Perturbative approach to Information Spreading in the deep MBL regime

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Perturbative Calculation of OTOC in the deep MBL regime

Joint work with Myeonghyun Kim (SNU)

Gwangju Institute of Science and Technology

Outline

- **Information spreading in Many-Body Localization (MBL)**
  - MBL vs. Anderson Localization
  - Out-of-Time-Ordered Commutator/Correlator (**OTOC**)
  - **Beyond the "l-bit" model or full diagonalization? Can MBL survive in 2D?**

- **Growth of OTOC in the disordered XXZ model**
  - Perturbation method in the weak hopping limit $(J \ll J_z \ll h)$

→ **Logarithmic light cone of OTOC is shown in 1D and tree-like lattices.**

→ **2D MBL? Logarithmic light cone seems to exist in 2D!**

XXZ model and "typical" ingredients of MBL

Heisenberg XXZ model in 1D

$$\hat{H} = -\sum_{i=1}^{L-1}\left[J\left(\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y\right) + J_z\hat{S}_i^z\hat{S}_{i+1}^z\right] + \sum_{i=1}^{L} h_i\hat{S}_i^z$$

interaction ***disorder*** $h_i \in [-h, h]$

- **Disorders**

  - onsite energy, hopping strength, quasi-periodicity

  - cf. disorder-free Stark MBL [PRL 2019; PNAS 2019; Nature 2021]

- **Interactions**

  - A *naive* picture of MBL = Anderson localization + Interactions

  - What's essential: "***dephasing***" → *Information spreading/scrambling*

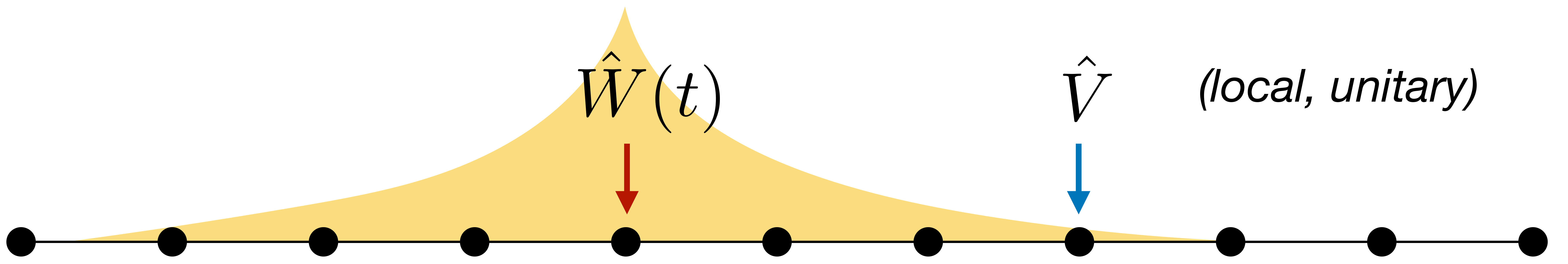
Role of interactions in many-body localization

- AL and MBL share lots of non-thermal features. (memory, ETH breaking, no particle transport, eigenstates with area-law entanglement)
- "Many-body" localization - **what is the role of "interactions"?**
- "Information" spreads, yet very slowly, over an MBL system. (entanglement entropy, **out-of-time-ordered commutator/correlator**)

Chaos	Anderson Localization	"Many-Body" Localization
linear in time	everything is frozen	linear in "**logarithmic**" time

Out-of-Time-Ordered Correlator / Commutator

a measure of "information" spreading



OTOCommutator $$C(t) = \frac{1}{2}\langle\, |\, [\hat{W}(t), \hat{V}(0)]\, |^2 \,\rangle = 1 - \mathrm{Re}[F(t)]$$

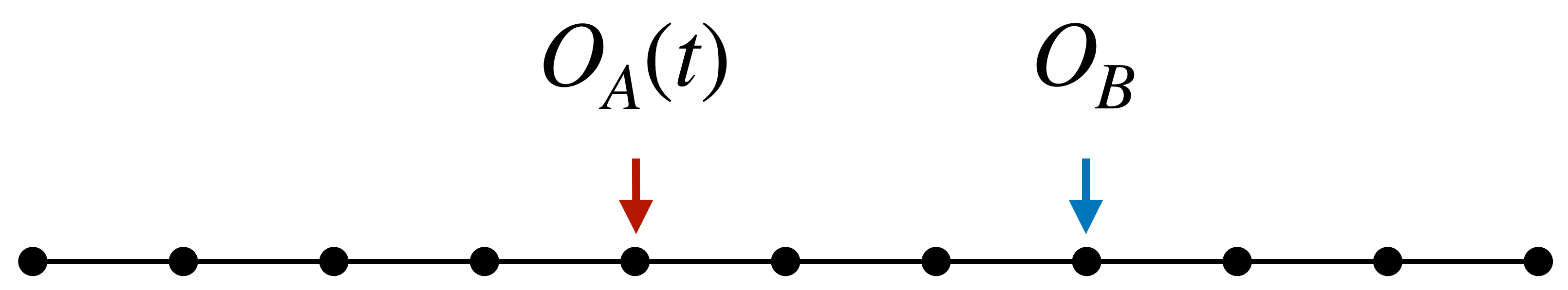
OTOCorrelator $$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger\hat{W}(t)\hat{V}\rangle$$

Freedom of choices

$\hat{\rho}$ ***measurement state***

$\hat{W}, \hat{V}$ ***local operators***

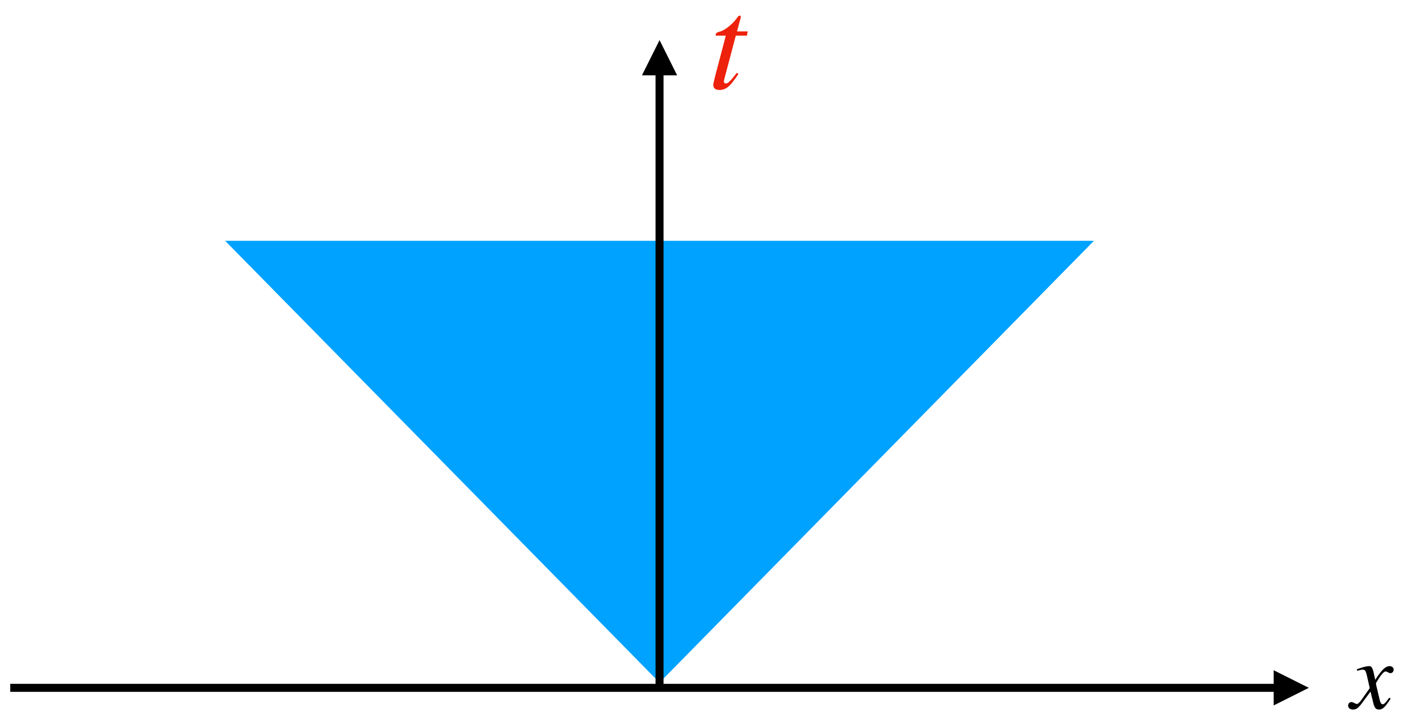
OTOC in chaotic and MBL systems



Chaotic systems: Lieb-Robinson bound

[E. B. Lieb and D. W. Robinson, Commun. Math. Phys. 28, 251 (1972)]

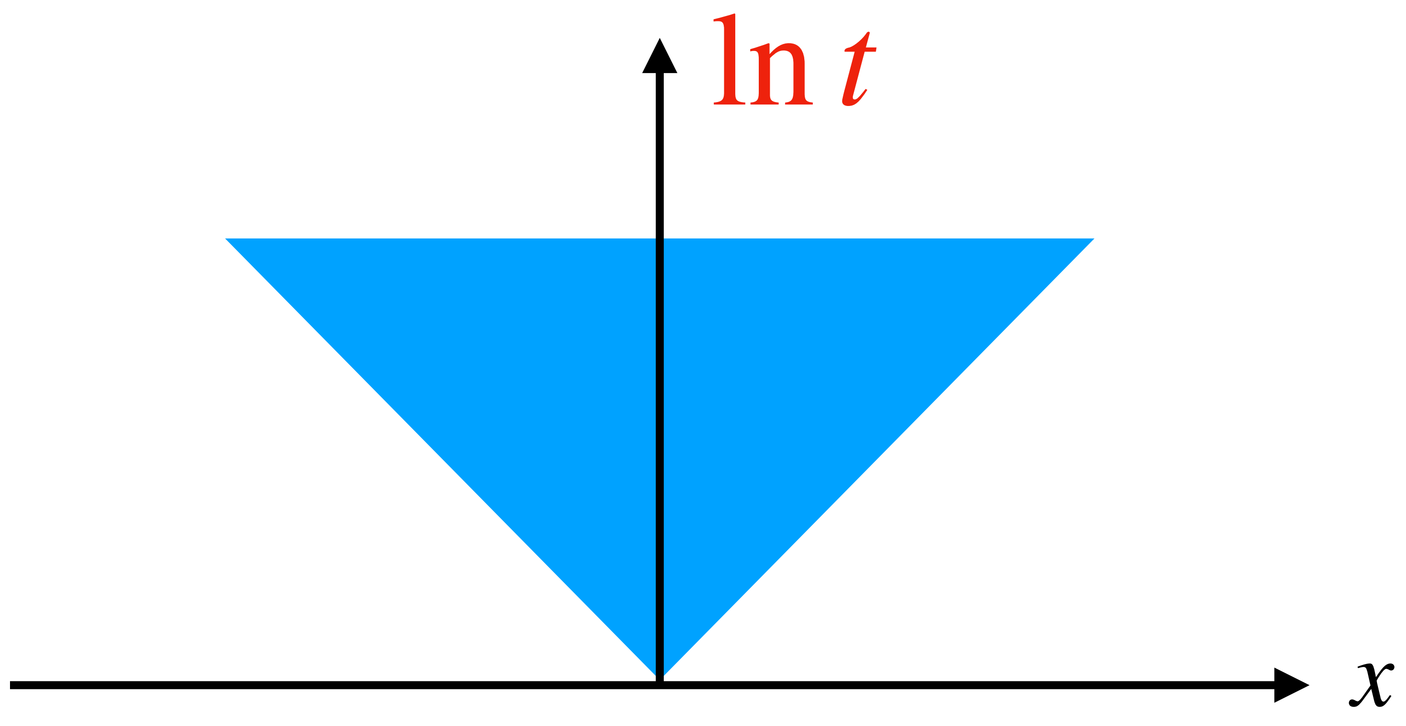
$$\|[O_A(t), O_B]\| \leq c\min(|A|, |B|)e^{-a(x-vt)}$$



MBL systems: *modified* Lieb-Robinson bound

[I. H. Kim, A. Chandran, and D. A. Abanin, arXiv:1412.3073]

$$\mathbb{E}_\mu\|[O_A(t), O_B]\| \leq ct|\partial A|e^{-\frac{x}{2\xi}}$$



← **exponentially decaying interactions between LIOMs**

(quasi-)Local integral of motion (LIOM)

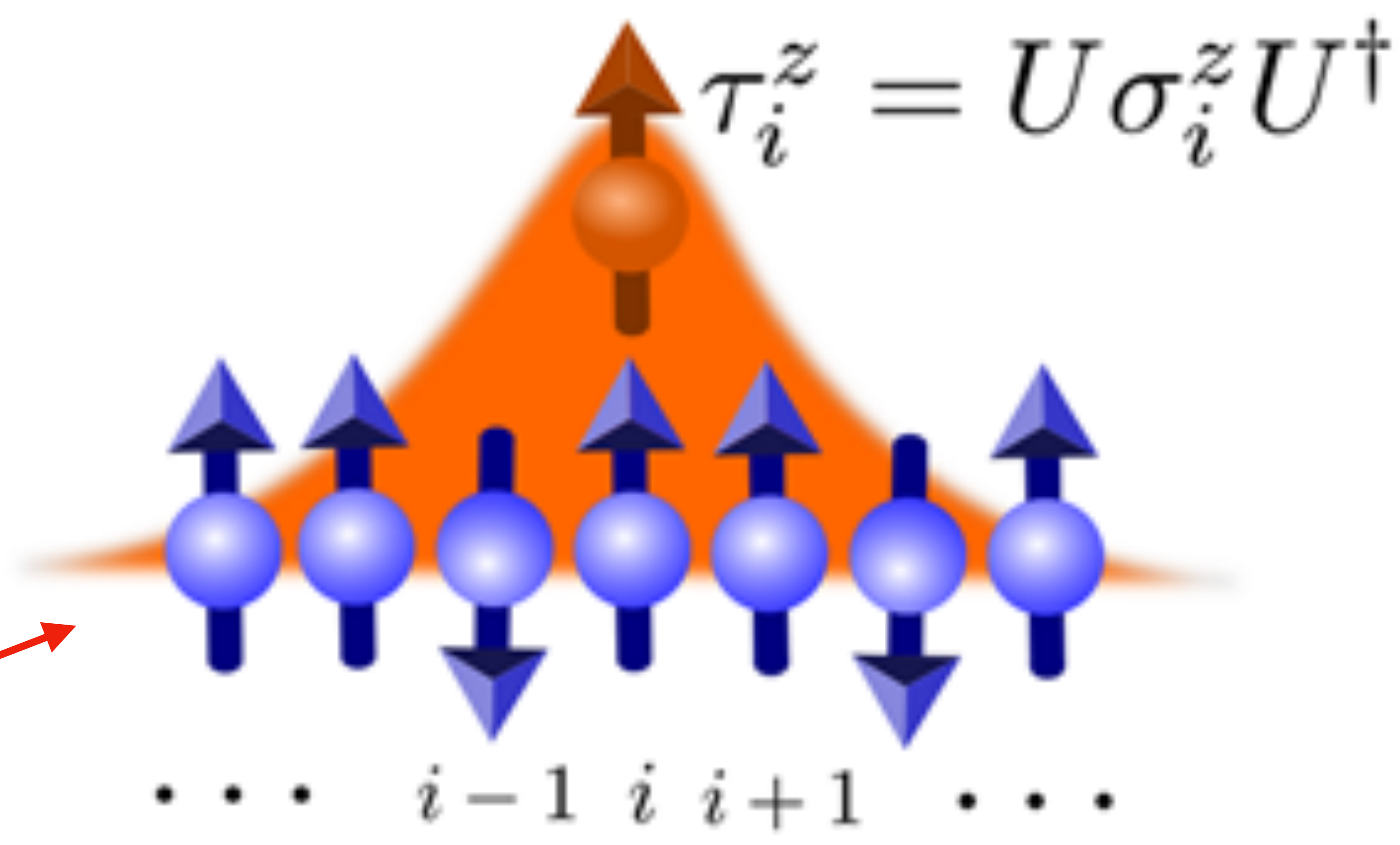
M. Serbyn et al., PRL 111, 127201 (2013)

