*_\psi \Rightarrow \Gamma^* \).

\( \Gamma^*_\psi \) represents a well-defined stochastic process:

\[ \int dq' \Gamma^*_\psi[q', t + dt|q, t] = 1, \]

\[ \Rightarrow \text{It follows from} \int dq' \Gamma[q, t + dt|q', t] \rho_\psi(q') = \rho_\psi(q) + (dt)H_K \rho_\psi(q). \]
Let us use
\[ \psi(q) = \phi(q) - \sigma \phi_A(q) = (1 - \sigma)\phi(q) + \sigma \phi(\epsilon q) \]

Note that \( \tilde{\Gamma}^+[q', t + dt|q, t] = \Gamma^*_{\psi}[\epsilon q', t + dt|\epsilon q, t] \) for \( \sigma = \frac{1}{4} \).
Let us use
\[ \psi(q) = \phi(q) - \sigma \phi_A(q) = (1 - \sigma)\phi(q) + \sigma \phi(\epsilon q) \]

Note that \( \tilde{\Gamma}^+ [q', t + dt | q, t] = \Gamma^* [\epsilon q', t + dt | \epsilon q, t] \) for \( \sigma = \frac{1}{4} \).

Generalized bDB EP:
\[
d\Delta S^\sigma_{\text{bDB}} \equiv \ln \frac{\Gamma [q', t + dt | q, t]}{\Gamma^* [\epsilon q', t + dt | \epsilon q, t]},
\]
\[
d\Delta S^\sigma_{\text{bDB}} = -dt \left\{ \left( \dot{p} - f^{\text{rev}}(q) + D \partial_p \phi(\epsilon q) + \sigma D \partial_p \phi_A(q) \right) \right.
\]
\[
\times D^{-1} \left( G \frac{p}{m} - f^{\text{ir}}(q) - D \partial_p \phi(\epsilon q) - \sigma D \partial_p \phi_A(q) \right) \right\}
\]
\[
+ dt \partial_p \left( G \frac{p}{m} - f^{\text{ir}}(q) - D \partial_p \phi(\epsilon q) - \sigma D \partial_p \phi_A(q) \right),
\]

Average EP rate: (Spinney & Ford (\( \sigma = 0 \)); \( \Delta \tilde{S}_{\text{bDB}} (\sigma = 1/4) \))
\[
\frac{d}{dt} \langle \Delta S^\sigma_{\text{bDB}} \rangle = \int dq \left\{ \frac{J_{p, \text{ir}}^s(\epsilon q)}{\rho^s(\epsilon q)} - \sigma D \partial_p \phi_A(q) \right\} D^{-1} \left\{ \frac{J_{p, \text{ir}}^s(\epsilon q)}{\rho^s(\epsilon q)} - \sigma D \partial_p \phi_A(q) \right\} \rho(q).
\]
Housekeeping EP / Parity asymmetry of $\rho^S(q)$

- **Total EP:**

\[ \Delta S_{\text{tot}} = \Delta S_{\text{excess}} + \Delta S_{\text{bDB}} + \Delta S_{\text{as}}, \]

\[ d\Delta S_{\sigma} = \ln \left[ \frac{\Gamma^*_{q,t+dt} | q', t} {\Gamma_{q,t+dt} | q, t} \right] \]

No IFT for $\Delta S_{\text{as}}$

\[ d\Delta S_{\sigma} = dt \left( \dot{p} \frac{\partial p}{\partial \phi_A(q)} + \dot{x} \frac{\partial x}{\partial \phi_A(q)} \right) + dt \sigma \frac{\partial p}{\partial \phi_A(q)} - dt \sigma \frac{\partial p}{\partial \phi_A(q)} \]

This vanishes when $\phi_A = 0$

Average EP rate:

\[ d\langle \Delta S_{\sigma} \rangle = \int dq \left[ \phi_A(q) \partial_t \rho(q) + \sigma \phi_A(q) \right] \]

\[ \frac{1}{2} j_{s,ir}(\epsilon q) \rho_s(\epsilon q) - \sigma \frac{\partial p}{\partial \phi_A(q)} \]
Housekeeping EP / Parity asymmetry of $\rho^s(q)$

- Total EP:
  \[ \Delta S_{tot} = \Delta S_{excess} + \Delta S_{bDB} + \Delta S_{as}, \]

