Enhancing synchronization in the growing systems of oscillators

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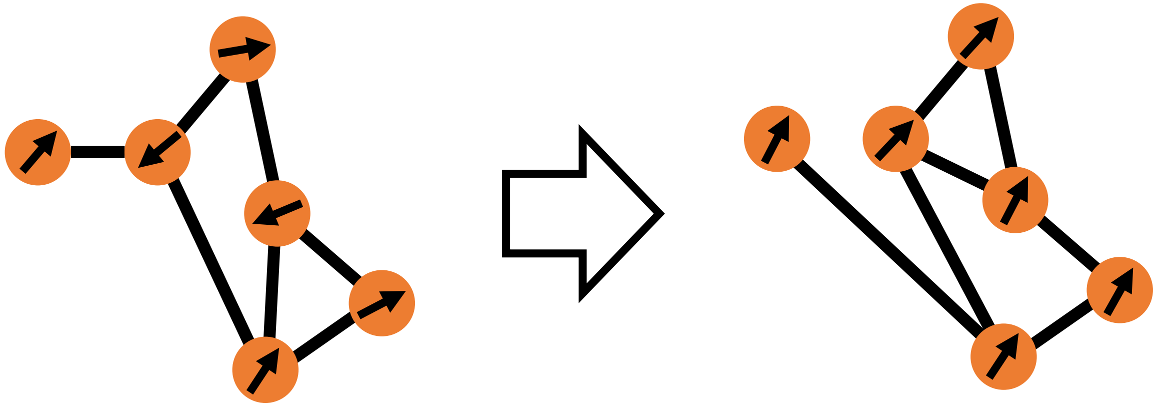
Introduction

- Synchronization
  - observed in various systems
    flashing fireflies, chirping crickets, clapping audiences, …
  - required for the proper function of artificial or living systems
    power-grid system, circadian rhythm, heart cells, neural networks, …
  - relevant questions
    What is the optimal condition for maximum synchronization?
    What is the best way to enhance the synchronization?

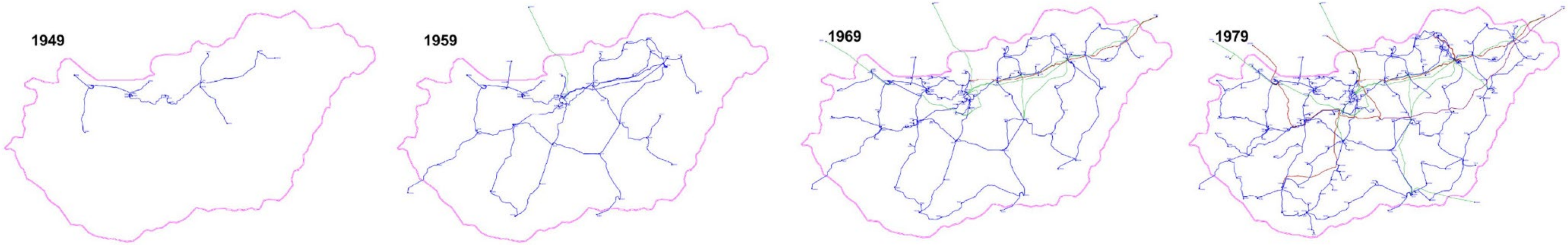
Motivation

- What is the optimal [interaction structure / oscillator allocation / individual property assignment]?



[Skardal, Taylor, Sun, PRL (2014)]
[Pinto, Saa, PRE (2015)]

- Real systems are growing → what is the effect of growth on synchronization?