D. A. Huse et al., PRB 90, 174202 (2014)

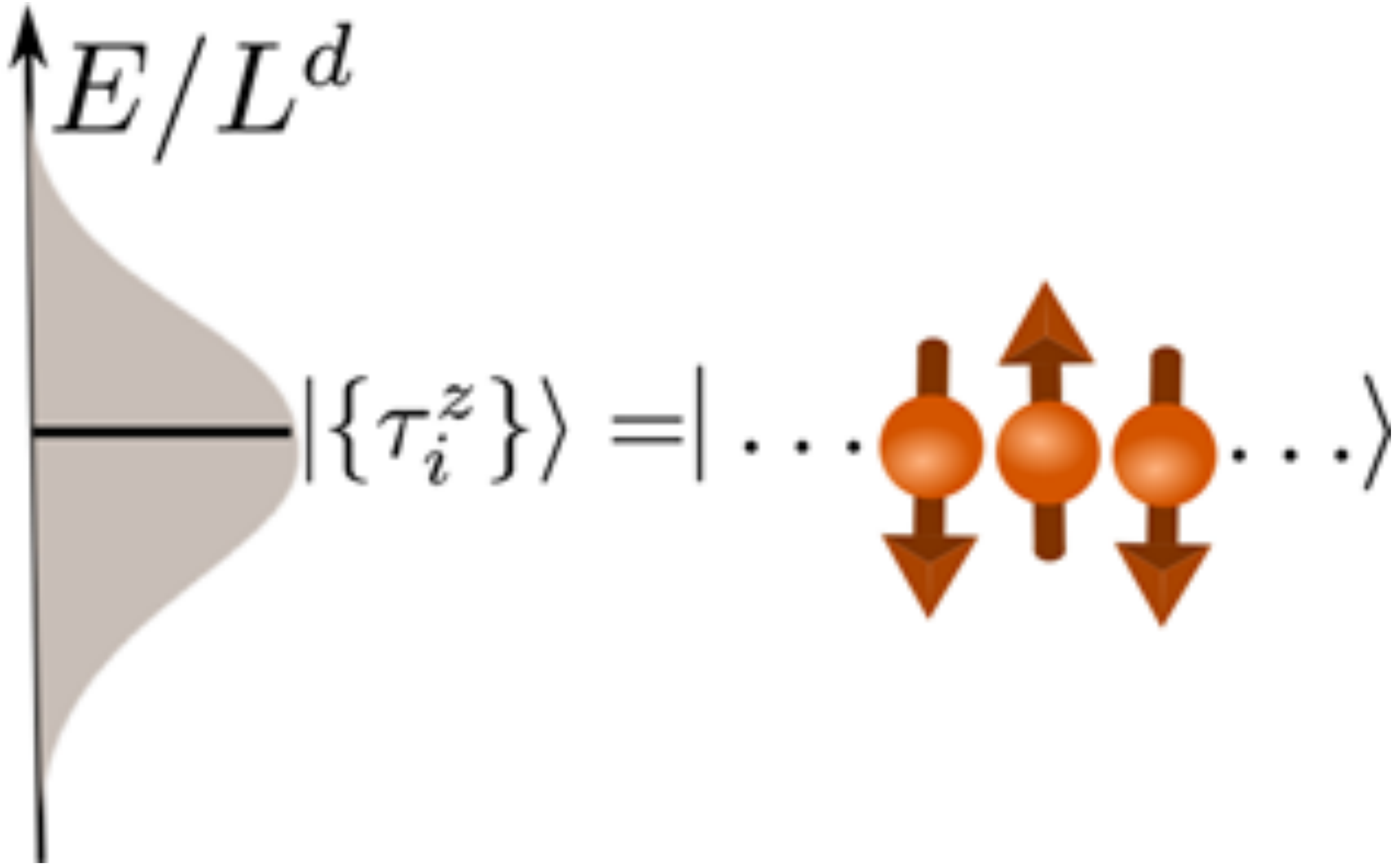
Review: Abanin & Papic, Ann. Phys. 529 (2017)

Review: Abanin et al., RMP 91, 021001 (2019)

Theoretical description of MBL with LIOM



physical spins



All-to-all interactions between LIOMs (l-bits)

Hamiltonian with physical bits (local interactions) $\longrightarrow$

$$H = \sum_i h_i \hat{\tau}_i^z + \sum_{\{i,j\}} J_{ij}\hat{\tau}_i^z\hat{\tau}_j^z + \sum_{\{i,j,k\}} K_{ijk}\hat{\tau}_i^z\hat{\tau}_j^z\hat{\tau}_k^z + \cdots$$

Effective l-bit model of MBL

RG: Vosk & Altman (2013); Pekker et al. (2014)

Non-perturbative proof: Imbrie (2016)

$$H = \sum_i h_i \hat{\tau}_i^z + \sum_{\{i,j\}} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{\{i,j,k\}} K_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots$$

with exponentially decaying interactions between LIOMs

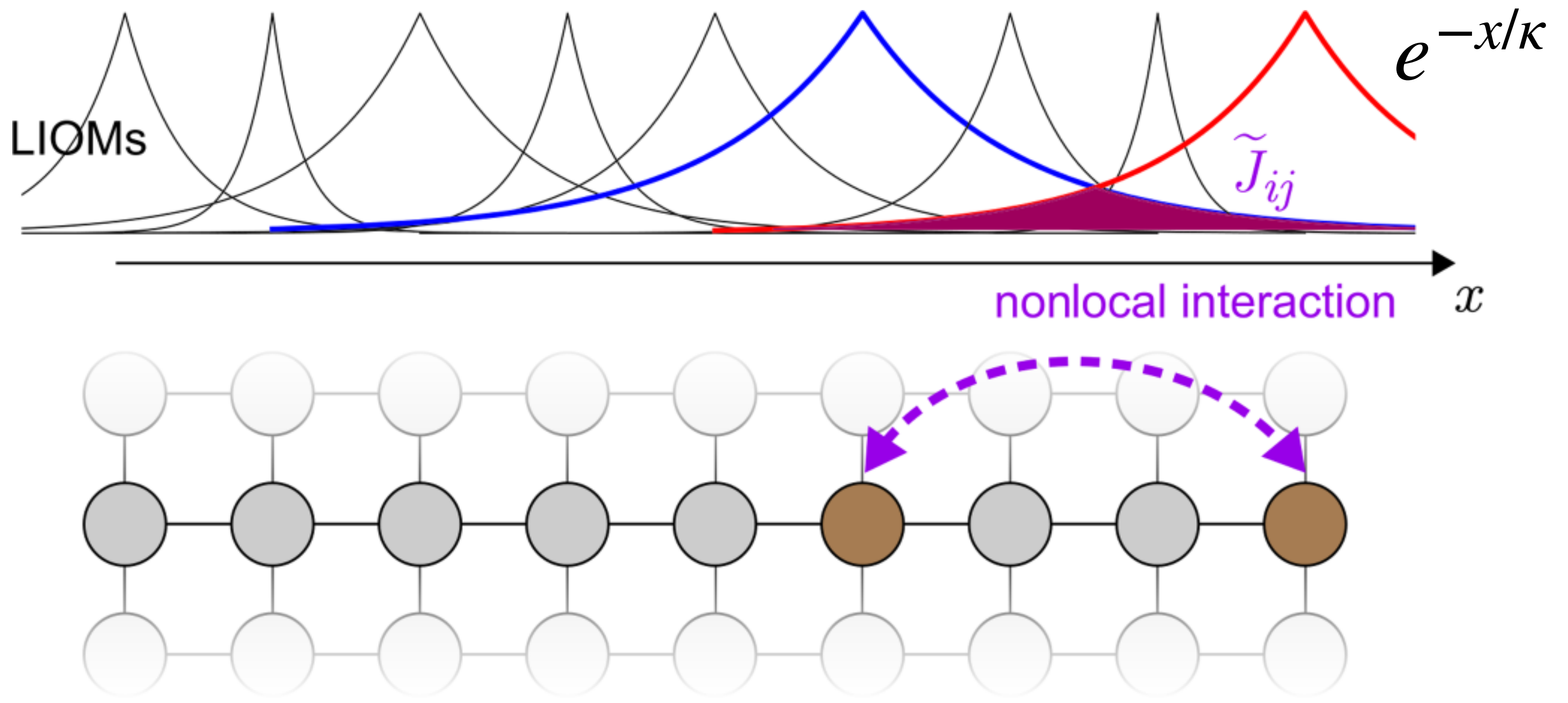


Figure from B. Chiaro et al., PRR 4, 013148 (2022)

$$\boxed{J_{\mathrm{MBL}}^{\mathrm{eff}} \sim \tilde{J} \exp(-x/\xi)}$$

Existence of LIOMs →

power-law relaxation of local observables

l-bit spin precessions; dephasing dynamics

Serbyn, Papic, Abanin, PRB 90, 174302 (2014)

logarithmic information spreading (EE)

Serbyn, Papic, Abanin, PRL 110, 260601 (2013)

Huse, Nandkishore, Oganesyan, PRB 90, 174202 (2014)

Logarithmic light cone of OTOC in the effective l-bit Hamiltonian

LIOM →

$$H = \sum_i h_i \hat{\tau}_i^z + \sum_{\{i,j\}} J_{ij} \hat{\tau}_i^z \hat{\tau}_j^z + \sum_{\{i,j,k\}} K_{ijk} \hat{\tau}_i^z \hat{\tau}_j^z \hat{\tau}_k^z + \cdots$$

$$\boxed{J_{ij}^{\text{eff}} \sim \tilde{J} \exp(-|i-j|/\xi)}$$

exponentially decaying effective interaction

OTOC (effective l-bit model; $\beta = 0$)

$$F(t) = \langle \hat{\tau}_i^x(t) \hat{\tau}_j \hat{\tau}_i^x(t) \hat{\tau}_j^x \rangle_{\beta=0} = \cos(4tJ_{ij}^{\text{eff}})$$

disorder average

$$\overline{F}(t) = \frac{\sin[4tJ\exp(-|i-j|\xi)]}{4tJ\exp(-|i-j|\xi)}$$

Logarithmic light cone

$$\boxed{t_0 = \frac{\pi}{4J} e^{-|i-j|/\xi}}$$

R. Fan et al., Sci. Bull. 62, 707 (2017)

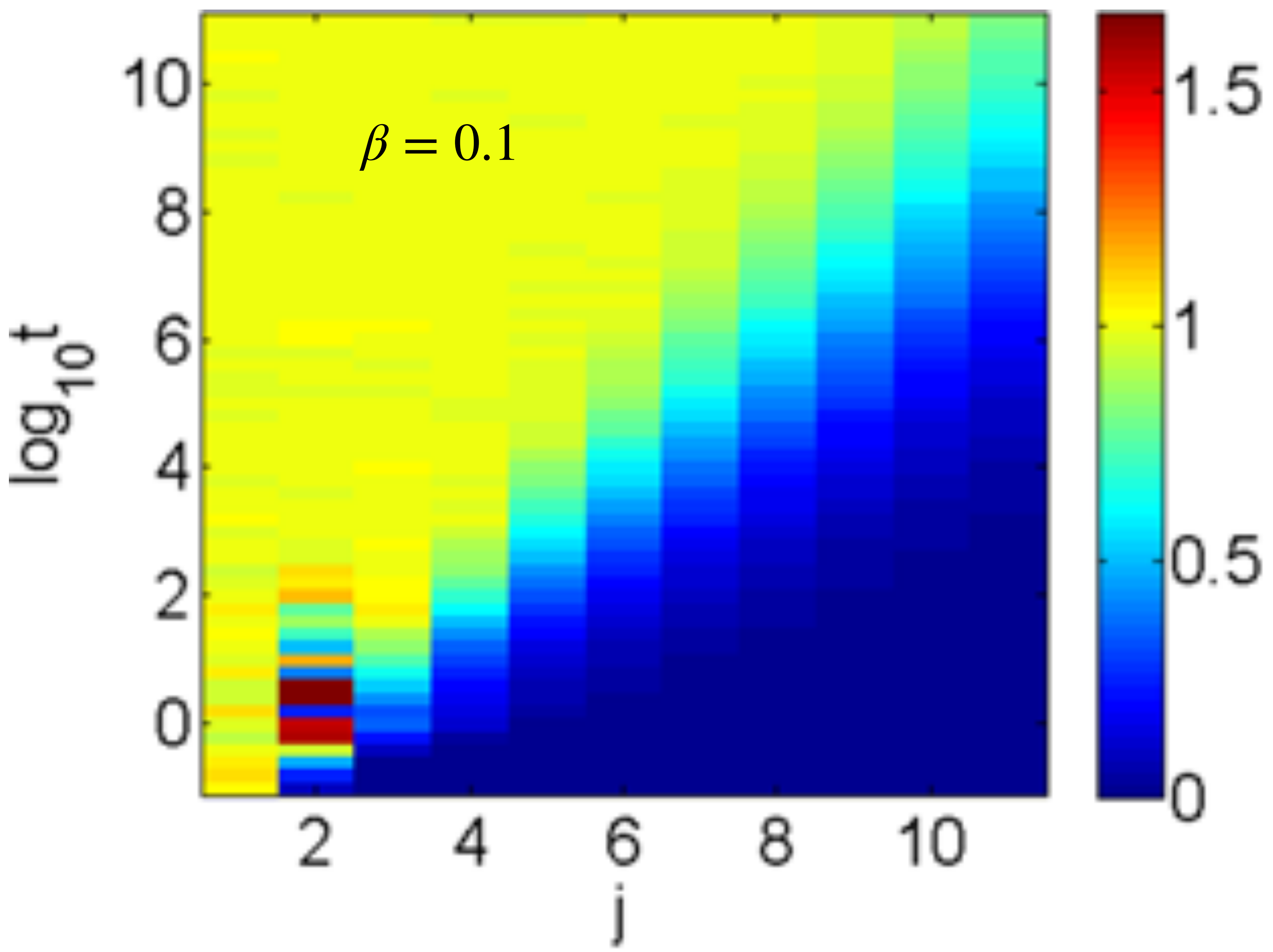
X. Chen et al., Ann. Phys. 1600332 (2016)