- Arrow EP:
  \[ d\Delta S_{as}^\sigma = \ln \frac{\Gamma^*[q, t + dt|q', t]}{\Gamma[q', t + dt|q, t]} \frac{\Gamma^*[\epsilon q', t + dt|\epsilon q, t]}{\Gamma[\epsilon q, t + dt|\epsilon q', t]} . \]

- No IFT for $\Delta S_{as}^\sigma$
Housekeeping EP / Parity asymmetry of $\rho_s(q)$

- Total EP:
  \[
  \Delta S_{\text{tot}} = \Delta S_{\text{excess}} + \Delta S_{\text{bDB}}^{\sigma} + \Delta S_{\text{as}}^{\sigma},
  \]

- \[
  d\Delta S_{\text{as}}^{\sigma} = \ln \left[ \frac{\Gamma^*[q, t + dt | q', t]}{\Gamma[q', t + dt | q, t]} \right] \left[ \frac{\Gamma^*[\epsilon q', t + dt | \epsilon q, t]}{\Gamma[\epsilon q, t + dt | \epsilon q', t]} \right].
  \]

- No IFT for $\Delta S_{\text{as}}^{\sigma}$

- \[
  d\Delta S_{\text{as}}^{\sigma} = dt \left( \dot{p} \partial_p \phi_A(q) + \dot{x} \partial_x \phi_A(q) \right) + dt \sigma D \partial_p \partial_p \phi_A(q)
  - dt \sigma \left( \partial_p \phi_A(q) \right) \left[ \dot{p} - G \frac{p}{m} - f(\epsilon q) + 2D \partial_p \phi(\epsilon q) + \sigma D \partial \phi_A(q) \right].
  \]

- This vanishes when $\phi_A = 0$
Housekeeping EP / Parity asymmetry of $\rho^s(q)$

- Total EP:
  \[
  \Delta S_{\text{tot}} = \Delta S_{\text{excess}} + \Delta S_{\text{bDB}}^\sigma + \Delta S_{\text{as}}^\sigma,
  \]

- No IFT for $\Delta S_{\text{as}}^\sigma$

- \[
  d\Delta S_{\text{as}}^\sigma = \ln \left[ \frac{\Gamma^*[q, t + dt|q', t] \Gamma^*[\epsilon q', t + dt|\epsilon q, t]}{\Gamma[q', t + dt|q, t] \Gamma[\epsilon q, t + dt|\epsilon q', t]} \right].
  \]

- \[
  d\Delta S_{\text{as}}^\sigma = dt(\dot{p} \partial_p \phi_A(q) + \dot{x} \partial_x \phi_A(q)) + dt \sigma D \partial_p \partial_p \phi_A(q)
  - dt \sigma (\partial_p \phi_A(q)) \left[ \dot{p} - G \frac{p}{m} - f(\epsilon q) + 2D \partial_p \phi(\epsilon q) + \sigma D \partial \phi_A(q) \right].
  \]

- This vanishes when $\phi_A = 0$

- Average EP rate:
  \[
  \frac{d}{dt} \langle \Delta S_{\text{as}}^\sigma \rangle = \int d\mathbf{q} \left[ \phi_A(q) \partial_t \rho(q) + \sigma (\partial_p \phi_A(q)) \left\{ \frac{2j_{\text{s,ir}}^s(\epsilon q)}{\rho^s(\epsilon q)} - \sigma D \partial_p \phi_A(q) \right\} \rho(q) \right].
  \]
\[ \Delta S_{\text{tot}} = \Delta S_{\text{excess}} + \Delta S_{\text{housekeeping}} \]
\[ = \Delta S_{\text{excess}} + \Delta S_{\text{bDB}}^\sigma + \Delta S_{\text{as}}^\sigma, \]

- Based on the generalized dual process with \( \psi(q) = \phi(q) - \sigma \phi_A(q) \)
- \( \Delta S_{\text{bDB}}^\sigma \) satisfies IFT
- \( \sigma = \frac{1}{4} \): \( \Delta S_{\text{bDB}}^\sigma = \Delta \tilde{S}_{\text{bDB}} \)
- \( \sigma = 0 \): \( \Delta S_{\text{bDB}}^\sigma = \Delta S_2 \) in Spinney and Ford
- \( \Delta S_{\text{as}}^\sigma \) due to asymmetry of SSD does not satisfy IFT
- \( \Delta S_{\text{as}}^\sigma \) is nontransient for \( \sigma \neq 0 \)
- Continuous variable case: zero irr. current & parity symmetry of SSD are mixed in the DB condition \( \Rightarrow \) no clear-cut \( \Delta S_{\text{bDB}} \) as in discrete case