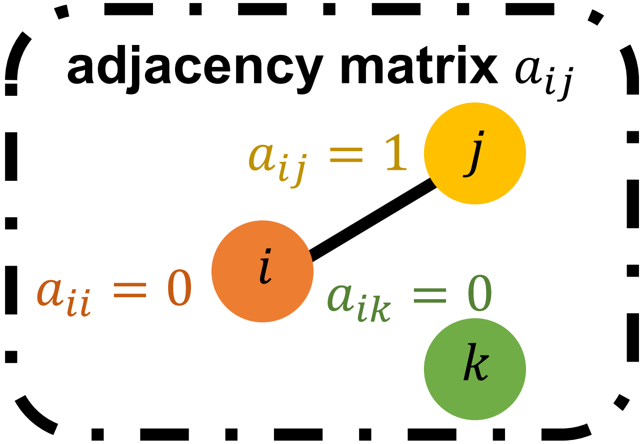


evolution of Hungarian power grid
[Hartmann and Sugar, Scientific Reports (2021)]

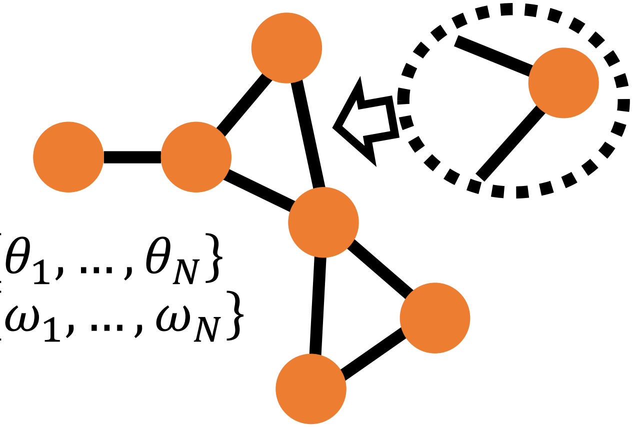
Model

- equation of motion for phase θ_i $(i = 1, \dots, N)$

$$\dot{\theta}_i(t) = \omega_i - K \sum_j a_{ij} \sin\left(\theta_i(t) - \theta_j(t)\right)$$

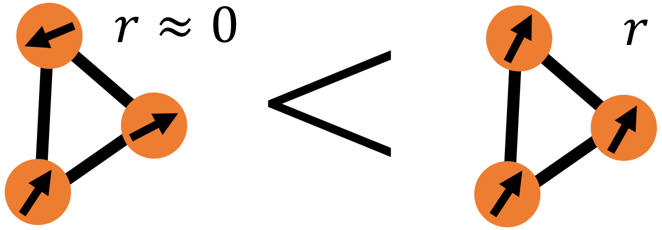


- growing process of network
  - New node $i = 0$ is added.
  - New links are connected under a given rule.
  - System relaxes to the stationary state.



- measure of synchronization: order parameter

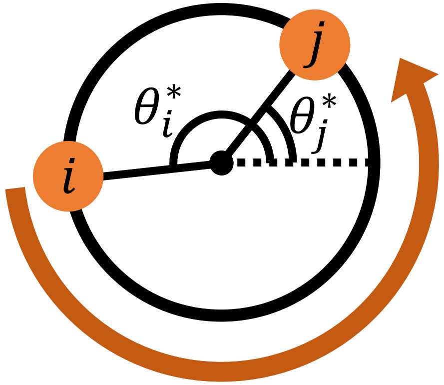
$$r = \left| \frac{1}{N+1} \sum_{j=0}^{N} e^{i\theta_j^*} \right|$$



What is the optimal natural frequency $\omega_0^* = \mathrm{argmax}\{r(\omega_0)\}$?

Method

- stationary state condition: $\theta_i(t) = \theta_i^* + \bar{\omega} t$



$$\bar{\omega} = \frac{1}{N+1}\sum_{i=0}^{N} \omega_i$$

$$\bar{\omega} = \omega_i - K \sum_j a_{ij} \sin(\theta_i^* - \theta_j^*)$$

$$\theta_i^* = \cdots \quad r = \left| \frac{1}{N+1} \sum_{j=0}^{N} e^{i\theta_j^*} \right| = r(\omega_0)$$

- strong coupling limit $K \gg \omega_i \rightarrow \sin(\theta_i^* - \theta_j^*) \approx \theta_i^* - \theta_j^*$
  - linearized dynamics

$$\boldsymbol{\theta}^* = \frac{1}{K} \mathbf{L}^\dagger (\boldsymbol{\omega} - \bar{\omega}\mathbf{1})$$