B. Swingle and D. Chowdhury, PRB 95, 060201(R) (2017)

Logarithmic light cone of OTOC beyond the l-bit Hamiltonian

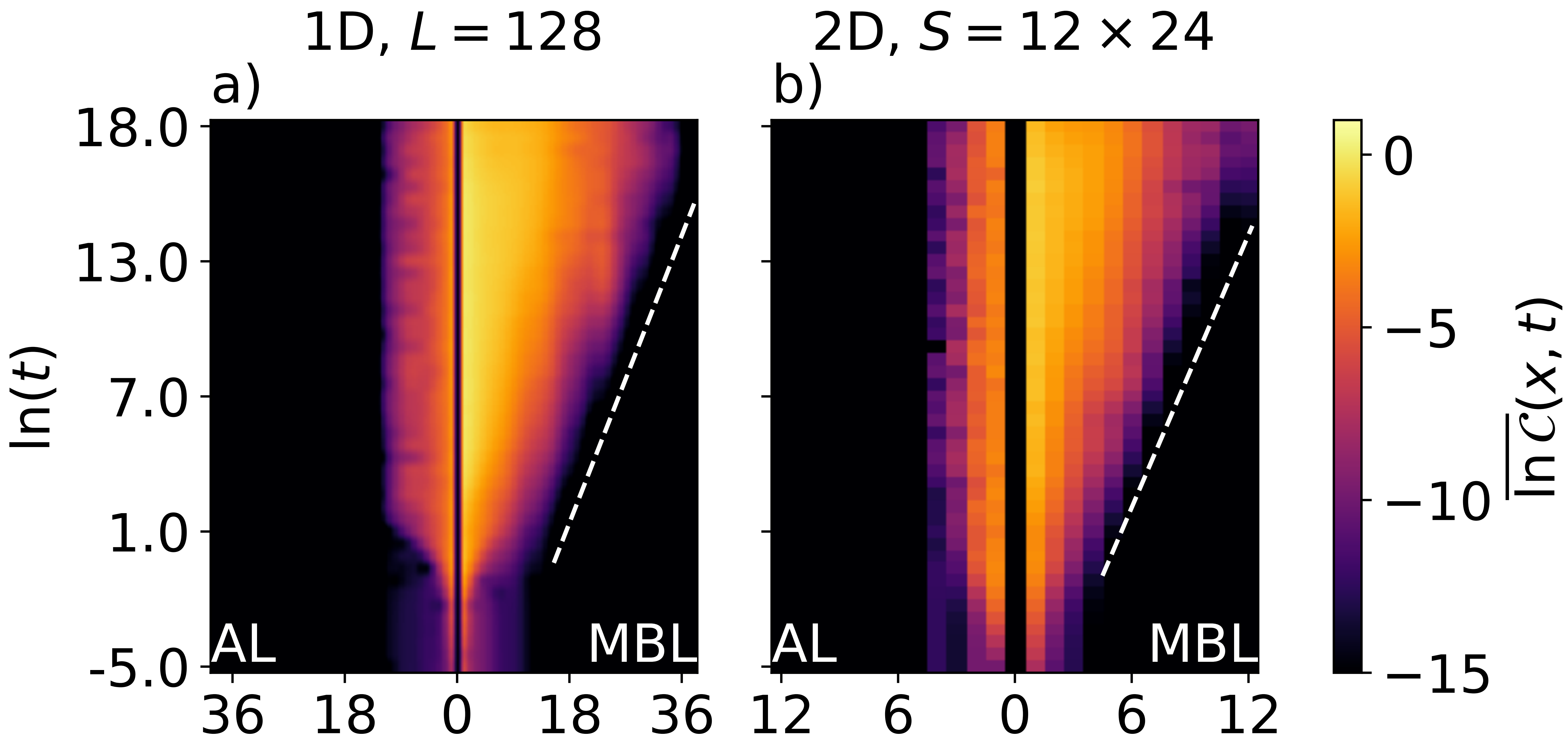
Exact (full) diagonalization on the disorderd XXZ chain

Y. Huang, Y.-L. Zhang & X. Chen, Ann. Phys. 529, 1600318 (2017)



Approximating LIOM in bosonic systems (weak interactions, strong disorder)

S. W. Kim, G. De Tomasi & M. Heyl, PRB 104, 144205 (2021)



Our goal

- We want to go beyond the system size limit of the exact diagonalization.
  - Practical limit of full diagonalization: $L \sim 12 - 14$; 1D only.
- We do not want to rely on the LIOMs and the effective l-bit model.
  - We want to stick with a "real" XXZ spin-1/2 Hamiltonian.

→ **Perturbation expansion of the OTOC frequency**

Construction of perturbation calculations

Weak hopping limit (strong interaction, strong disorder) $J \ll J_z \ll h$ **(natural energy unit : J_z)**

$$\hat{H} = -\frac{J}{2}\sum_{i=1}^{L-1}(\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y\hat{\sigma}_{i+1}^y) - \frac{J_z}{2}\sum_{i=1}^{L-1}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + \sum_{i=1}^{L} h_i\hat{\sigma}_i^z \equiv \hat{H}_0 + \hat{H}'$$

Perturbation (hopping)

$$\hat{H}' = \sum_{i=1}^{L-1}\hat{V}_i$$

$$\left[V_i = -J(\hat{\sigma}_i^+\hat{\sigma}_{i+1}^- + \hat{\sigma}_i^-\hat{\sigma}_{i+1}^+)\right]$$

Unperturbed Hamiltonian (Ising chain)

$$H_0 = -\frac{J_z}{2}\sum_{i=1}^{L-1}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + \sum_{i=1}^{L} h_i\hat{\sigma}_i^z$$

Non-degenerate eigenvalues are assumed.

Construction of perturbation calculations

$\hat{W}(t) = \hat{\sigma}_i^x(t)$ $\qquad\qquad$ $\hat{V} = \hat{\sigma}_j^x$

OTOC:

Deep MBL regime

$$C_\alpha(i,j;t) = 1 - \langle \hat{\sigma}_i^x(t)\,\hat{\sigma}_j^x\,\hat{\sigma}_i^x(t)\,\hat{\sigma}_j^x \rangle = 1 - \sum_{\beta,\gamma,\delta} s_{\alpha\beta\gamma\delta}\cos\left(\omega_{\alpha\beta\gamma\delta} t\right) \approx \boxed{1 - \cos\left(\omega_{\alpha\beta\gamma\delta} t\right)}$$

$\rightarrow$ $s_{\alpha\beta\gamma\delta} = \langle \alpha | \hat{\sigma}_i^x | \beta \rangle \langle \beta | \hat{\sigma}_j^x | \gamma \rangle \langle \gamma | \hat{\sigma}_i^x | \delta \rangle \langle \delta | \hat{\sigma}_j^x | \alpha \rangle \approx 1$

Every Fock (unperturbed) state $|\alpha^{(0)}\rangle$ fixes $|\beta^{(0)}\rangle$, $|\gamma^{(0)}\rangle$, $|\delta^{(0)}\rangle$ for a given (i,j).

$\rightarrow$ $\omega_{\alpha\beta\gamma\delta} = E_\alpha - E_\beta + E_\gamma - E_\delta \approx \boxed{\Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta}$

: We have to compute this.

*****No information spreading with the unperturbed Hamiltonian**

Rayleigh-Schrödinger perturbation expansion (numerical)

For our case of $V_{nn} = 0$,

Only even-order terms survive.

Fast computation possible with sparse matrix-vector multiplications

$$E^{(2n)} = \left\langle E^{(0)} \middle| \hat{V} \prod_{i=1}^{2n-1} \left(\hat{P}\hat{V}\right) \middle| E^{(0)} \right\rangle$$

$$+ \sum_{m=1}^{n-1} (-1)^m \sum_{k=m}^{n-1} \left[\sum_{\{k_1,\dots,k_m\}}' \left(\prod_{j=1}^{m} E^{(2k_j)} \right) \right] \left[\sum_{\{\alpha_1,\dots,\alpha_{2n-2k-1}\}}'' \left\langle E^{(0)} \middle| \hat{V} \prod_{j=1}^{2n-2k-1} \left[\hat{P}^{1+\alpha_j}\hat{V}\right] \middle| E^{(0)} \right\rangle \right]$$

projection operator

$$\hat{P} = \frac{1}{E^{(0)} - \hat{H}_0}$$

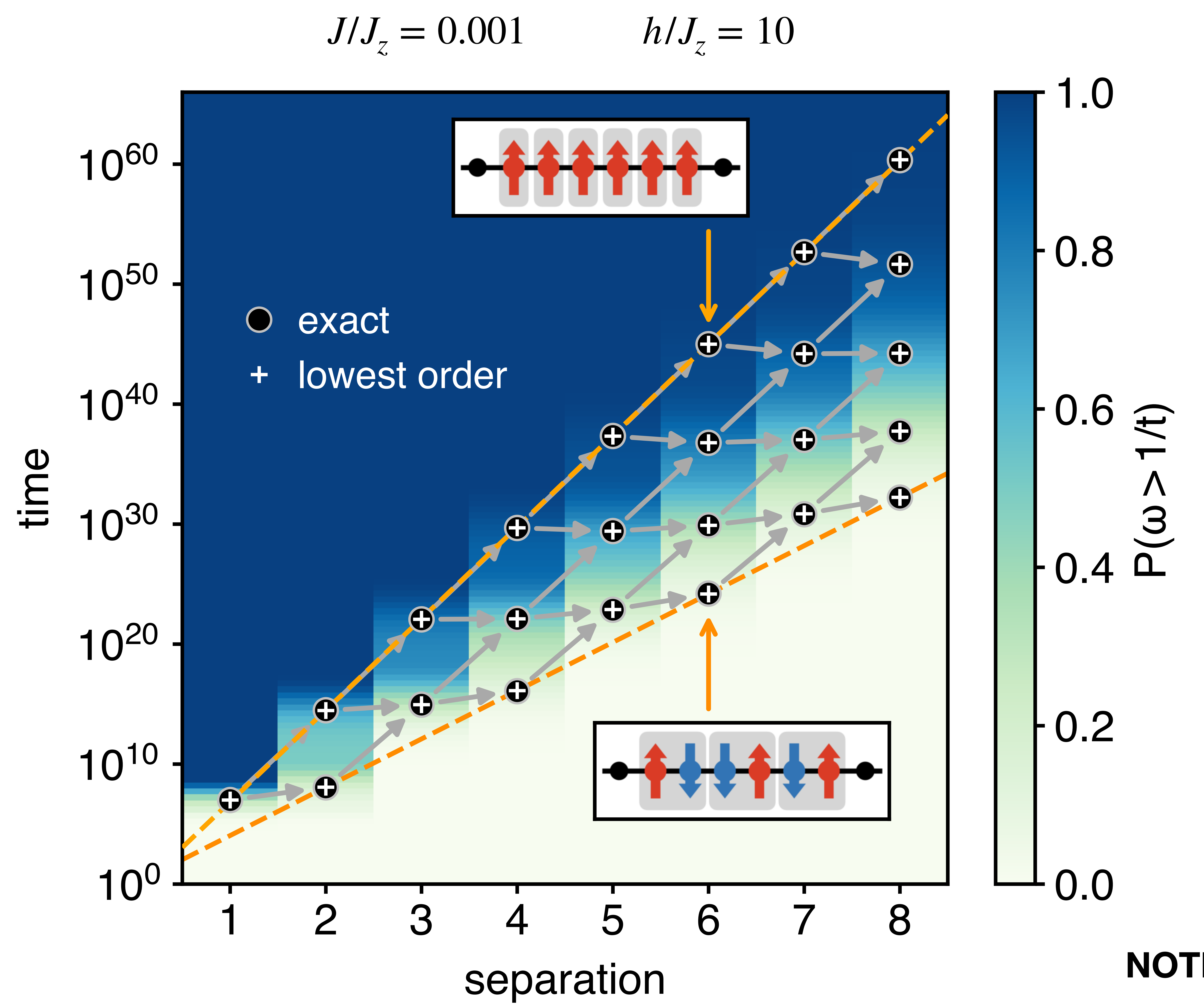
$$\sum_{\{k_1,\dots,k_m\}}'$$

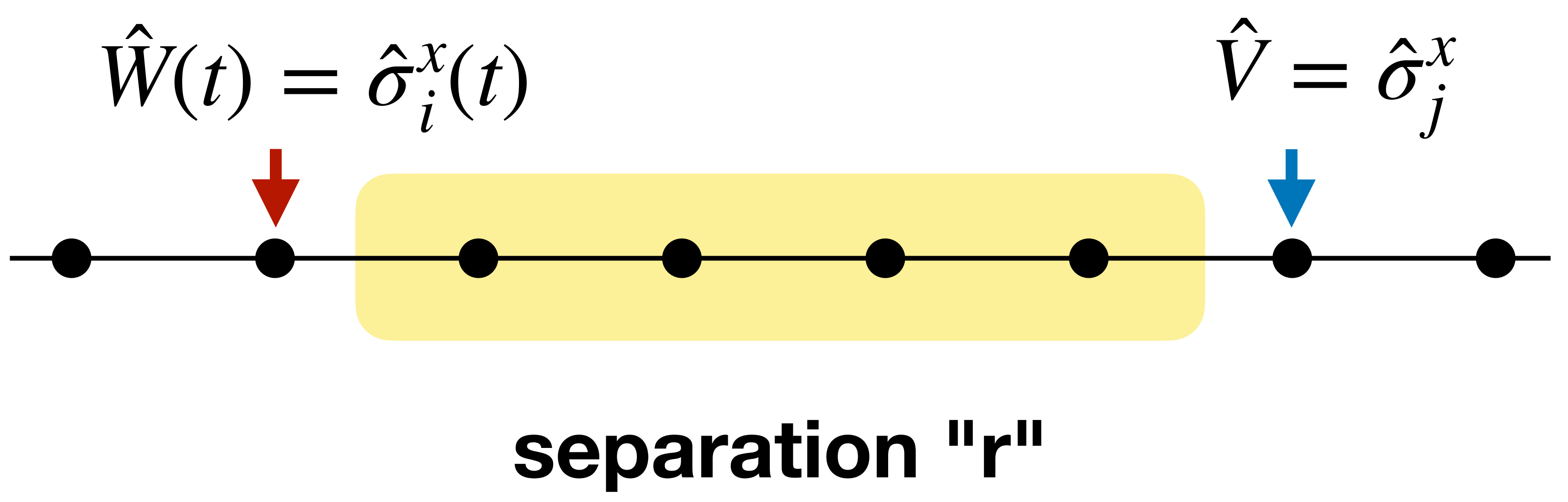
sum over all permutations of $\{k_1, \dots, k_m\}$ *subject to* $\sum k_j = k$ *and* $k_j \geq 1$