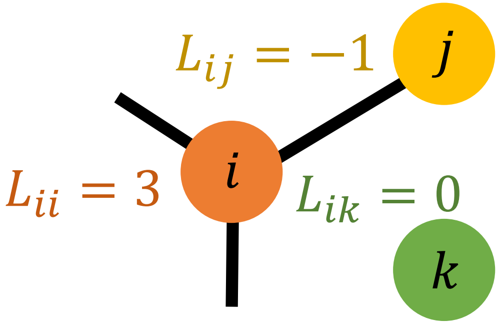
  - order parameter

$$r \approx 1 - \frac{1}{2(N+1)K^2} (\boldsymbol{\omega} - \bar{\omega}\mathbf{1})^{\mathrm{T}} (\mathbf{L}^\dagger)^2 (\boldsymbol{\omega} - \bar{\omega}\mathbf{1})$$

≡ **objective function**

Laplacian matrix

$$L_{ij} = \left(\textstyle\sum_j a_{ij}\right)\delta_{ij} - a_{ij}$$



Moore-Penrose inverse $\mathbf{L}^\dagger$

Method

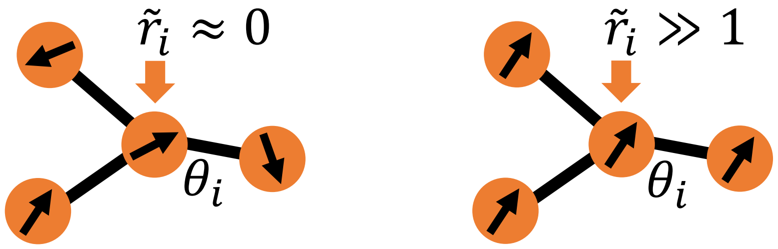
- global order parameter

$$r \overset{K\gg 1}{\approx} 1 - \frac{1}{2(N+1)K^2}(\boldsymbol{\omega} - \bar{\omega}\mathbf{1})^{\mathrm{T}}\left(\mathbf{L}^{\dagger}\right)^2(\boldsymbol{\omega} - \bar{\omega}\mathbf{1})$$

- total local order parameter

$$\tilde{r} = \sum_i \tilde{r}_i\,, \qquad \tilde{r}_i \equiv \sum_j a_{ij}\cos\left(\theta_i(t) - \theta_j(t)\right)$$

$$\tilde{r} \overset{K\gg 1}{\approx} 1 - \frac{1}{K^2}(\boldsymbol{\omega} - \bar{\omega}\mathbf{1})^{\mathrm{T}}\mathbf{L}^{\dagger}(\boldsymbol{\omega} - \bar{\omega}\mathbf{1})$$



- objective functions

$$O_p = (\boldsymbol{\omega} - \bar{\omega}\mathbf{1})^{\mathrm{T}}\left(\mathbf{L}^{\dagger}\right)^p(\boldsymbol{\omega} - \bar{\omega}\mathbf{1})$$

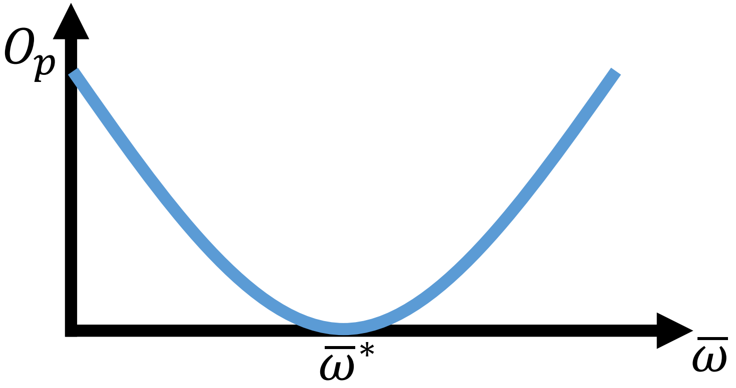
for $p = 1$, adjusted Lyapunov function (ALF)
for $p = 2$, synchrony alignment function (SAF)

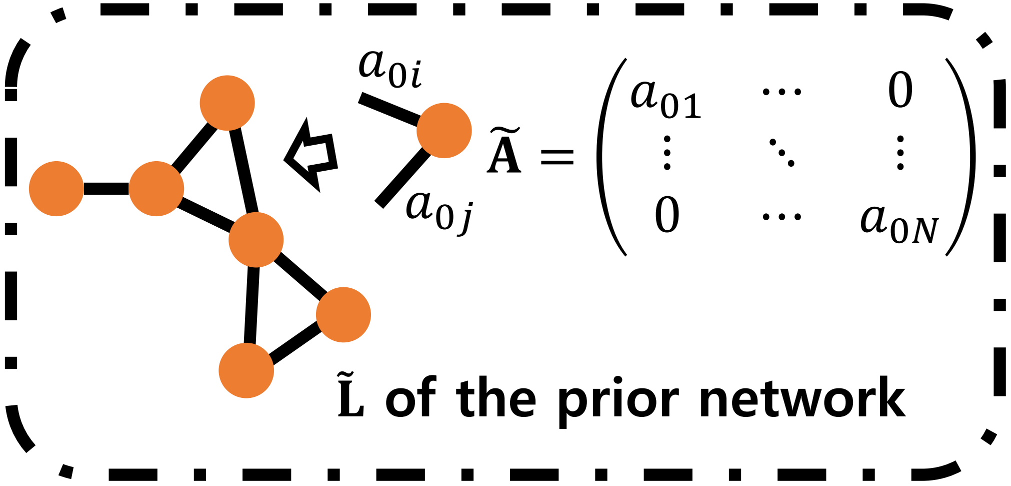
Main result

- change of variables $\{\omega_0, \omega_1, \ldots, \omega_N\} \to \{\overline{\omega}, \omega_1, \ldots, \omega_N\}$

$$O_p(\overline{\omega}) = \sum_{i,j=1}^{N} [\tilde{\mathbf{E}}_p]_{ij}\, \overline{\omega}^2 + 2 \sum_{i,j=1}^{N} [\tilde{\mathbf{E}}_p]_{ij} \omega_j\, \overline{\omega} + \sum_{i,j=1}^{N} [\tilde{\mathbf{E}}_p]_{ij} \omega_i \omega_j$$

$$\tilde{\mathbf{E}}_p = \left(\tilde{\mathbf{L}} + \widetilde{\mathbf{A}}\right)^{-1} \left(\left(\tilde{\mathbf{I}} - \frac{1}{N+1} \tilde{\mathbf{J}} \right) \left(\tilde{\mathbf{L}} + \widetilde{\mathbf{A}}\right)^{-1} \right)^{p-1}$$





- main results

$$\overline{\omega}^* = \sum_{i=1}^{N} [\tilde{\boldsymbol{c}}_p]_i \omega_i, \qquad [\tilde{\boldsymbol{c}}_p]_i = \frac{\sum_{j=1}^{N} [\tilde{\mathbf{E}}_p]_{ij}}{\sum_{i,j=1}^{N} [\tilde{\mathbf{E}}_p]_{ij}}$$

$[\tilde{\boldsymbol{c}}_p]_i$: interaction structure

ω_i: individual property

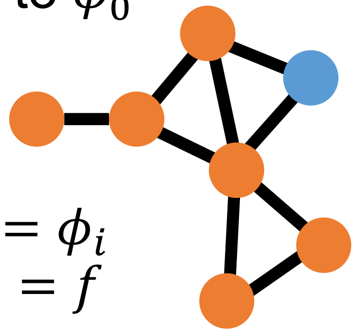
contribution of node i = structural contribution × intrinsic contribution

Interpretation of $\tilde{\boldsymbol{c}}_p$

- What is the physical interpretation of the structure-dependent parts $\tilde{\boldsymbol{c}}_p$?