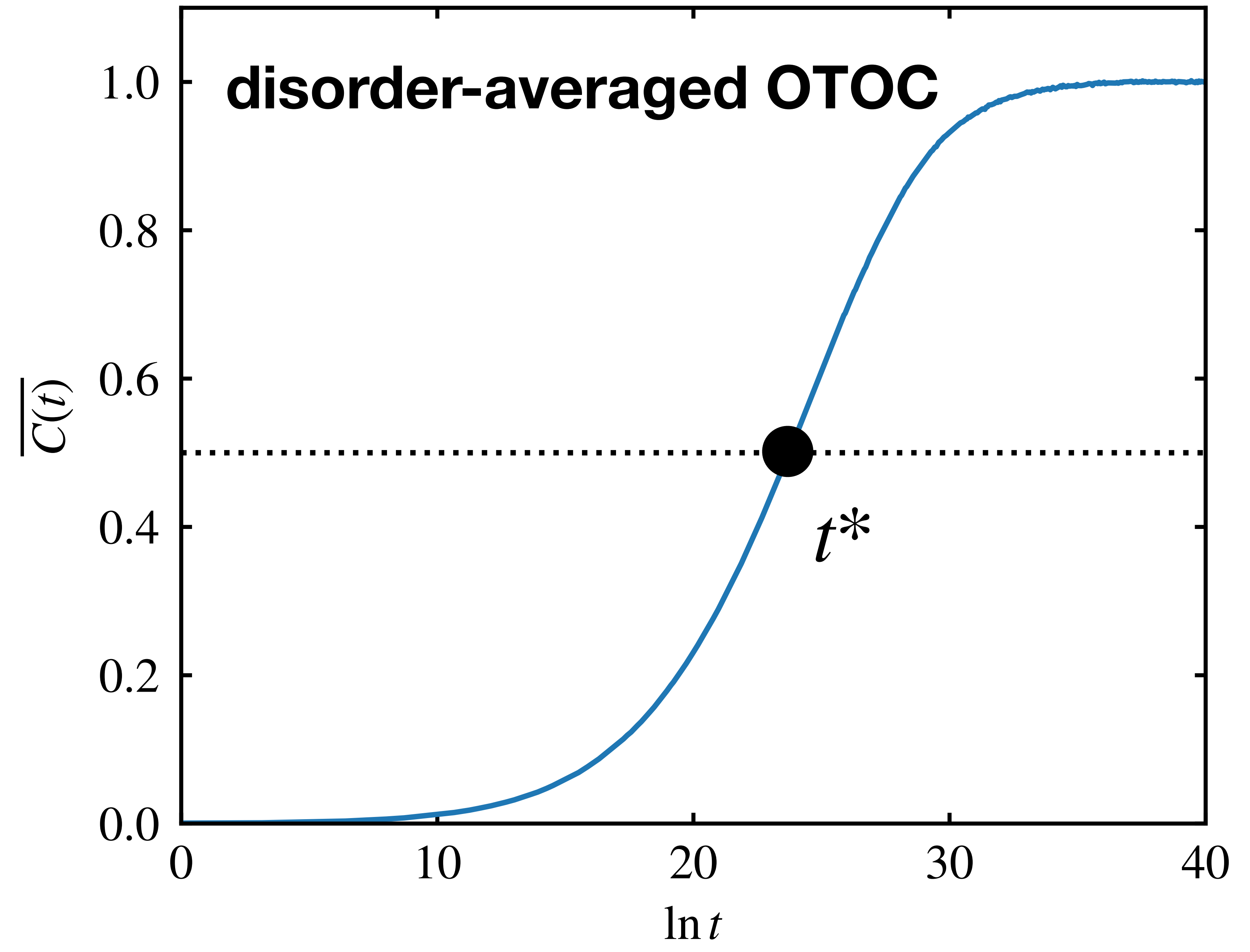
$$\sum_{\{\alpha_1,\dots,\alpha_{2n-2k-1}\}}''$$

sum over all permutations of $\{\alpha_1, \dots, \alpha_{2n-2k-1}\}$ *subject to* $\sum \alpha_j = m$ *and* $\alpha_j \geq 0$

Structure of the light cone (1D)







NOTE: multiprecision library is used. (500 digits)

OTOC frequency : the non-zero lowest order term

Perturbation expansion of the frequency

$$H_0 = -\frac{J_z}{2}\sum_{i=1}^{L-1}\hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + \sum_{i=1}^{L} h_i\hat{\sigma}_i^z \qquad \hat{H}' = -J\sum_{i=1}^{L-1}\hat{V}_i$$

$$\omega_\alpha(i,j) = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta \longleftarrow$$

We only need the non-zero lowest order of J!

If $\omega_\alpha(i,j) = \omega_\alpha(r \equiv |i-j|-1) = aJ^{n(r)}$ **and thereby** $C_\alpha(r,t) = 1 - \cos[aJ^{n(r)}t]$**,**

$$\boxed{\omega t_* = \text{constant} \longrightarrow \ln t_* \sim n(r) \quad \text{: light cone}}$$

Allowed lowest orders

r	2	3	4	5	6	7	8	9	...
n(r)	**2**, **4**	**4**, **6**	**4**, 6, **8**	**6**, 8, **10**	**6**, 8, 10, **12**	**8**, 10, 12, **14**	**8**, 10, 12, 14, **16**	**10**, 12, 14, 16, **18**	...

Non-zero lowest order

$$\omega_\alpha(i,j) = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$

Both ends do not contribute.

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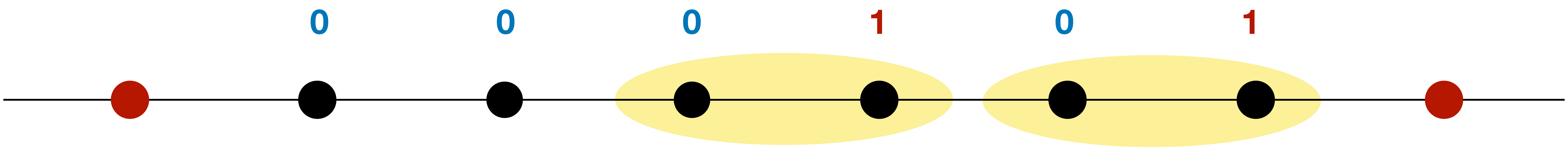
For a given Fock state $|\alpha\rangle$,



lowest order **κ**(r) = 2 × [r - maximum count of (01) or (10) pairs]

minimal setting of V operators bridging the two sites i and h

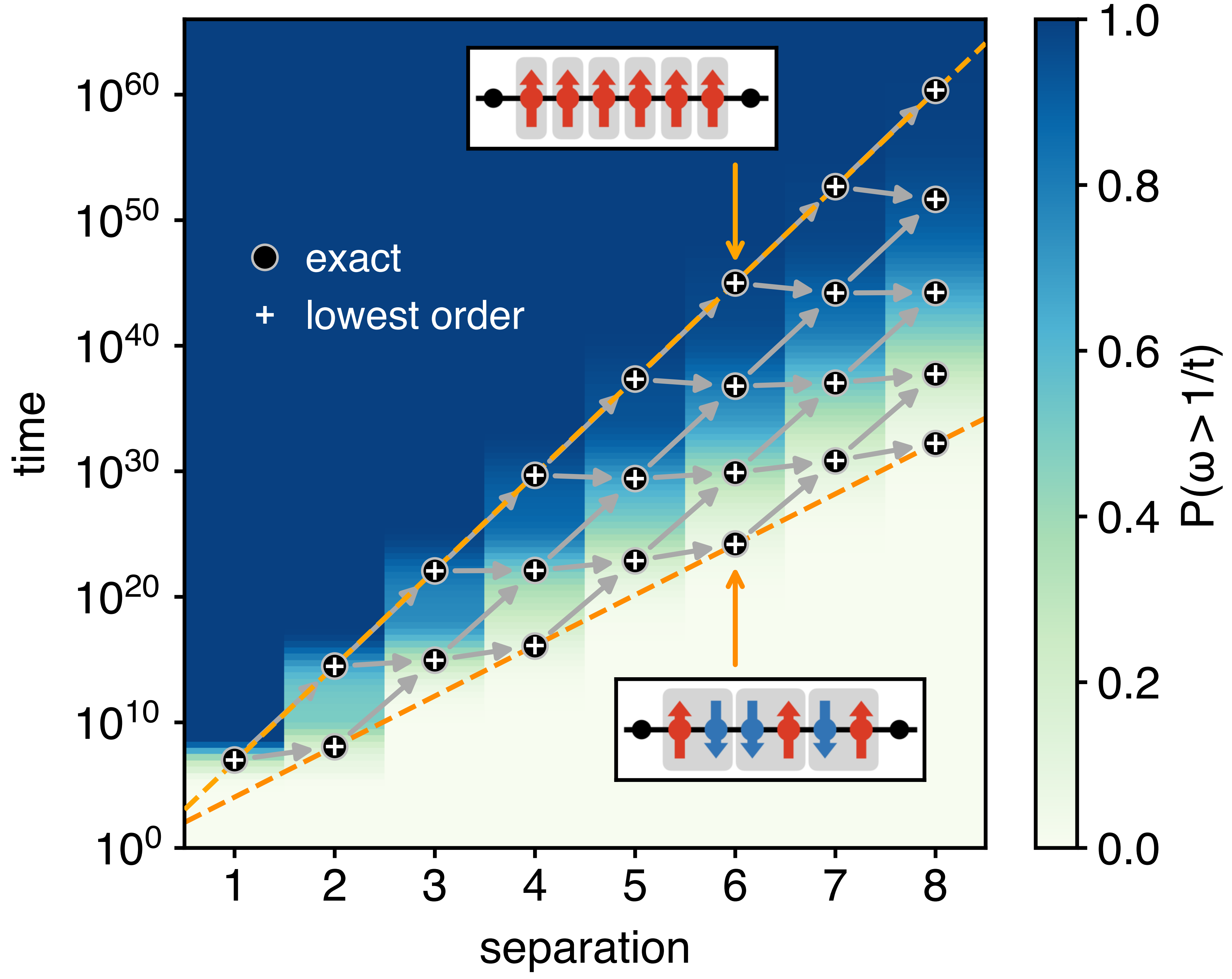
Bounds of the allowed lowest order

Max. of the allowed lowest order = **2r**

Min. of the allowed lowest order = **r** (if r is even) or **r+1** (if r is odd)

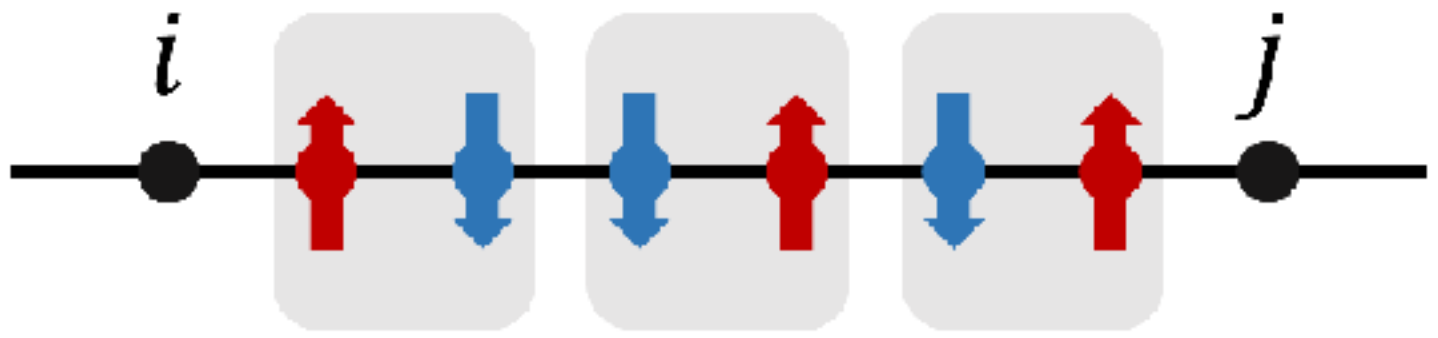
$$\omega_\alpha(r) = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$

$$\equiv J^{n(r)} F(r, h)$$



"Upper" light cone: ferromagnetic state

"Lower" light cone: filled with up-down pairs



"Upper" light cone (FM; slowest spreading)

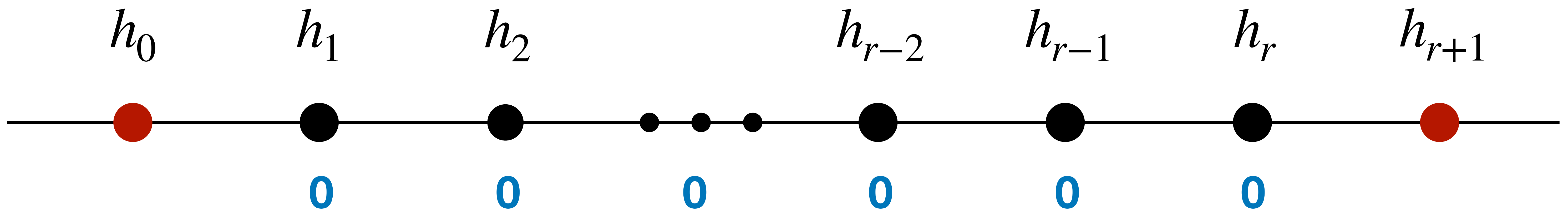
closed-form expression

$$\omega = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$

(nonzero lowest order)

$$= \left(\frac{J}{2}\right)^{2r} \cdot 2J_z \cdot \left[\frac{1}{\left(\prod_{i=1}^{r-1} F_i\right)^2 F_r (F_r - J_z)} + \frac{1}{G_1(G_1 - J_z)\left(\prod_{i=2}^{r} G_i\right)^2} + \sum_{k=1}^{r-1} \frac{F_k + G_{k+1}}{\left(\prod_{i=1}^{k} F_i \cdot \prod_{j=k+1}^{r} G_j\right)^2 \cdot (F_k + G_{k+1} - J_z)} \right]$$

$$F_i \equiv h_0 - h_i \qquad\qquad G_i \equiv h_{r+1} - h_i$$



$\Delta E_\alpha = 0$ **0** ***cannot move!*** **0**

"Upper" light cone (FM; slowest spreading)