- We consider a fictitious system
  - All oscillators are identical except the node 0.
  - $[\tilde{\boldsymbol{c}}_1]_i$ is proportional to the relative phase ϕ_i^* with respect to ϕ_0^*

$$[\tilde{\boldsymbol{c}}_1]_i \propto \phi_i^* - \phi_0^*$$



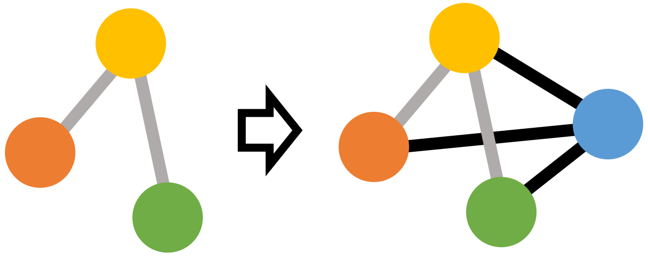
- meaning of $\left[\tilde{\boldsymbol{c}}_p\right]_i$
  - low $\left[\tilde{\boldsymbol{c}}_p\right]_i$ = high synchronization affinity of node i with respect to node 0

$$\omega_0^* = \sum_{i=1}^{N} \underline{\left((N+1)\left[\tilde{\boldsymbol{c}}_p\right]_i - 1\right)} \omega_i \,,$$

Result – simple cases

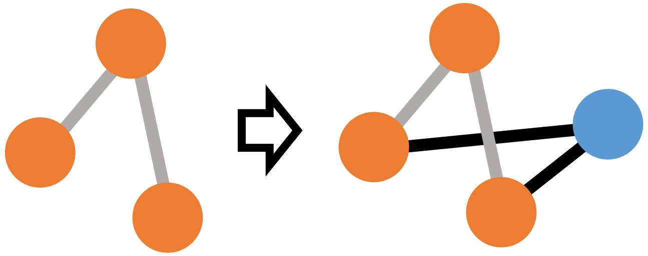
- case I: one to all ($\mathbf{A} = \mathbf{I}$)
  - The optimal frequency is the average of the rest.

$$[\tilde{\boldsymbol{c}}_p]_i = \frac{1}{N} \rightarrow \bar{\omega} = \frac{1}{N}\sum_{i=1}^{N} \omega_i \rightarrow \omega_0^* = \frac{1}{N}\sum_{i=1}^{N} \omega_i$$



- case II: identical oscillators ($\omega_{i \geq 1} = \omega$)
  - The optimal frequency is identical to the others.

$$\sum_{i=1}^{N} [\tilde{\boldsymbol{c}}_p]_i = 1 \rightarrow \bar{\omega} = \sum_{i=1}^{N} [\tilde{\boldsymbol{c}}_p]_i \omega_i = \omega \rightarrow \omega_0^* = \omega$$



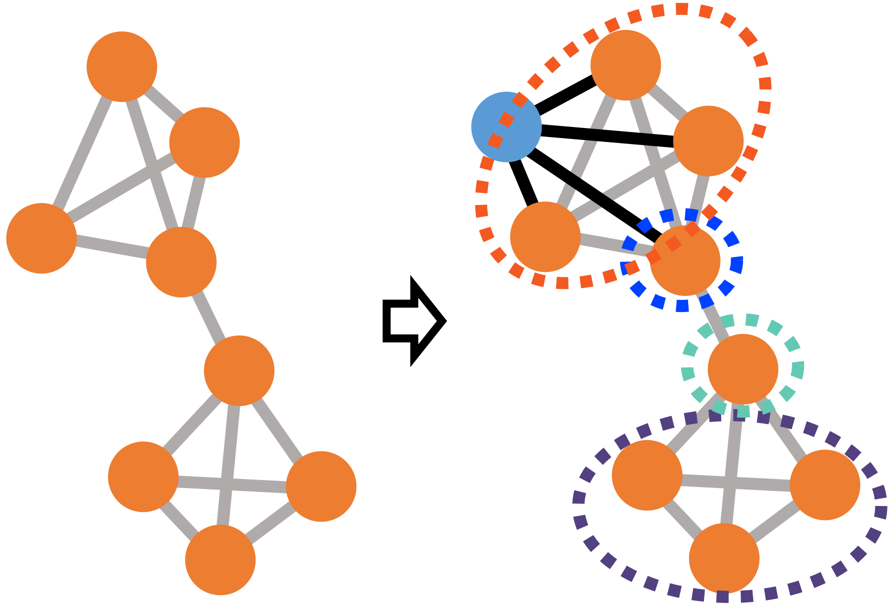
- If the frequencies or connections are homogeneous,
  the optimal frequency equals to the average frequency.

Result – two-clique network

- case III: weakly connected cliques
  - Nodes in the opposite community give larger contribution
  - Neighbors have less influence: counter intuitive

$$\tilde{\boldsymbol{c}}_1 \sim \frac{1}{N^2}\begin{pmatrix} 2 \\ \vdots \\ 3 \\ N-3 \\ \vdots \\ N-2 \end{pmatrix} + \mathcal{O}\left(\frac{1}{N^3}\right)$$

$$\tilde{\boldsymbol{c}}_2 \sim \frac{1}{N^4}\begin{pmatrix} 4 \\ \vdots \\ N^2-3N+14 \\ N^3-N^2+2N-10 \\ \vdots \\ N^3-3 \end{pmatrix} + \mathcal{O}\left(\frac{1}{N^5}\right)$$



Result – random growing network

- random initial frequency $\omega_{i\geq 1} \sim \mathcal{N}(0,1)$ and random connection with $p = 0.2$

Global Average (GA)

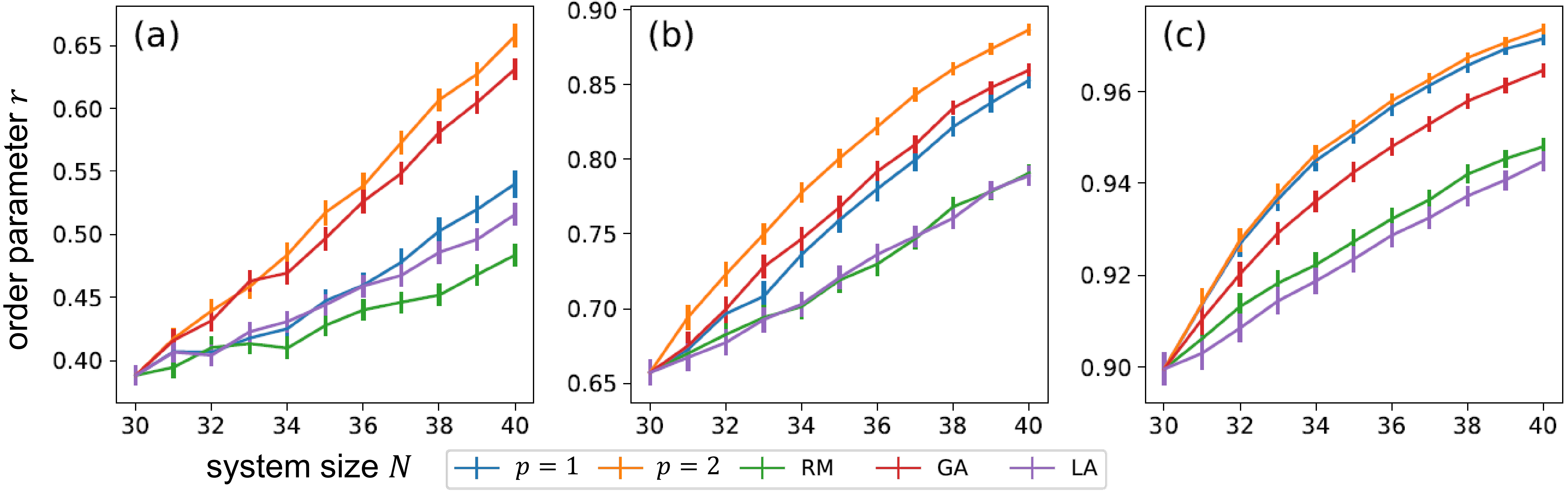
$$\omega_0 = \frac{1}{N}\sum_{i=1}^{N}\omega_i \leftarrow O_{p=0}$$