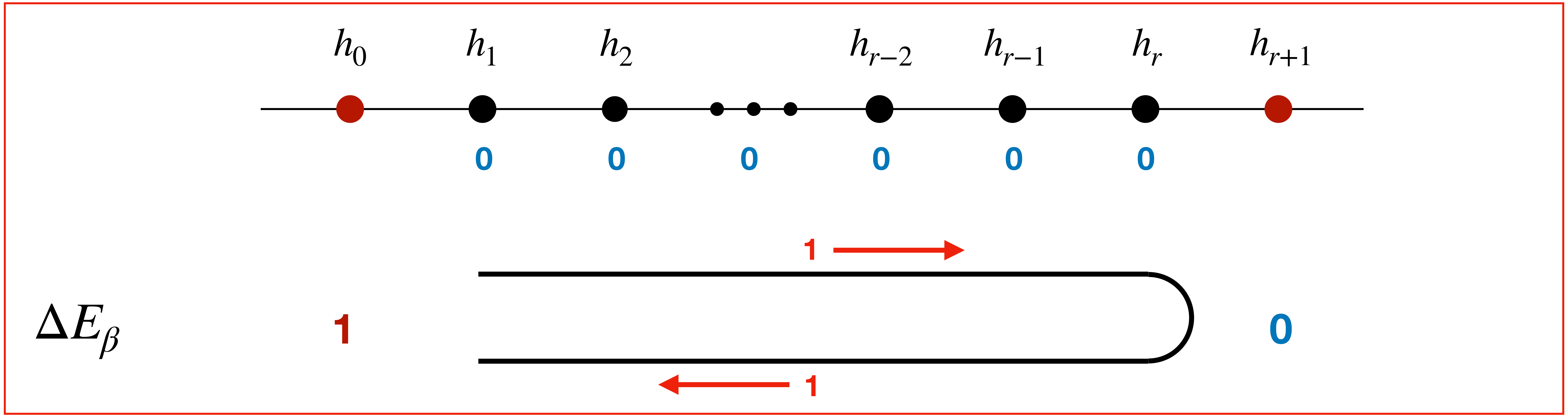
Analytically computable!

$$\omega = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$ **(nonzero lowest order)**

$$= \left(\frac{J}{2}\right)^{2r} \cdot 2J_z \cdot \left[\frac{1}{\left(\prod_{i=1}^{r-1} F_i\right)^2 F_r (F_r - J_z)} + \frac{1}{G_1 (G_1 - J_z) \left(\prod_{i=2}^{r} G_i\right)^2} + \sum_{k=1}^{r-1} \frac{F_k + G_{k+1}}{\left(\prod_{i=1}^{k} F_i \cdot \prod_{j=k+1}^{r} G_j\right)^2 \cdot (F_k + G_{k+1} - J_z)} \right]$$

$F_i \equiv h_0 - h_i$

$G_i \equiv h_{r+1} - h_i$



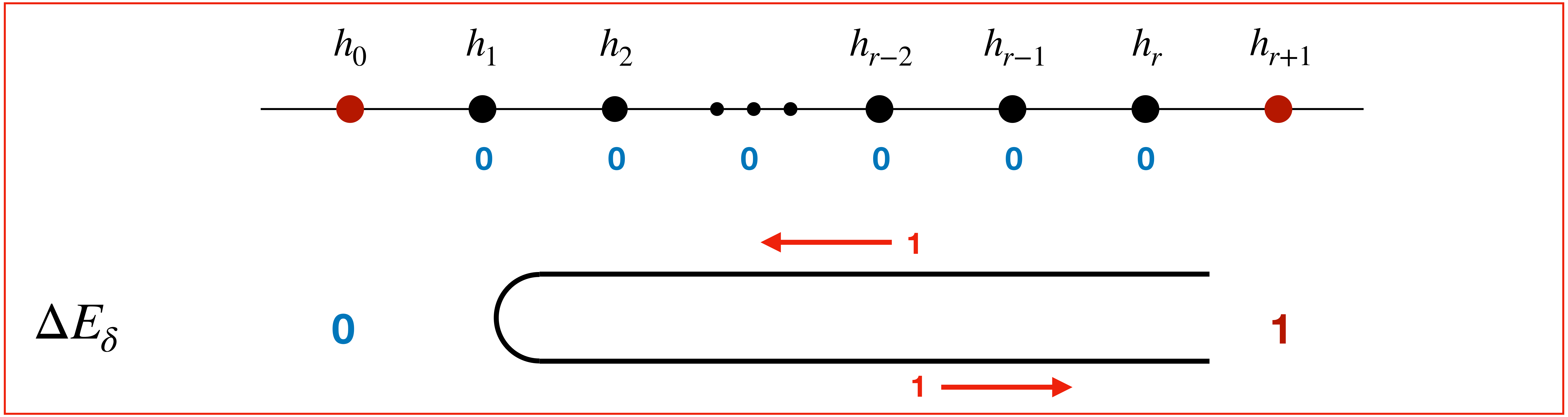
"Upper" light cone (FM; slowest spreading)

Analytically computable!

$$\omega = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$ **(nonzero lowest order)**

$$= \left(\frac{J}{2}\right)^{2r} \cdot 2J_z \cdot \left[\frac{1}{\left(\prod_{i=1}^{r-1} F_i\right)^2 F_r (F_r - J_z)} + \frac{1}{G_1(G_1 - J_z)\left(\prod_{i=2}^{r} G_i\right)^2} + \sum_{k=1}^{r-1} \frac{F_k + G_{k+1}}{\left(\prod_{i=1}^{k} F_i \cdot \prod_{j=k+1}^{r} G_j\right)^2 \cdot (F_k + G_{k+1} - J_z)} \right]$$

$$F_i \equiv h_0 - h_i \qquad G_i \equiv h_{r+1} - h_i$$



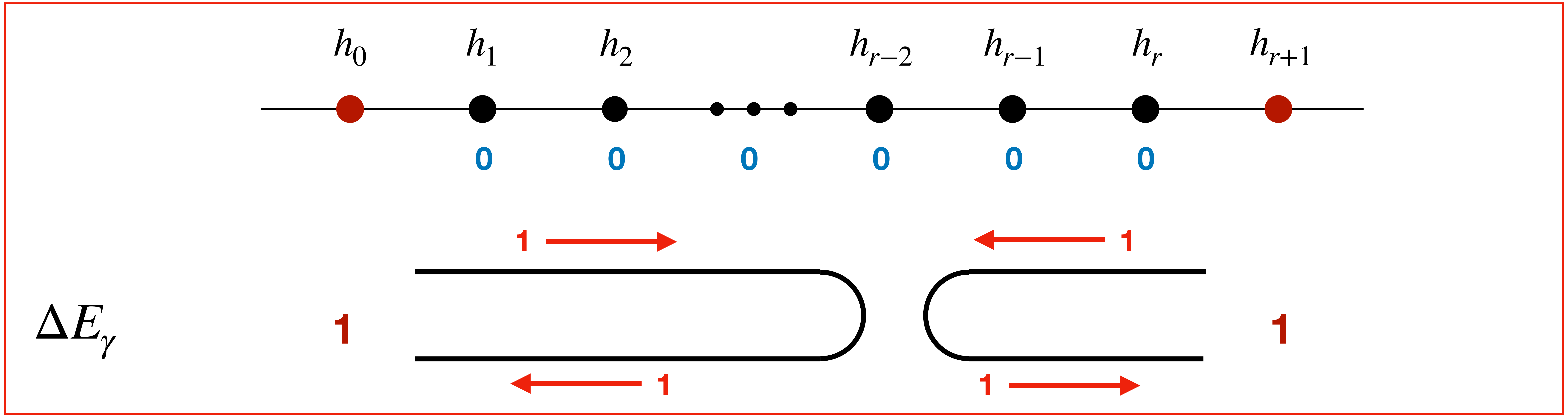
"Upper" light cone (FM; slowest spreading)

Analytically computable!

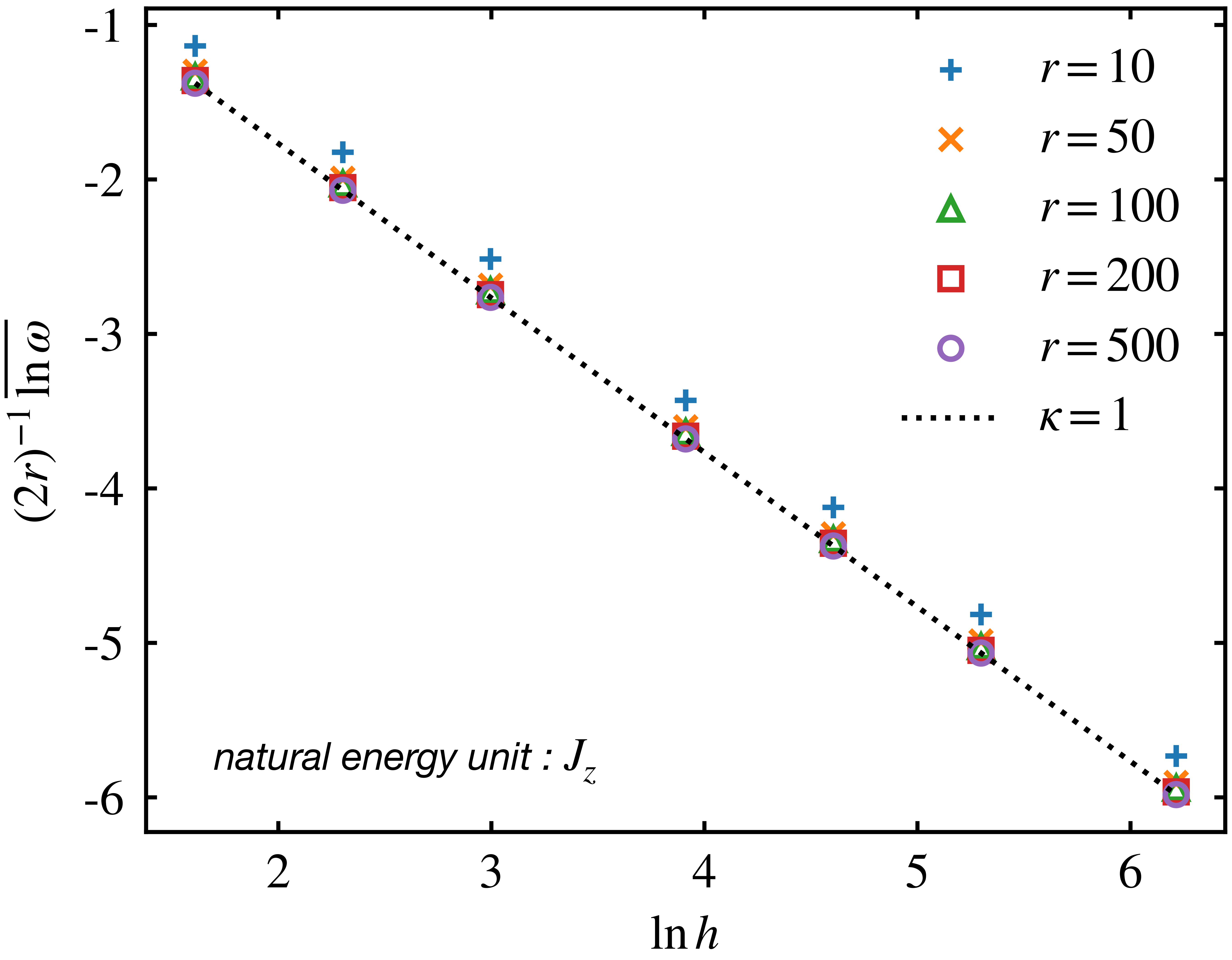
$$\omega = \Delta E_\alpha - \Delta E_\beta + \Delta E_\gamma - \Delta E_\delta$$ **(nonzero lowest order)**

$$= \left(\frac{J}{2}\right)^{2r} \cdot 2J_z \cdot \left[\frac{1}{\left(\prod_{i=1}^{r-1} F_i\right)^2 F_r (F_r - J_z)} + \frac{1}{G_1 (G_1 - J_z) \left(\prod_{i=2}^{r} G_i\right)^2} + \sum_{k=1}^{r-1} \frac{F_k + G_{k+1}}{\left(\prod_{i=1}^{k} F_i \cdot \prod_{j=k+1}^{r} G_j\right)^2 \cdot (F_k + G_{k+1} - J_z)} \right]$$

$$F_i \equiv h_0 - h_i \qquad\qquad G_i \equiv h_{r+1} - h_i$$



"Upper" light cone : scaling behavior



$$\overline{\ln \omega} \propto \ln[(ah)^{-2r}]$$

regardless of "r"

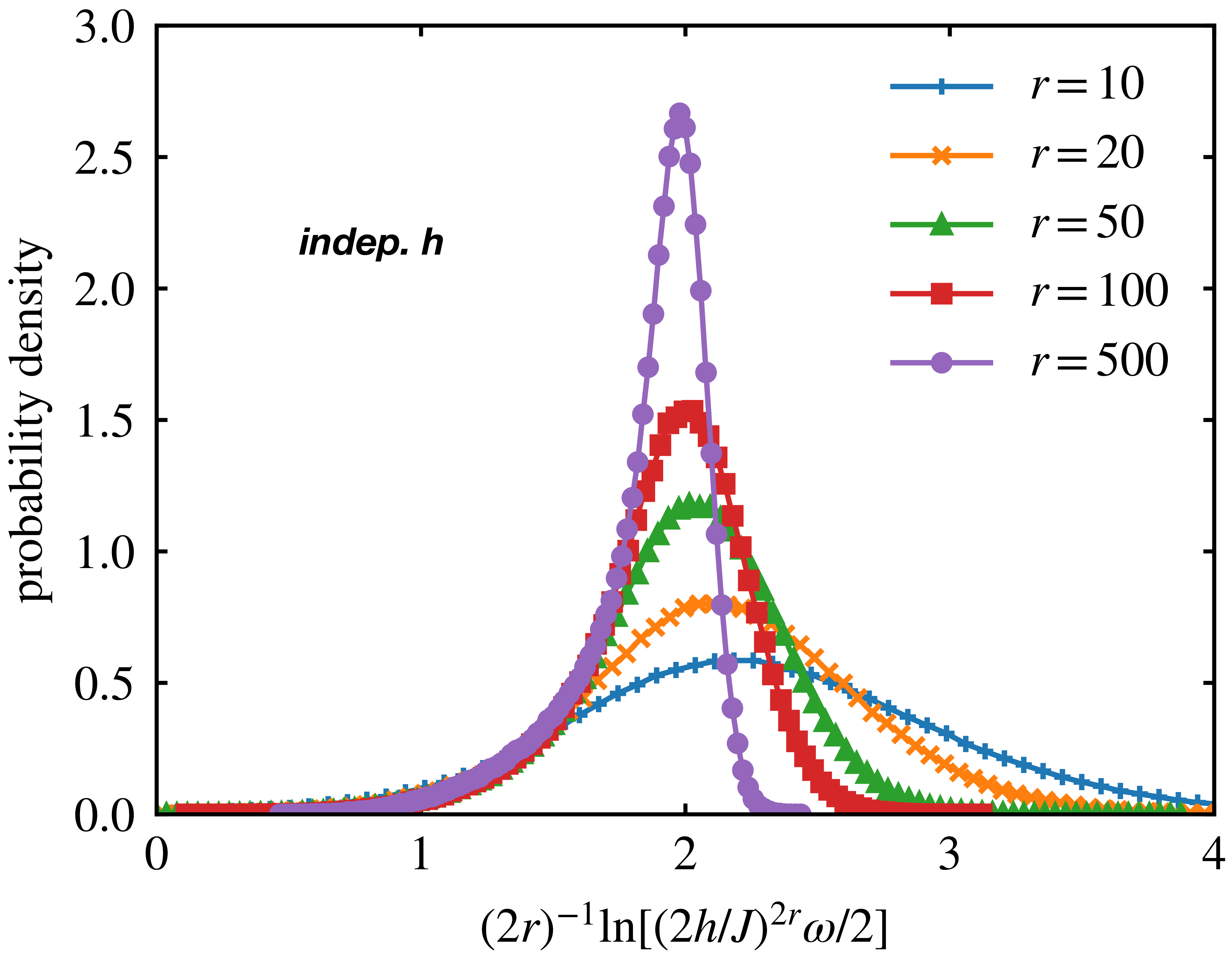
At a large *h*,

$$\omega_{\text{FM}}(r,h) \rightarrow 2\left(\frac{J}{2h}\right)^{2r} \exp(2r\underline{\tilde{\omega}})$$

normalized random variable

NOTE: multiprecision library is used. (500 digits)

"Upper" light cone : distribution of $\tilde{\omega}$



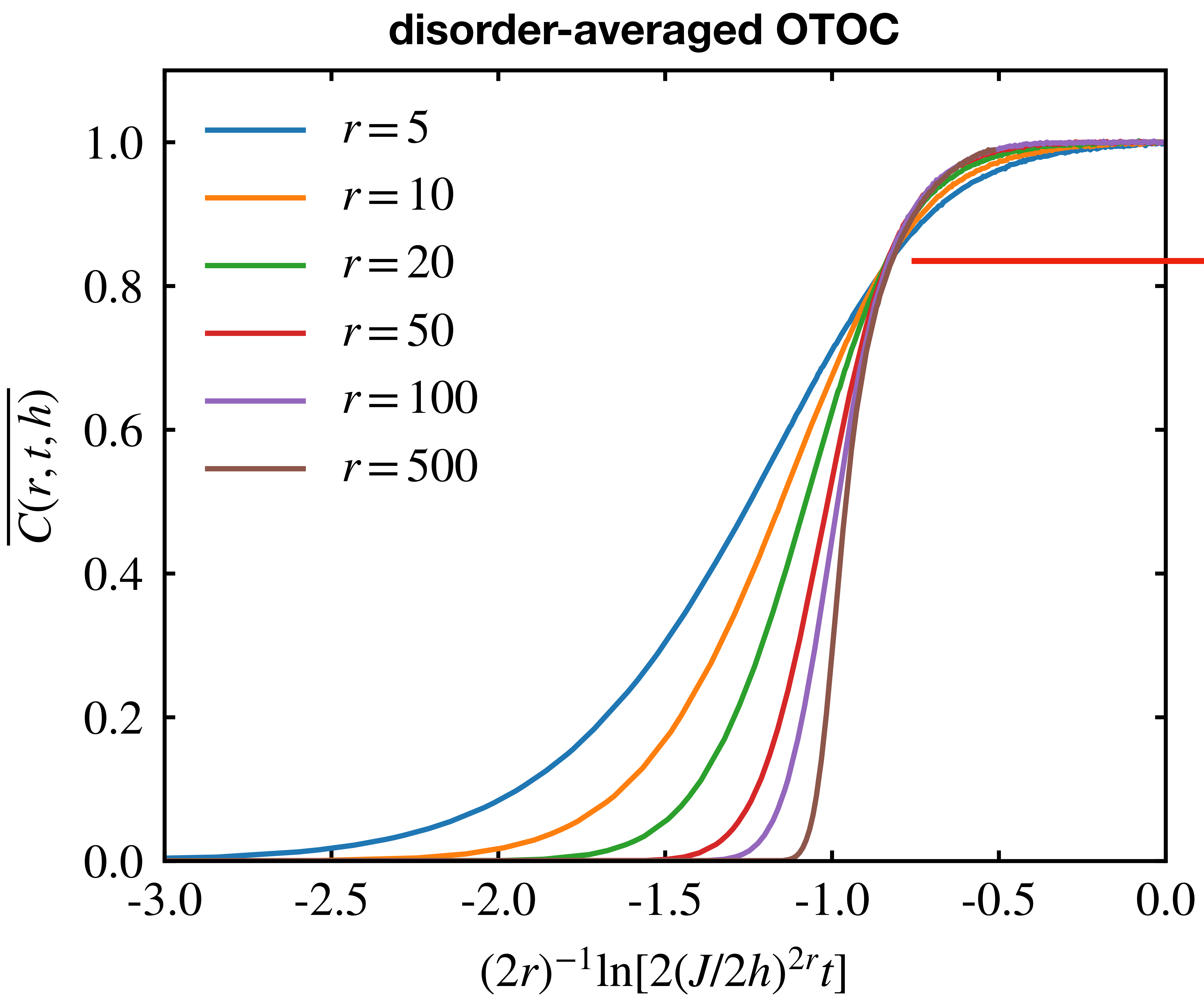
Increasingly sharp as *r* increases.

Most probable value is meaningful.

Growth of OTOC becomes sharp in a relative scale.

$$\frac{\delta[\ln t]}{r}$$

"Upper" light cone : distribution of $\tilde{\omega}$



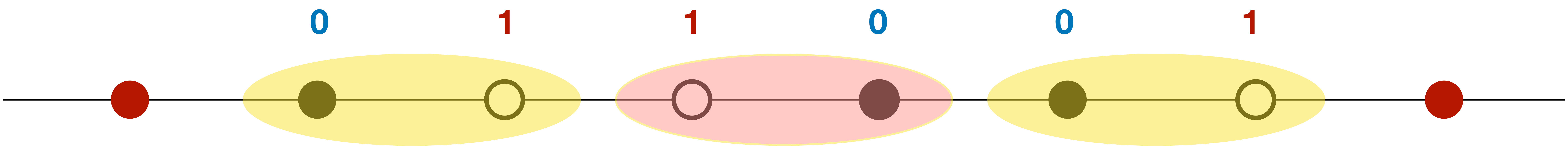
$$\ln[2(J/2h)^{2r}t_*] \propto -2r$$

$$\ln t_* \propto r$$

logarithmic light cone!

"Lower" light cone (AFM; fastest spreading)

Original XXZ Hamiltonian : $\hat{H} = -\frac{J}{2}\sum_i (\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y\hat{\sigma}_{i+1}^y) - \frac{J_z}{2}\sum_i \hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + \sum_i h_i\hat{\sigma}_i^z$

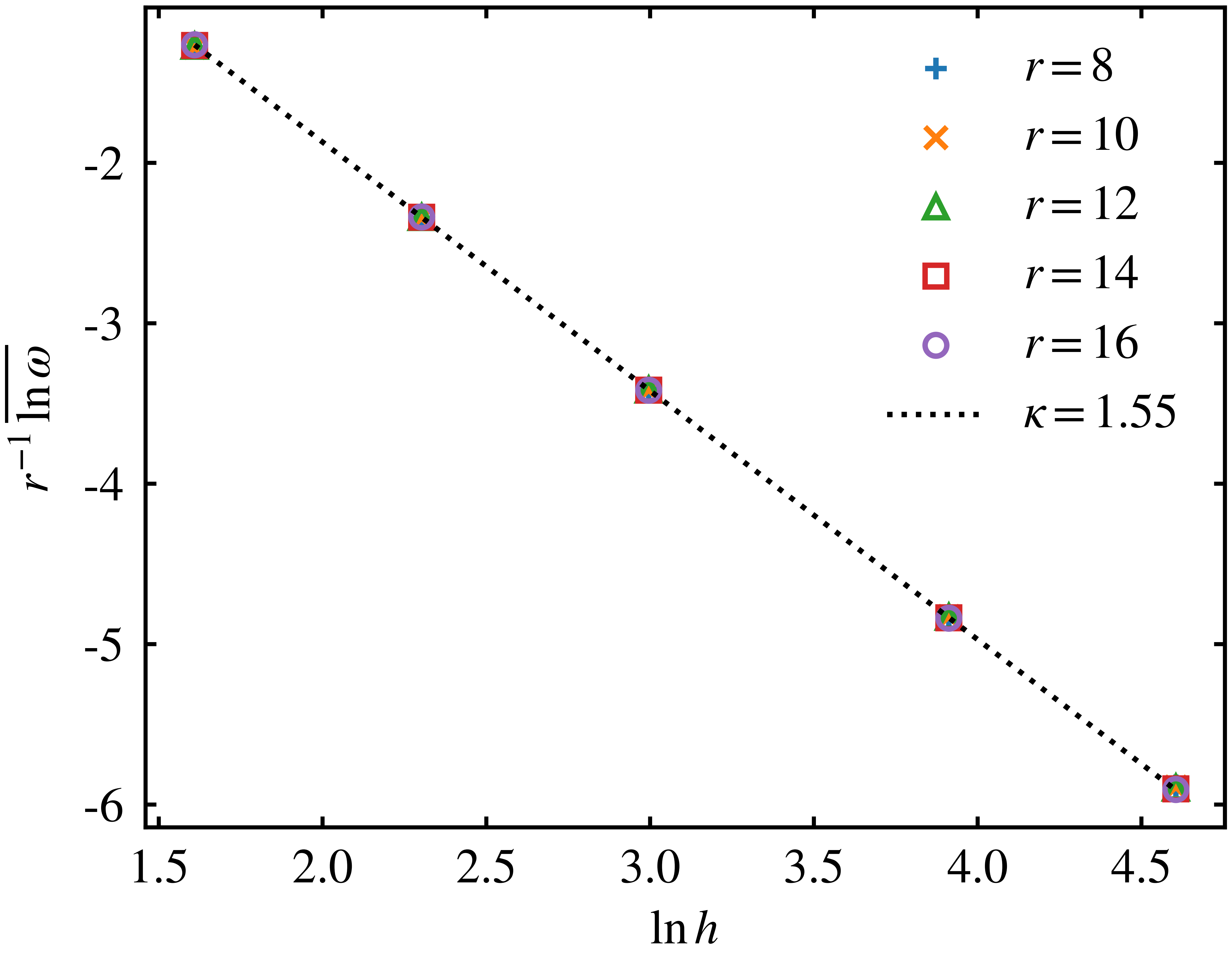


$|\Uparrow\rangle$ $|\Downarrow\rangle$ $|\Uparrow\rangle$

with just a half of the chain length : $l' = l/2$

$$\hat{H}_{\mathrm{eff}} = -\frac{J_z}{2}\sum_{i=0}^{l'} \hat{\sigma}_i^z\hat{\sigma}_{i+1}^z + \sum_{i=1}^{l'} \Delta_i\hat{\sigma}_i^z + h_0\hat{\sigma}_0^z - h_{l'+1}\hat{\sigma}_{l'+1}^z + J\sum_{i=1}^{l'} \hat{\sigma}_i^x$$

"Lower" light cone : scaling behavior



$$\overline{\ln\omega} \propto \ln[a^r h^{-\kappa r}]$$

$$\kappa \approx 1.55$$ **(3/2 ?)**

At a large *h*,

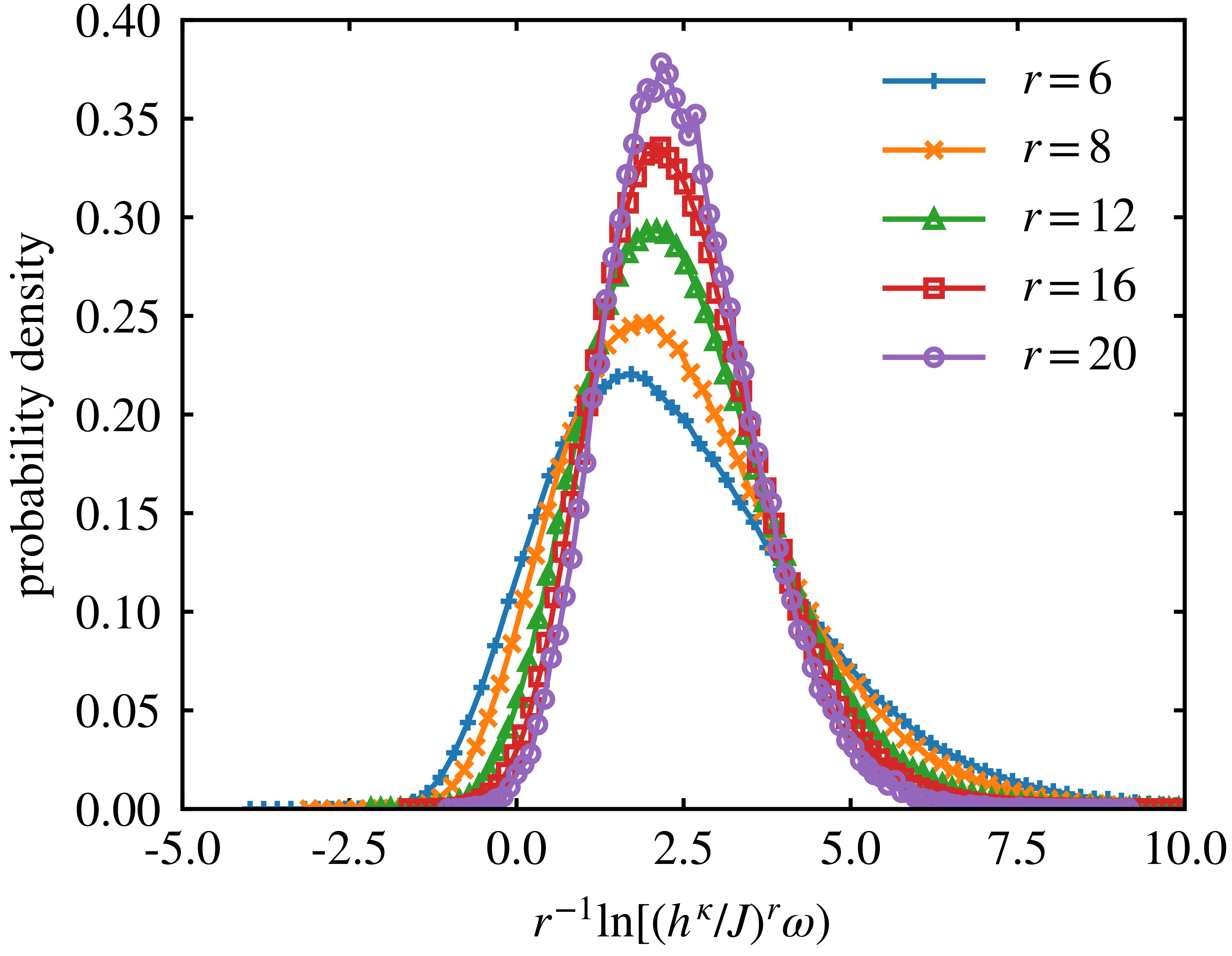
$$\omega_{\mathrm{AF}}(r,h) \to \left(\frac{J}{h^{\kappa}}\right)^{r} \exp(r\underline{\tilde{\omega}})$$