Local Average (LA)

$$\omega_0 = \frac{1}{k_0}\sum_{i=1}^{N}a_{0,i}\omega_i$$

Random Method (RM)

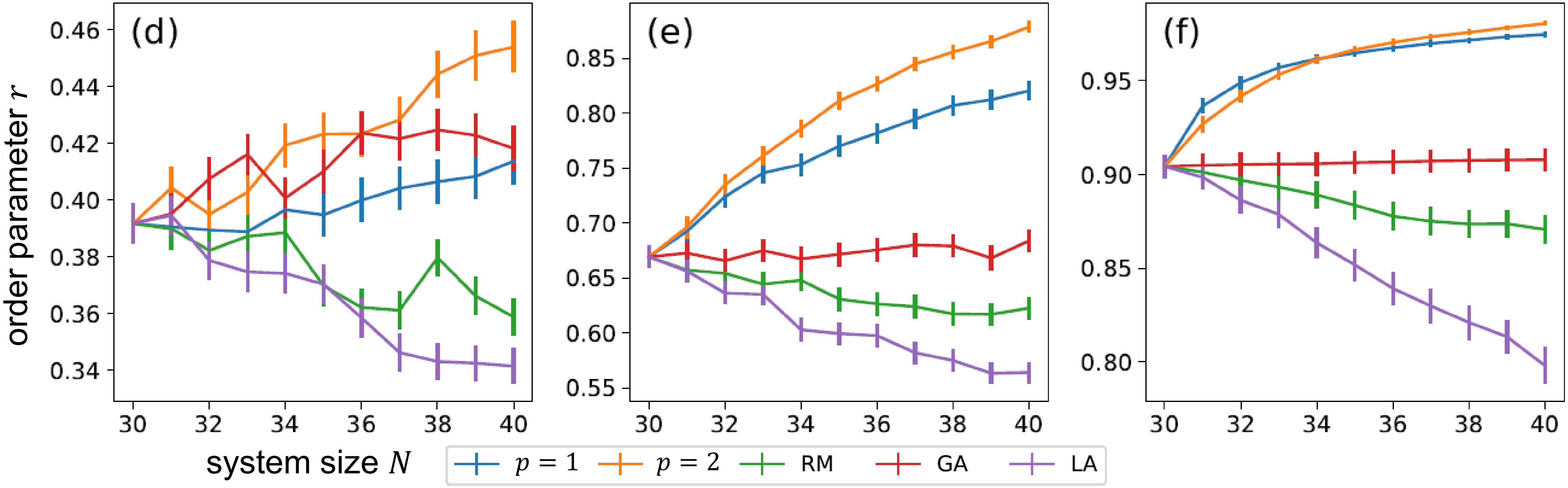
$$\omega_0 \sim \mathcal{N}(0,1)$$



Result – Barabasi-Albert network

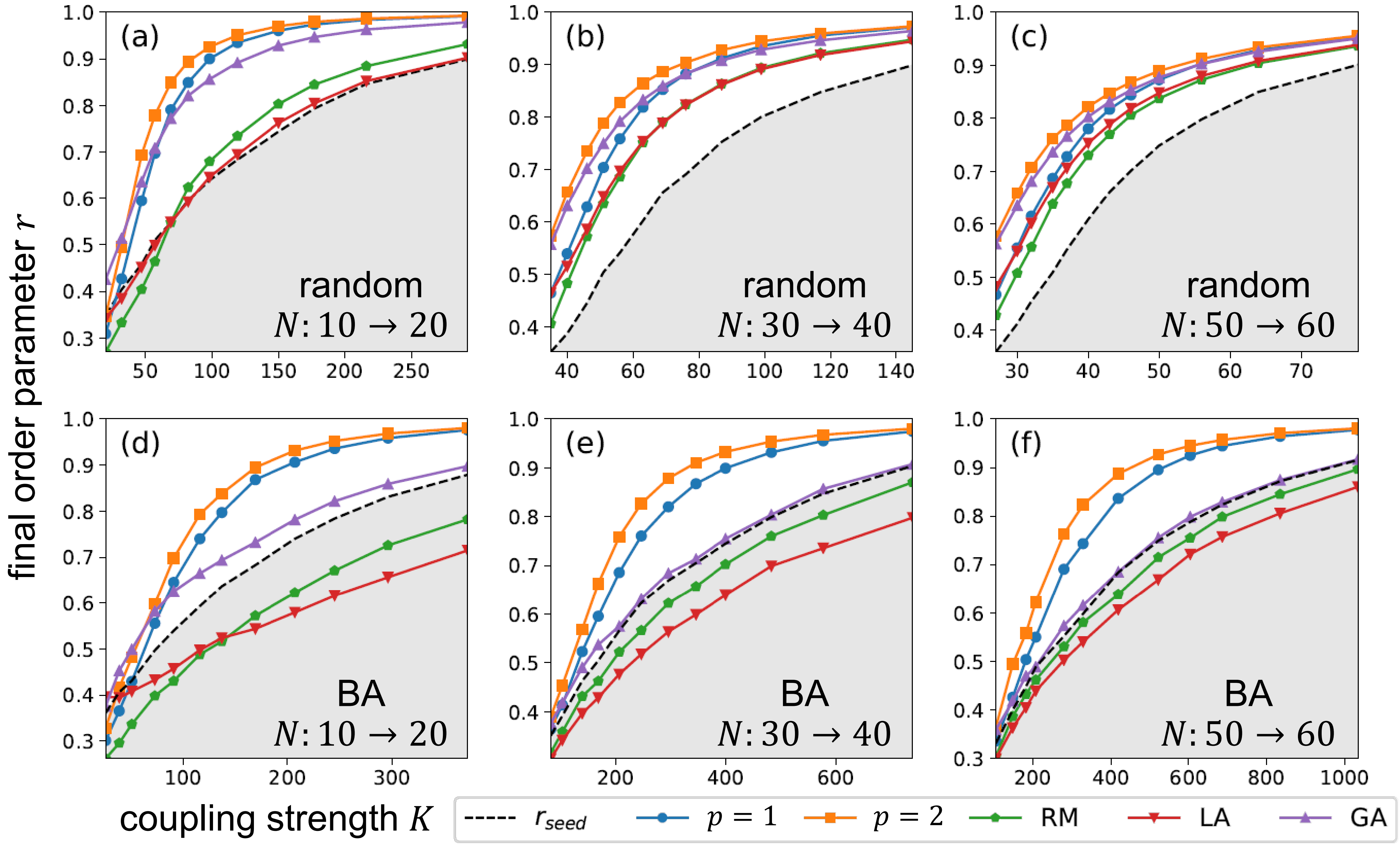
- random initial frequency $\omega_{i\geq 1} \sim \mathcal{N}(0,1)$ and preferential attachment rule

Global Average (GA)	Local Average (LA)	Random Method (RM)
$\omega_0 = \frac{1}{N}\sum_{i=1}^{N}\omega_i \leftarrow O_{p=0}$	$\omega_0 = \frac{1}{k_0}\sum_{i=1}^{N}a_{0,i}\omega_i$	$\omega_0 \sim \mathcal{N}(0,1)$



Result – beyond linear regime

- order parameter in the increased system vs. coupling constant



Summary

- We can enhance synchronization in growing systems.
- The stronger enhancement takes place near the optimal natural frequency.
- We found approximations of the optimal frequency in the strong coupling limit.
- They outperform other average-based strategies in a wide range of coupling strength.
- The heterogeneous interaction leads average-based strategies to underperform.

Thank you for your attention!