normalized random variable

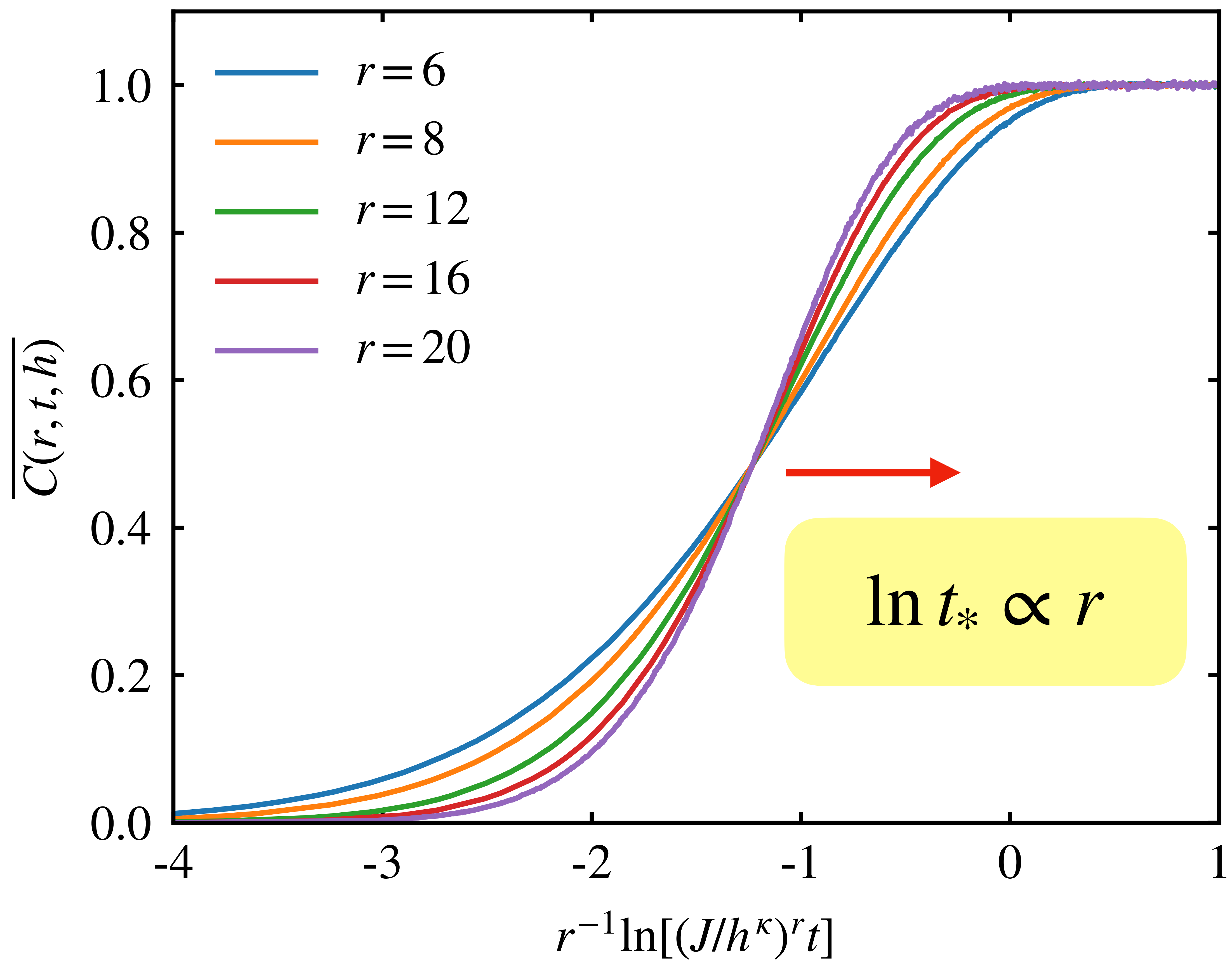
NOTE: multiprecision library is used. (500 digits)

"Lower" light cone : well-define, logarithmic

Increasingly sharp distribution

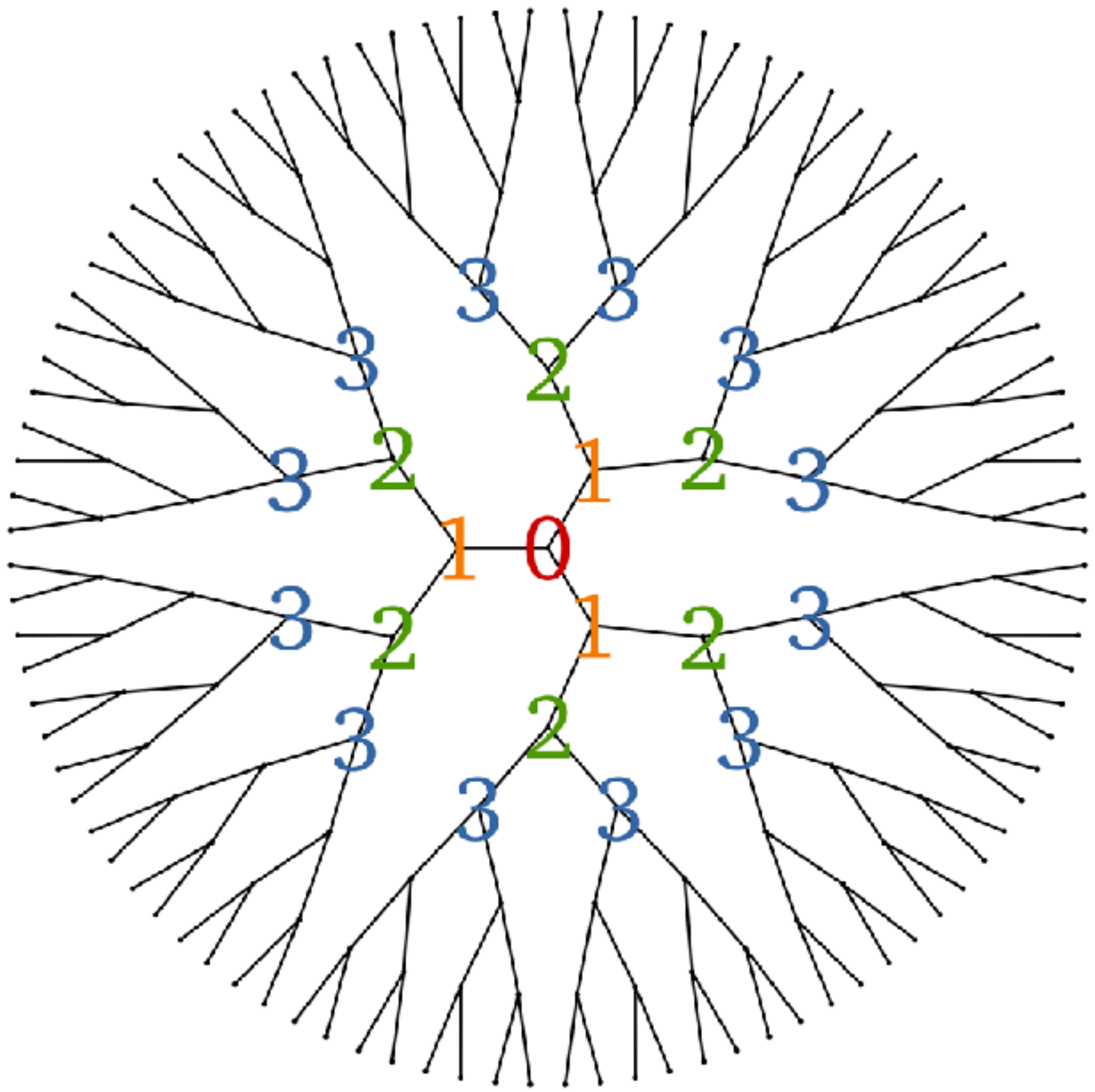


disorder-averaged OTOC in a scaled log-time



Logarithmic spreading in other lattice geometry?
higher dimensions?

The perturbation calculations can be generalized to all tree-like geometry.



For example, Bethe lattices.

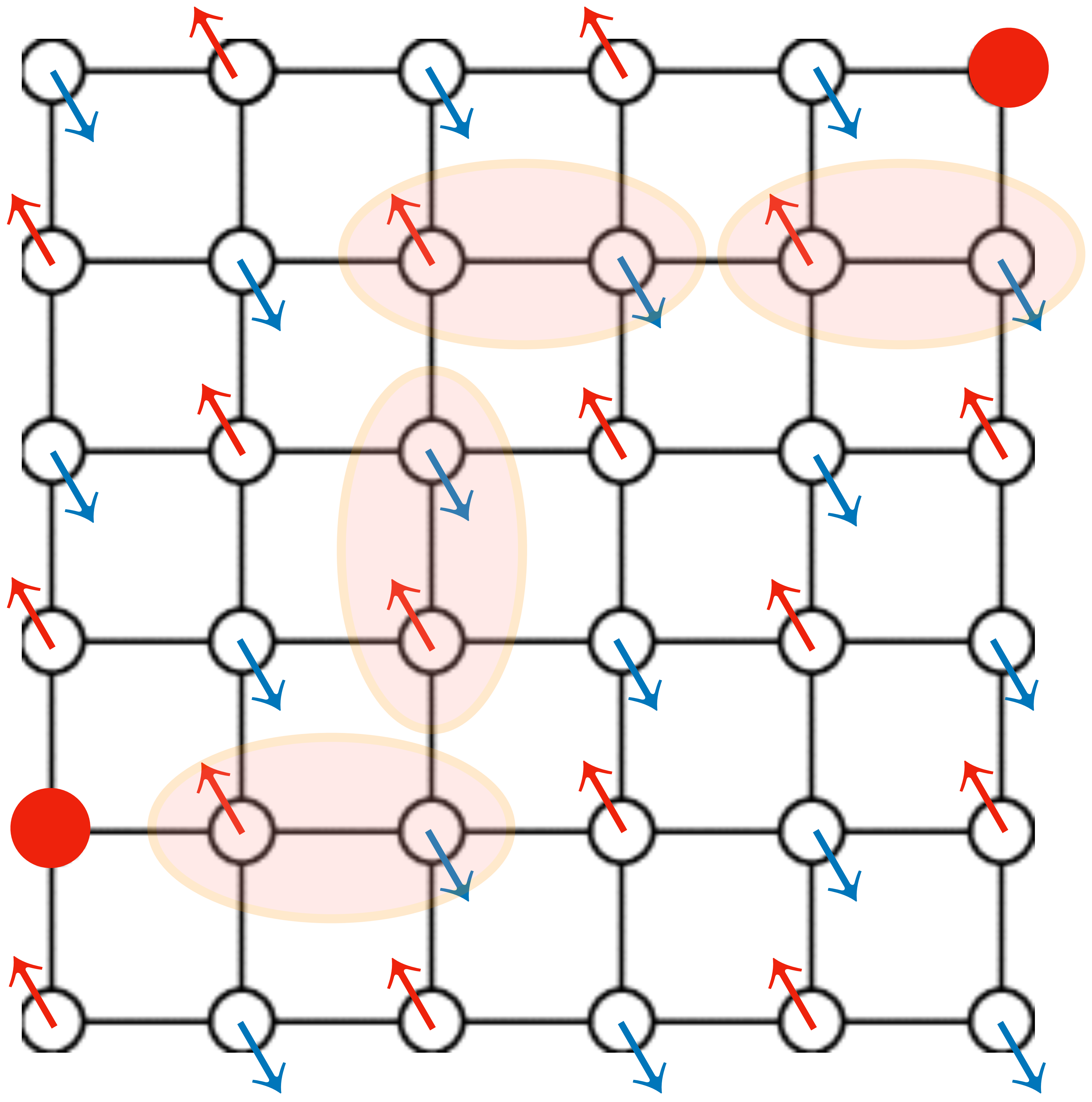
Nothing changes in the perturbation expansion.

$\longrightarrow$ **Logarithmic light cone!**

What about then, in 2D square lattices?

Logarithmic light cone in 2D : AFM

AFM



$$C_\alpha(i,j;t) = 1 - \sum_{\beta,\gamma,\delta} s_{\alpha\beta\gamma\delta} \cos\left(i\omega_{\alpha\beta\gamma\delta} t\right) \approx 1 - \cos\left(\omega_{\alpha\beta\gamma\delta} t\right)$$

No loop allows in the lowest-order term!

$$\omega = \sum_{k \in \text{shortest paths}} \omega_k$$

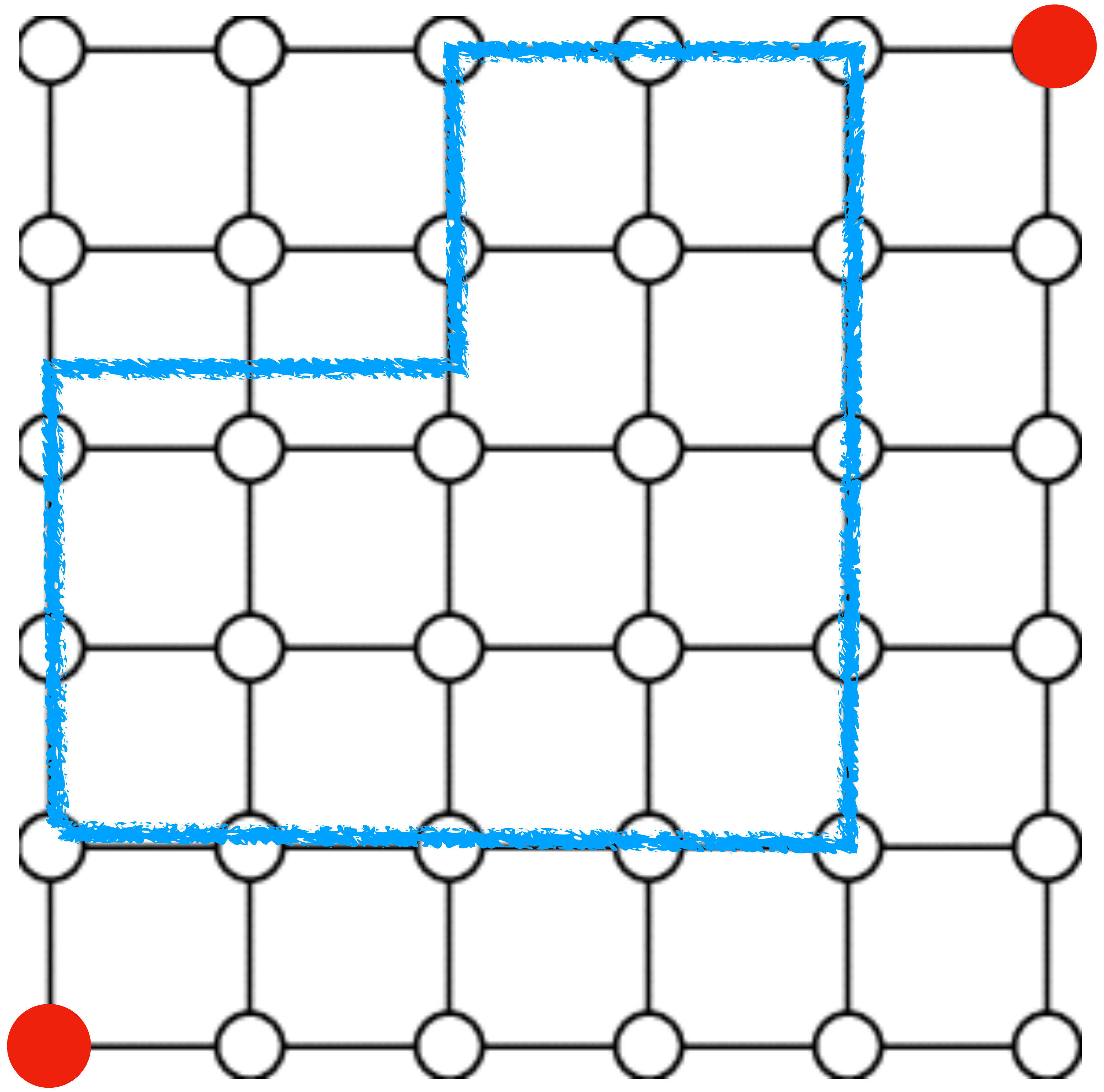
$$\ln[\text{number of shortest paths}] \sim L$$

If there is no disorder correlation between the paths (but there should be),

$$\ln \omega \sim L$$

Logarithmic light cone in 2D : FM

Analytic expression can be obtained.



Now we have loop contributions.

My pure "speculation" :)

The loop terms that have plus and minus signs in the random disorder ensemble would be canceled out among themselves.

$$\overline{\omega} \approx \sum_{k \in \text{shortest paths}} \overline{\omega_k}$$

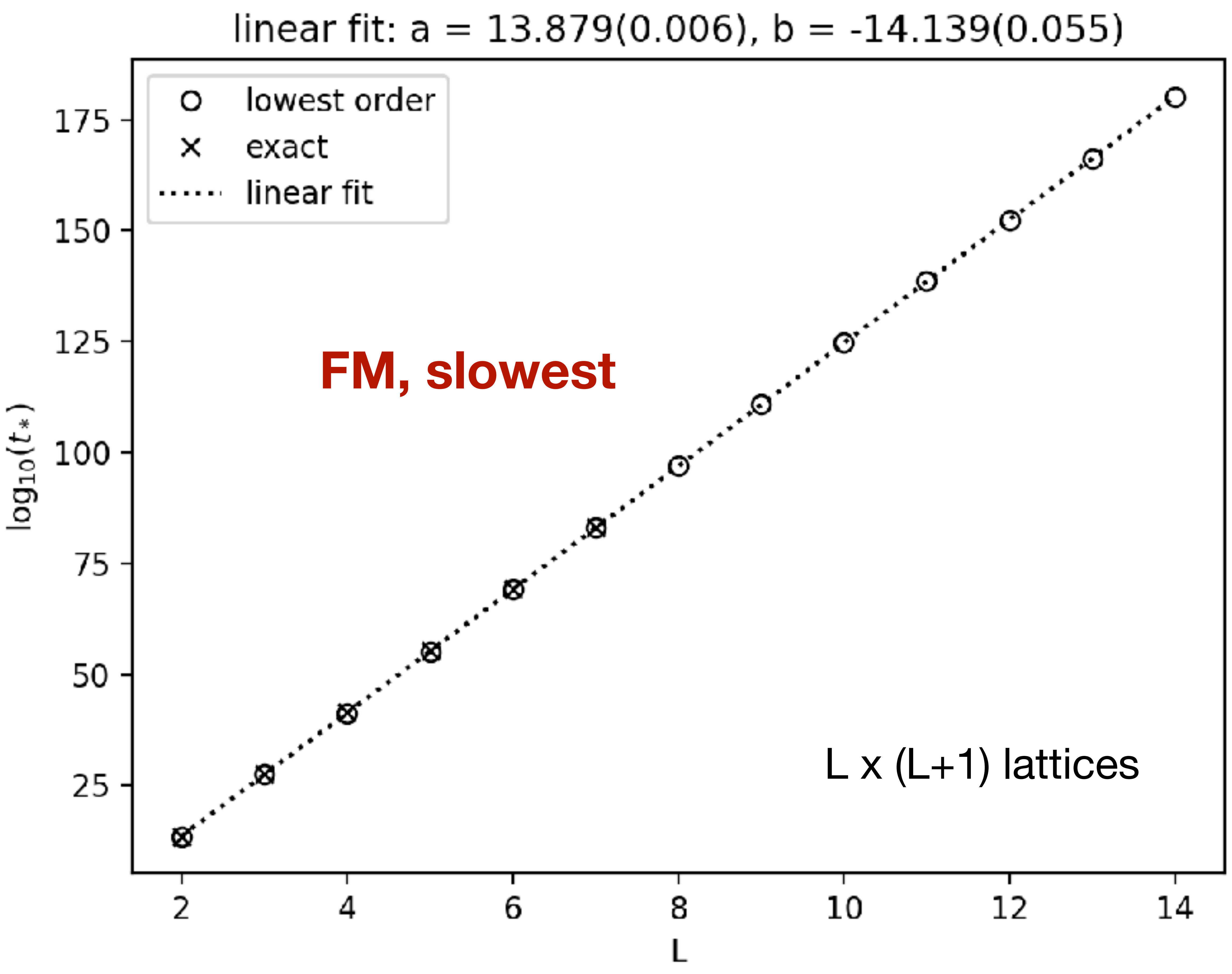
$$\downarrow$$

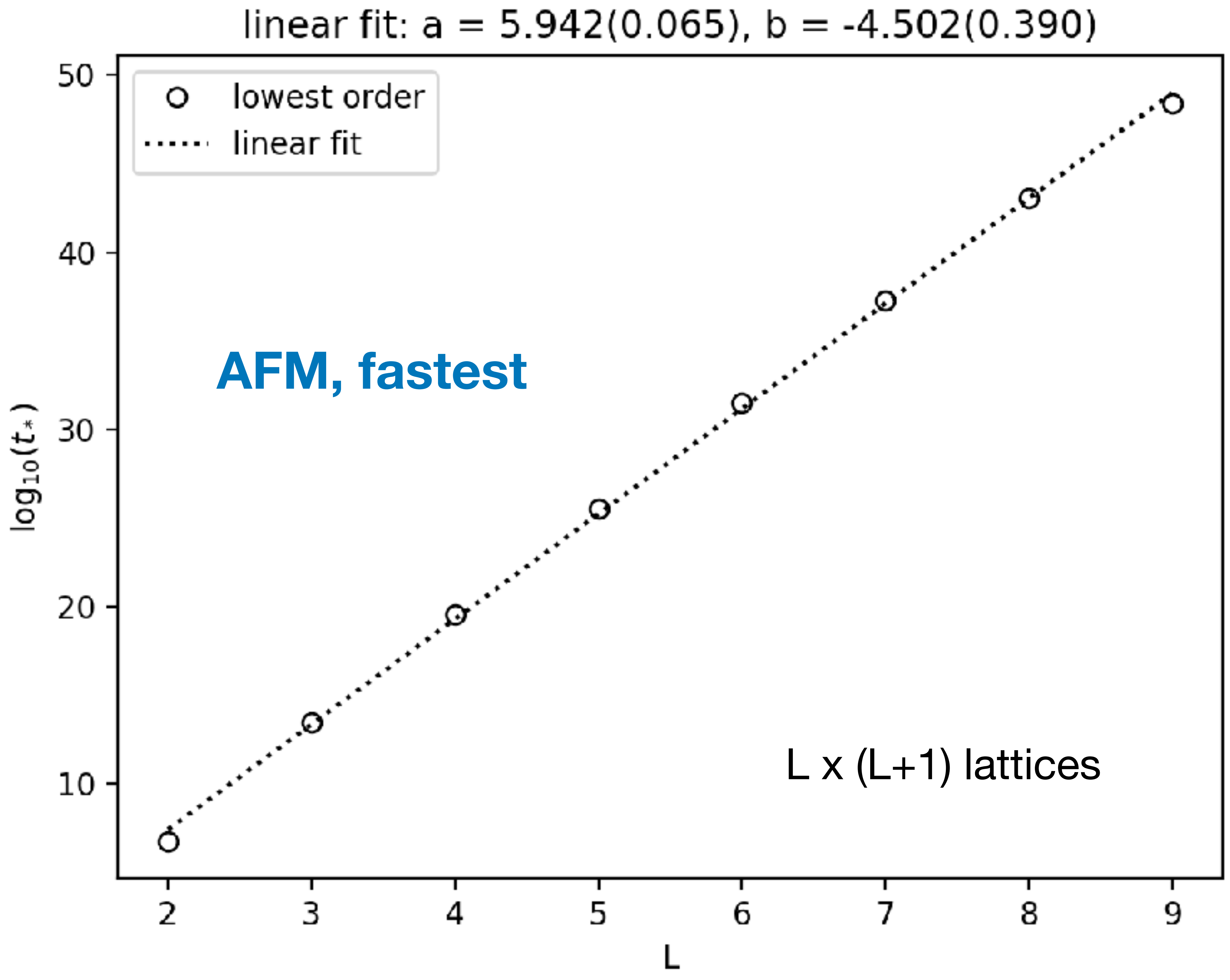
$$\ln \omega \sim L$$

Information spreading in strongly disordered 2D XXZ model

The light cone still looks logarithmic in 2D!

2D MBL?

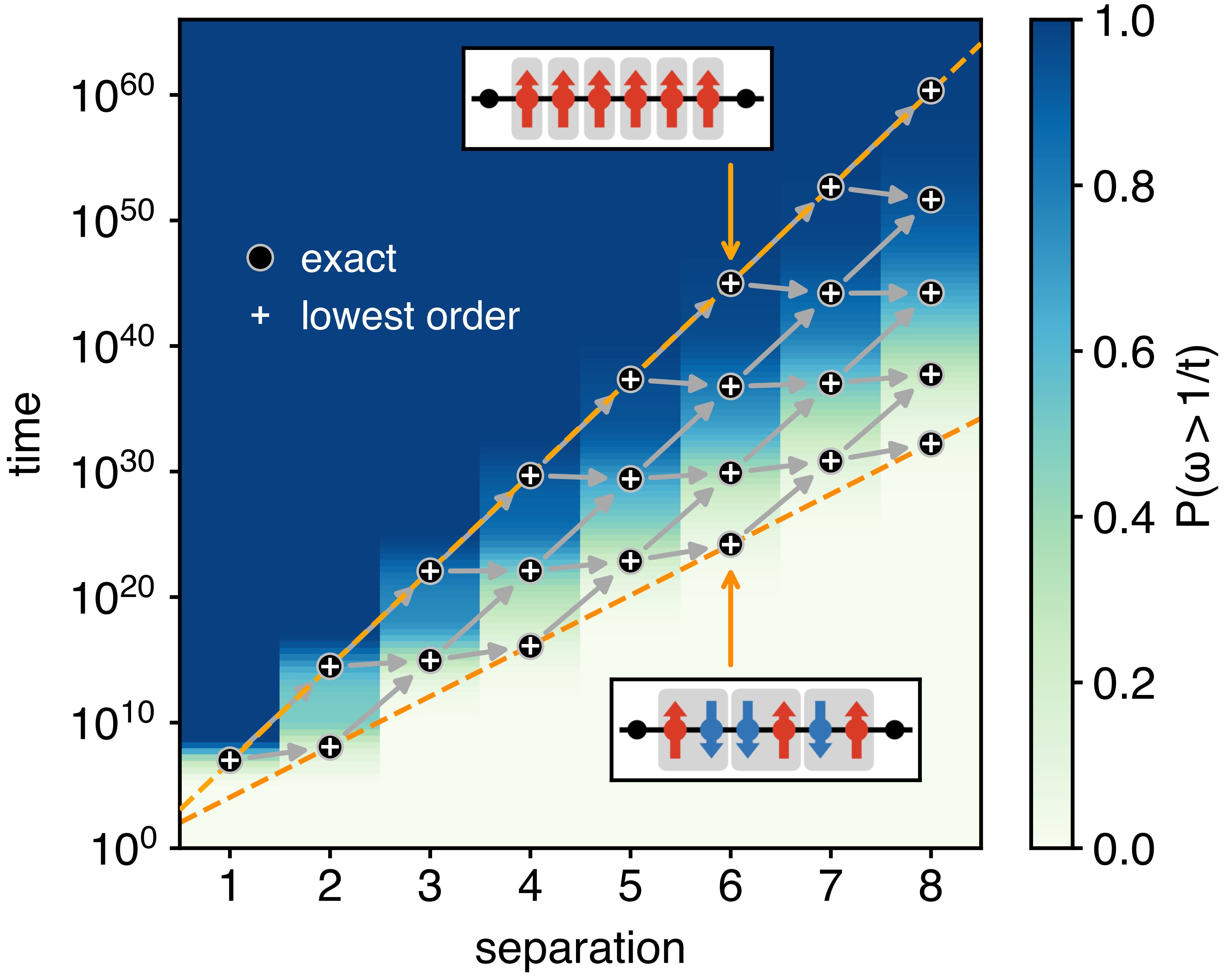




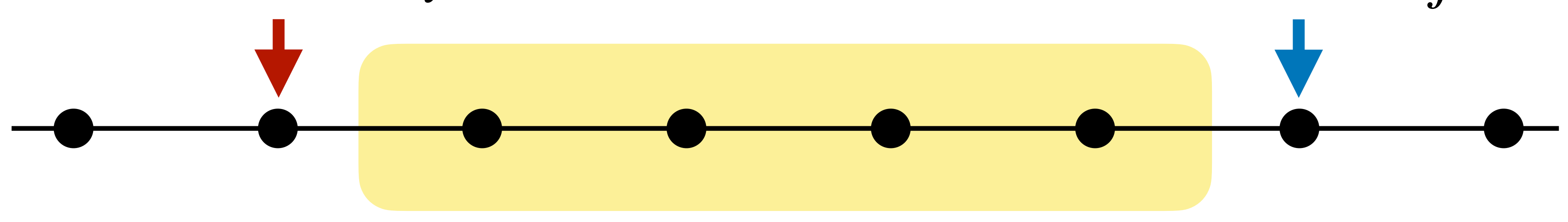
Summary

- We have developed the perturbative method to compute the OTOC for the MBL phase of the *strongly disordered* XXZ chain in the *weak-hopping* limit.

- Lowest-order perturbation calculations reveal the logarithmic light cone in 1D and strongly suggests the similar slow logarithmic information spreading in 2D.



$\hat{W}(t) = \hat{\sigma}_i^x(t)$ $\hat{V} = \hat{\sigma}_j^x$



separation "*r*"

$$C_\alpha(r, t, h) \approx 1 - \cos[\omega_\alpha(r, h) \cdot t]$$

$$[\omega_{\mathrm{FM}}] \sim \left(\frac{J}{h}\right)^{2r} \qquad [\omega_{\mathrm{AF}}] \sim \left(\frac{J}{h^\kappa}\right)^{r}$$